Minimax Duality of Gaussian Vector Broadcast Channels

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Abstract — **This paper establishes a connection between the uplink-downlink duality of the Gaussian vector multiple-access channel and broadcast channel and the Lagrangian duality in minimax optimization. This new minimax duality allows the optimal transmit covariance matrix and the least-favorable noise for the broadcast channel to be characterized in terms of the dual variables. Further, it allows uplink-downlink duality to be generalized to broadcast channels with arbitrary linear constraints. In particular, it shows that the dual of a broadcast channel with individual per-antenna power constraint is a multiple-access channel with a diagonal uncertain noise.**

I. Introduction

The sum capacity of a Gaussian vector broadcast channel $\mathbf{Y_i} = H_i\mathbf{X} + \mathbf{Z_i}$ is the solution to the following minimax problem [1] [2] [3]:

$$C_{BC} = \max_{S_{xx}} \min_{S_{zz}} \frac{1}{2} \log \frac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|}, \tag{1}$$

where the maximization is subject to a power constraint $\text{trace}(S_{xx}) \leq P$ and the minimization is subject to the constraint that the diagonal blocks of S_{zz} are identity matrices. (Here, $H^T = [H_1^T \cdots H_K^T]$.) Curiously, this sum capacity is also equal to the sum capacity of a dual multiple access channel, $\mathbf{Y'} = H_i^T\mathbf{X'_i} + \mathbf{Z'_i}$, where the roles of inputs and outputs are reversed and the channel matrix is transposed [2] [3]:

$$C_{MAC} = \max_{\Sigma_{xx}} \frac{1}{2} \log |H^T\Sigma_{xx}H + I|. \tag{2}$$

The maximization here is over the block-diagonal covariance matrix Σ_{xx} with a sum power constraint $\text{trace}(\Sigma_{xx}) \leq P$. The purpose of this paper is to illustrate that uplink-downlink duality is equivalent to Lagrangian duality in minimax optimization. This new interpretation not only gives rise to fast numerical algorithms for computing the broadcast channel sum capacity, but also allows uplink-downlink duality to be generalized to broadcast channels with arbitrary linear covariance constraints.

II. Main results

Theorem 1 *The following two minimax problems*

$$\max_{S_{xx}} \min_{S_{zz}} \quad \frac{1}{2} \log \frac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|} \tag{3}$$

$$\text{subject to} \quad \text{tr}(S_{xx}Q_x) \leq 1 \quad \text{and} \quad \text{tr}(S_{zz}Q_z) \leq 1 \tag{4}$$

and

$$\max_{\Sigma_{xx}} \min_{\Sigma_{zz}} \quad \frac{1}{2} \log \frac{|H^T\Sigma_{xx}H + \Sigma_{zz}|}{|\Sigma_{zz}|} \tag{5}$$

$$\text{subject to} \quad \text{tr}(\Sigma_{xx}\Psi_x) \leq 1 \quad \text{and} \quad \text{tr}(\Sigma_{zz}\Psi_z) \leq 1 \tag{6}$$

are duals of each other in the following sense:

- The optimal dual variable λ_x in the maximization part of (3) is the optimal dual variable ν_z in the minimization part of (5).
- The optimal dual variable λ_z in the minimization part of (3) is the optimal dual variable ν_x in the maximization part of (5).
- The optimal (S_{xx}, S_{zz}) in (3) is related to the constraints of (5) by $(S_{xx}, S_{zz}) = (\nu_z\Psi_z, \nu_x\Psi_x)$.
- The optimal $(\Sigma_{xx}, \Sigma_{zz})$ in (5) is related to the constraints of (3) by $(\Sigma_{xx}, \Sigma_{zz}) = (\lambda_z Q_z, \lambda_x Q_x)$.
- The optimum solution of (3) is the same as the optimum solution of (5).

This minimax duality is established based on the Karush-Kuhn-Tucker (KKT) conditions for the minimax optimization problem. When applied to the Gaussian broadcast channel, this new interpretation of duality reveals that the optimal transmit covariance matrices in the dual multiple-access channel are in fact the dual optimization variables in the broadcast channel. Thus, uplink-downlink duality is equivalent to Lagrangian duality in optimization.

The derivation of duality via minimax optimization also allows uplink-downlink duality to be generalized to Gaussian vector broadcast channels with arbitrary linear input constraints. In particular, for a multi-antenna broadcast channel with individual per-antenna power constraint, the dual channel turns out to be a multiple-access channel with uncertain noise:

Theorem 2 *The sum capacity of a Gaussian multi-antenna broadcast channel with individual per-antenna transmit power constraints $S_{xx}(i,i) \leq P_i$ is the same as the sum capacity of a dual multiple-access channel with a sum power constraint and with a diagonal and uncertain noise:*

$$\begin{aligned} \min_{\Sigma_{zz}} \max_{\Sigma_{xx}} \quad & \frac{1}{2} \log \frac{|H^T\Sigma_{xx}H + \Sigma_{zz}|}{|\Sigma_{zz}|} \\ \text{s.t.} \quad & \Sigma_{xx}, \Sigma_{zz} \text{ are diagonal} \\ & \text{tr}(\Sigma_{xx}) \leq 1 \text{ and } \sum_i P_i \Sigma_{zz}(i,i) \leq 1, \end{aligned} \tag{7}$$

As the dual variables are in a lower dimensional space, the duality relation leads to the efficient numerical computation of sum capacity for the broadcast channel with per-antenna power constraints.

References

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