# On Sampling-controlled Stochastic Approximation[1]


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**Abstract**

In the general area of optimization under uncertainty, there are a large number of applications which require finding the 'best' values for a set of control variables or parameters and for which the only data available consist of measurements prone to random errors. Stochastic approximation provides a method of handling such noise or randomness in data; it has been widely studied in the literature and used in several applications. In this paper, we examine a new class of stochastic approximation procedures which are based on carefully controlling the number of observations or measurements taken before each computational iteration. This method, which we refer to as Sampling-controlled Stochastic Approximation, has advantages over standard stochastic approximation such as requiring less computation and the ability to handle bias in estimation. We address the growth rate required of the number of samples and prove a general convergence theorem for this new stochastic approximation method. In addition, we present applications to optimization and also derive a sampling-controlled version of the classic Robbins-Munro algorithm.



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# 1 Introduction

Stochastic approximation refers to a general technique for augmenting deterministic iterative algorithms in order to handle noise in the inputs. In the case of optimization, for example, there are several algorithms [26, 33] that cause a collection of control variables to iteratively move towards an optimum value using values of some function (inputs) at each step. When these values cannot be analytically computed and, instead, must be *estimated*, stochastic approximation presents a viable method of using *measurements* (subject to random errors) in order to reach the optimum.

Ever since the introduction of the now classic Robbins-Munro algorithm [31], the stochastic approximation technique has found several applications [11, 21, 27] from bio-assay [43] to theories of learning [22] and has received wide attention in the literature [2, 3, 7, 9, 12, 16, 21, 25, 35, 43]. An important theoretical concern in several research efforts is the convergence of the resulting algorithms to 'desired' values. This is accomplished, in the usual manner of stochastic approximation, by prescribing a sequence of *decreasing stepsizes* [21], one for each computational step of the algorithm. In this case it can be shown that, for a wide class of algorithms, convergence to the desired value is obtained with probability one [3, 7, 21, 25]. We refer to this method of augmenting an algorithm as *stepsize-controlled iteration.*

In this paper, we examine a new class of stochastic approximation procedures that achieve algorithmic convergence through repeated sampling of inputs while using a *fixed* stepsize. This procedure of adapting an algorithm in order to deal with noise in the inputs has advantages over the standard method of decreasing stepsizes such as requiring less computation [35] and handling inherent bias in measurements. In terms of computational advantage, it has been experimentally observed [35] that in several situations the new procedure performs better than stepsize-controlled methods. Since the procedure relies on increased sampling before each successive algorithmic step, there are fewer algorithmic computations made in a real-time than in a stepsize-controlled method, which usually computes with each sample taken. In addition, the new stochastic approximation scheme allows convergence results under somewhat different and, in some respects, weaker assumptions than those required for decreasing stepsize algorithms.

The ability of a stochastic approximation scheme to handle bias is particularly important in queueing systems and simulation methodology [9, 11, 12, 24, 27] since several estimators associated with queues are biased [14, 18, 39]. The stochastic approximation technique studied in this paper, referred to as *sampling-controlled iteration* was introduced by [34, 35] in the context of load balancing for computer systems. The convergence result in [34] required several strong conditions and an unnecessarily high rate of sampling. A similar approach has been independently proposed in [42], although in [42], both repeated sampling and decreasing stepsizes are used and, in addition to stronger assumptions, a convergence result is shown using a nonstandard definition of convergence.

When stochastic approximation is viewed as a combination of estimation and algorithmic iteration, our research may be viewed as addressing the practioner's problem of deciding between cautious but infrequent algorithmic iteration (with more accurate large-sample estimates) and frequent iteration (with smaller, 'noisier' estimates). Similar ideas that weigh the number of samples of inputs against decreasing stepsizes have been introduced by other research efforts in stochastic approximation [5, 16] as well as in other areas [1]. In [1], a fixed number of samples was used for an algorithm in the context of learning theory and only limited experimental results were shown to demonstrate superiority over related algorithms using single samples. In [5], a fixed number of iterations was considered (consequently, there was no convergence result) and the optimal selection of stepsizes was studied in order to minimize the mean-squared-error for the particular case of the Robbins-Munro algorithm [31]. These results, together with other examples of algorithmic speedup, are summarized in [16].

Our main contribution is a general convergence result. We devote some attention to the set of assumptions needed for convergence. One consequence is a convergence theorem with much weaker conditions than the one in [34]. In particular, the number of samples used at each iteration grows much more slowly. We demonstrate the utility of our method with applications to gradient-based optimization algorithms as well as an illustrative derivation of a sampling-controlled version of the classic Robbins-Munro algorithm. Our proofs, which are based on elementary upper bound results from Large Deviation Theory [41], are simpler than corresponding proofs for standard stochastic approximation.

In the next section, we consider a simple general recursion on a single variable in order to motivate and discuss our ideas. In Section 3 we present our convergence results for a general class of iterative algorithms on a multidimensional control variable. Finally, we present some applications in Section 4 before making our concluding remarks in Section 5.

## 2 Sampling-controlled Stochastic Approximation

For motivational purposes, we first examine the following simplified iteration on a single control variable:

$$x^{(n+1)} = x^{(n)} + ah(x^{(n)}) \tag{1}$$

where $\lim_{n\to\infty} x^{(n)} = x^*$; $n$ denotes the iteration number, $h$ is a real-valued function and $a$ is a real-valued stepsize. We refer to the above iteration as the *deterministic version* of the algorithm $h$. This framework, we note, is representative of several root-finding iterative algorithms over a single variable (such as Newton's Method [33]) and, in these cases, $h$ is computed analytically. In several applications, the evaluation step $h(x^{(n)})$ cannot be performed readily because it is computationally prohibitive or analytical formulae are simply not available. We consider the case in which the quantity $h(x^{(n)})$ is estimated based on random samples constituting measurements taken from the system of interest.

For each $x \in \mathbb{R}$ let $\left\{Y^k(x), k = 1, 2, ...\right\}$ be a family of random variables and define

$$\bar{Y}^L(x) = \frac{1}{L}\sum_{k=1}^{L} Y^k(x), \qquad L = 1, 2, ...$$

to be a point estimate [32] of $Y(x)$ based on $L$ independent, identically distributed (i.i.d.) random samples, $Y^1(x), ..., Y^L(x)$. Typically, $Y(x)$ represents a system quantity that is relatively easy to estimate whereas $h(x)$ is a more complex system property that is expressed as a function of simpler quantities such as $Y(x)$. Then, in order to estimate $h$, it becomes necessary to use an estimate, $g\left(\bar{Y}^L(x)\right)$, constructed from the estimate $\bar{Y}^L(x)$ of $Y(x)$, where $g$ is a function on the range of $\bar{Y}^L(x)$.

Consider the case in which $E\left[g\left(\bar{Y}^L(x)\right)\right] = h(x)$ and where we replace the fixed stepsize in equation (1) with decreasing stepsizes that satisfy the standard conditions [21, 31]: $a_n > 0, a_n \to 0$ and $\sum_{i=1}^{\infty} a_i = \infty$. Then, it is possible to show that [3, 7, 21, 25], under certain conditions, for fixed $L > 0$, the recursion defined by:

$$X^{(n+1)} = X^{(n)} + a_n g\left(\bar{Y}^L(X^{(n)})\right) \tag{2}$$

converges to $x^*$, i.e., $\lim_{n\to\infty} X^{(n)} = x^*$ *a.s.* (the notation *a.s.* is used to denote *almost surely* or convergence with probability one [32]). We refer to an iteration (related to $h$) involving random variables, such as $\bar{Y}^L(X^{(n)})$, as the *stochastic version* of algorithm $h$ and to the particular form of the stochastic version in (2) as the *stepsize-controlled* version of $h$. The role of the decreasing stepsizes is to reduce the noise introduced by estimating $h(X^{(n)})$ via $g\left(\bar{Y}^L(X^{(n)})\right)$. This is the approach taken in standard stochastic approximation [21] and we now distinguish it from our method of stochastic approximation.

In this paper, we focus on a particular class of *deterministic* recursions of the type in equation (1) in which algorithmic convergence to the desired value is obtained with a *fixed* stepsize. We note that there is a large class of useful numerical and optimization algorithms [26, 28, 33] in several recent applications [4, 10, 35, 37] which satisfy the above property. There are some deterministic recursions which require a decreasing stepsize and whose corresponding stochastic versions use the same decreasing stepsize in order to remove noise [21]; these recursions are not considered here. Deterministic recursions of the type in equation (1) use decreasing stepsizes only for the purpose of removing noise. This observation is central to our method of stochastic approximation: we replace noise removal through decreasing stepsizes with noise removal via increasing the number of samples taken per iteration.

In several applications, for example the load balancing problem in [34], the estimator $g\left(\bar{Y}^L(x)\right)$ may be biased [32], i.e., for any fixed $L$, $E\left[g\left(\bar{Y}^L(x)\right)\right] \neq h(x)$. It is known that most estimators associated with queueing systems suffer from this small-sample bias [14, 18, 39]. This bias is usually due to the nonlinearity of the function $g$ and the correlation present in a regenerative cycle of a queue. In these cases, although the sequence $X^{(n)}$ defined by equation (2) converges, it does not converge to the desired point, i.e., $\lim_{n\to\infty} X^{(n)} = x' \neq x^*$. More seriously, in the case of optimization problems such as load balancing [34], the limit $x'$ may not even be a *feasible* solution [34], i.e., may not satisfy given constraints on $x'$. Most often, $g\left(\bar{Y}^L(x^{(n)})\right)$ represents a measurement taken from a system after $x^{(n-1)}$ has been changed to $x^{(n)}$. In several situations [40], $h$ is a desired steady-state expectation and when $L$ is too small, the system is not given enough 'settling' time [40], thereby causing the estimator $g\left(\bar{Y}^L(x^{(n)})\right)$ to record effects of transient behavior. In this case, which is also representative of many queueing systems, we have $E\left[g\left(\bar{Y}^L(x^{(n)})\right)\right] \neq h(x^{(n)})$.

With this motivation, we consider an alternate form of stochastic approximation, in which the stepsize, $a$, remains fixed and, instead, the number of samples taken before iterating, $L(n)$, varies with the iteration number, $n$, and increases to infinity. We define the following *stochastic* iteration:

$$X^{(n+1)} = X^{(n)} + ag\left(\bar{Y}^{L(n)}(X^{(n)})\right) \tag{3}$$

where $L(n)$ is a sequence of positive integers such that $L(n) \to \infty$. Intuitively, as $L(n) \to \infty$, we would expect that for large $n$ the procedure tends to behave like its deterministic counterpart, equation (1). Now, if $g\left(\bar{Y}^{L(n)}(x)\right)$ is a *strongly consistent* estimator [32] of $h(x)$, i.e.,

$$\lim_{n\to\infty} g\left(\bar{Y}^{L(n)}(x)\right) = h(x) \;\; a.s.$$

(but not necessarily unbiased), then, as $n$ gets larger, the effects of bias are slowly removed. In this manner, we might obtain the desired convergence, $\lim_{n\to\infty} X^{(n)} = x^*$ $a.s.$.

In the next section, we prove a general convergence theorem for a multidimensional recursion of the type in equation (3). We derive sufficient conditions on the growth of $L(n)$ based on assumptions made about the estimators and the algorithm that guarantee almost sure convergence to a desired point, $x^*$.

# 3 Convergence

## 3.1 Definitions

In order to prove our convergence result we use the following notation:

- For each $x$ in a set $D \subset \mathbb{R}^K$ let $\{Y^1(x), Y^2(x), Y^3(x), ...\}$ be independent and identically distributed random variables taking values in $\mathbb{R}^p$ with finite common mean $Y(x)$ and define the sample averages $\{\bar{Y}^m(x), m = 1,2,3,...\}$ by

$$\bar{Y}^m(x) = \frac{1}{m}\sum_{k=1}^{m} Y^k(x), \qquad m = 1,2,3,... \tag{4}$$

  We will assume that $Y(x)$ is uniformly bounded for $x \in D$. Then, the Strong Law of Large Numbers [32] implies

$$\lim_{m\to\infty} \bar{Y}^m(x) = Y(x) \;\; a.s.$$

- For some continuous mapping $g : \mathbb{R}^p \to \mathbb{R}^K$, we define the mapping $h : D \to \mathbb{R}^K$ by

$$h(x) = g(Y(x)), \quad x \in D$$

  Then, since $g$ is continuous,

$$\lim_{m\to\infty} g\left(\bar{Y}^m(x)\right) = h(x) \;\; a.s.$$

- Finally, consider the following *fixed-stepsize* deterministic algorithm that produces the $\mathbb{R}^K$-valued iterates $\{x^{(1)}, x^{(2)}, ...\}$ through the recursion

$$x^{(n+1)} = x^{(n)} + aH\left(h(x^{(n)}), x^{(n)}\right) \tag{5}$$

  where the Borel mapping $H : \mathbb{R}^K \times D \to \mathbb{R}^K$ may or may not be continuous. We note that discontinuity is present in several algorithms of interest [10, 34] and is sometimes due to the fact that $x^{(n)}$ must be forced to remain in $D$, which is itself usually determined by inequality constraints.

We focus on deterministic recursions such that $\exists a > 0$ for which it is true that $\forall x^{(0)} \in D$:

$$\lim_{n\to\infty} x^{(n)} = x^* \tag{6}$$

where $x^*$ is the desired point of convergence. We will show that the *sampling-controlled stochastic* version of equation (5),

$$X^{(n+1)} = X^{(n)} + aH\left(g\left(\bar{Y}^{L(n)}(X^{(n)})\right), X^{(n)}\right) \tag{7}$$

converges to $x^*$ for all $X^{(0)} \in D$. In comparing equations (5) and (7), we observe that the function $h(x^{(n)})$ in equation (5) has been replaced by the corresponding estimator $g\left(\bar{Y}^{L(n)}(X^{(n)})\right)$ based on $L(n)$ i.i.d. samples at iteration $n$.

## 3.2 Assumptions and Results

We will assume throughout the paper that the process $X^{(n)}$ remains in a compact set $D \subset \mathbb{R}^K$. We note that it is typically necessary that $D$ be assumed compact in order to satisfy assumption $A3$, described below. This may have to be accomplished through some type of constraint mechanism [10, 33, 35], which we will assume to be incorporated into the function $H$. It is worth mentioning that in the stochastic case, additional care may be needed to ensure that $X^{(n)}$ remains in $D$ [35].

We observe that the process $X^{(n)}$ may visit some point in $D$ more than once, i.e., $X^{(n_1)} = X^{(n_2)} = x$ for some $x$ and $n_1 < n_2$. In this case, definition (4) should be understood to mean that, at $n_2$, different samples, independent of those taken at $n_1$ are used in the construction of the estimate $\bar{Y}^{L(n_2)}(X^{(n_2)})$. This could be accounted for in the notation by defining the estimate used at time $n$ to be

$$\bar{Y}^{L(n)}(X^{(n)}) = \frac{1}{L(n)} \sum_{k=j(n)+1}^{j(n)+L(n)} Y^k(X^{(n)}), \qquad \text{where } j(n) = \sum_{i=1}^{n} L(i).$$

We avoid the unnecessary complication of carrying around the extra notation in each such estimate with the implicit understanding that we will be using independent samples at different times $n$.

We will assume that the random variables $Y^k(x)$ have been constructed on a common probability space $(\Sigma, \mathcal{F}, P)$ in such a way that for $X^{(n)}$ defined through equation (7), if we define

$$S_n = \left(X^{(m)}, Y^k(X^{(m)}),\ k = 1, 2, ..., L(m);\ m = 0, 1, ..., n\right)$$

then, for each $n$ and Borel subsets $B_k$, $k = 1, 2, ..., L(n+1)$ of $R^p$,

$$\begin{aligned} &P\left[Y^k(X^{(n+1)}) \in B_k,\ k = 1, 2..., L(n+1) \,\middle|\, S_n, X^{(n+1)}\right] \\ &= P\left[Y^k(X^{(n+1)}) \in B_k,\ k = 1, 2..., L(n+1) \,\middle|\, X^{(n+1)}\right] \end{aligned}$$

The existence of the common probability space can be guaranteed by placing suitable measurability conditions on the distributions of $Y^k(x)$. We emphasize that the random variables $Y^k(x)$ need not be interpreted as as functions of $x$, but as random variables whose distributions are determined by $x$. For example, in the routing problem of section 4.2, the moments of $Y^k$ are functions of $x$.

**Assumptions:**

*A*1. *Existence of moment generating functions.* We assume that

$$\sup_{x \in D} E\left[\exp\left\langle \alpha, Y^j(x) - Y(x)\right\rangle\right]$$

is finite in some open neighborhood of $\alpha = (\alpha_1, ..., \alpha_p) = 0$ with a derivative at $\alpha = 0$.

*A*2. *Continuity of g.* We assume that the function $g$ is uniformly continuous in the domain of interest, i.e., in the union of the range of $Y^j(x)$ for all values of $x$.

*A*3. *Stability of convergence under perturbations.* We assume that the deterministic version given by equation (5) satisfies the following stability property. Given $\delta > 0$ there exists $N = N(\delta) < \infty$ and $\epsilon = \epsilon(\delta) > 0$ such that if the $\mathbb{R}^K$-valued iterates $z^{(n)}$ are returned through the perturbed recursion

$$z^{(n+1)} = z^{(n)} + aH\left(h(z^{(n)}) + \epsilon^n, z^{(n)}\right), \qquad n = 0, 1, ..., 2N-1 \text{ with } z^{(0)} \in D \tag{8}$$

and if $\sup_{0 \le n \le 2N} |\epsilon^n| < \epsilon$ (using any suitable norm, the sup norm, for example) then, for all $z^{(0)} \in D$,

$$\left|z^{(n)} - x^*\right| \le \delta, \qquad N \le n \le 2N$$

Intuitively, this last assumption characterizes the convergence behavior of the deterministic version under infinitesimally small perturbations by requiring the trajectory of the perturbed version, equation (8), to behave nearly as well as the unperturbed version, equation (5), for a certain fixed number of iterations ($N$ to $2N$), for small enough perturbations (smaller than $\epsilon$). As discussed in the next section, *A*3 allows for the possibility of the function $H$ being discontinuous.

**THEOREM 1:** There exists a lower semi-continuous convex function $L_Y(\beta) : \mathbb{R}^p \to [0, \infty]$ with the properties

(i) $L_Y(\beta) = 0$ if and only if $\beta = 0$.
(ii) The sets $\{\beta : L_Y(\beta) \le s\}$ are compact for all $s \in [0, \infty)$.
(iii) Given $\nu > 0$ and $s > 0$ there exists $M < \infty$ such that uniformly in $x \in D$ and closed $A$ satisfying $\inf_{\beta \in A} L_Y(\beta) \ge s$,

$$P\left[(\bar{Y}^m(x) - Y(x)) \in A\right] \le \exp\left(-(s - \nu)m\right)$$

for all $m \ge M$.

**PROOF:** (see Appendix).

**THEOREM 2:** There exists a lower semi-continuous function $L_g(x, \gamma) : \mathbb{R}^K \to [0, \infty]$ convex in $\gamma$ with the properties

(i) $L_g(x,\gamma) = 0$ if and only if $\gamma = 0$.
(ii) The sets $\{\gamma : L_g(x,\gamma) \le s\}$ are compact for each $x \in D$ and all $s \in [0,\infty)$.
(iii) Given $\nu > 0$ and $s > 0$ there exists $M < \infty$ such that uniformly in $x \in D$ and closed $A$ satisfying $\inf_{\gamma \in A} L_g(x,\gamma) \ge s$,

$$P\left[g\left(\bar{Y}^m(x)\right) - g\left(Y(x)\right) \in A\right] \le \exp\left(-(s-\nu)m\right)$$

for all $m \ge M$.
(iv) The function $L_g(x,\gamma)$ can be expressed in terms of $L_Y(\beta)$:

$$L_g(x,\gamma) = \inf\left\{L_Y(\theta - Y(x)) : \theta \in g^{-1}(g(Y(x)) + \gamma)\right\}, \quad x \in D, \gamma \in \mathrm{I\!R}^K$$

**PROOF:** (See Appendix).

**THEOREM 3:** Under assumptions $A1 - A3$, the stochastic version, equation (7), converges to $x^*$, i.e.,

$$\lim_{n\to\infty} X^{(n)} = x^* \ \ a.s.$$

if $L(n) = c_n \log n$ where $c_n \to \infty$ as $n \to \infty$.

**PROOF:** (See Appendix).

### 3.3 Discussion

The proof of our convergence result may be interpreted by some simple intuitive arguments. First, we recall a few facts from large deviation theory. For fixed $x$, the estimator $\bar{Y}^m(x)$ converges to the mean $Y(x)$ via the Strong Law of Large Numbers. In this case elementary Large Deviation Theory (Cramer's Theorem in [41]) provides a simple upper bound on the probability that $\bar{Y}^m(x)$ is found in any closed set $A$ not containing $Y(x)$ after $m$ samples:

$$P\left[\bar{Y}^m \in A\right] \le \exp\left(-c(A)m\right) \tag{9}$$

where $c(A) > 0$ (the large deviation rate [41]) is a number derived from $A$. Conversely, the Borel-Cantelli lemma [32] may be used to show that if an estimator satisfies equation (9), i.e., possesses a large deviation property, then it converges. Also, large deviation properties extend to continuous functions of $\bar{Y}^m(x)$, as in Theorem 2.

Next we observe that the recursion given by equation (7) may be viewed as an *estimator* for $x^*$ and hence if a large deviation property could be derived for $X^{(n)}$, the Borel-Cantelli lemma could be used in proving convergence. This would in fact be quite simple to achieve if the functions $H$ were also continuous. However, just as several algorithms satisfy this continuity property, many of those considered in the literature do not [10]. Hence we take a slightly different approach, which requires our use of the stability of convergence assumption. We have explicitly separated our study of the behavior of the estimator, $g$, from that of the algorithm, $H$, which may be discontinuous.

In deriving our results we have made use of assumption $A1$ for Theorem 1, $A2$ for Theorem 2 and $A3$ for Theorem 3 and we now discuss some of the implications of our assumptions. Assumption $A1$ is a statement about the behavior of the moments of $Y^m$ and is stronger than assuming simply the existence of the moments. However, we note that stronger assumptions have often been made in the case of stepsize-controlled iteration [7, 31], such as the existence of the moment generating function in $A1$ for all $\alpha$. In the case of gradient estimators for queues [30, 38], such as the one described in section 4.1 of this paper, we believe that if the queue is recurrent and if the moments of the length and number in a busy period satisfy the assumption, then the moments of the gradient estimator, being polynomially-bounded functions of the busy period variables, will also satisfy the assumption.

A necessary and sufficient condition for the smoothness assumption in $A1$ may be formulated as follows. Let

$$J(x,\alpha) = \log E\left[\exp\left\langle \alpha, Y^j(x) - Y(x)\right\rangle\right]$$

This is a convex function that is analytic in a neighborhood of $\alpha = 0$. Using $Y(x) = E\left[Y^j(x)\right]$, one may show $J(x,\alpha) \geq 0$ and $J(x,\alpha) = 0$ iff $\alpha = 0$. We now show that $A1$ holds iff there is a convex function $\bar{J}(\alpha)$ with $\bar{J}(0) = 0$ that is differentiable at $\alpha = 0$ and satisfies $J(x,\alpha) \leq \bar{J}(\alpha)$ for all $x \in D$ and $\alpha \in \mathbb{R}^p$. If this is the case then

$$0 \leq \sup_x J(x,\alpha) = \log \sup_x E\left[\exp\left\langle \alpha, Y^j(x) - Y(x)\right\rangle\right] \leq \bar{J}(\alpha)$$

which implies the differentiability. For example, if the $Y^j(x)$ are Gaussian with variances given by the symmetric matrices $a(x)$, then we may take

$$\bar{J}(\alpha) = \frac{1}{2}\alpha' a \alpha$$

where $a$ is any symmetric matrix satisfying $a(x) \leq a$ for all $x \in D$. If the moment generating functions exist, then the centering caused by subtracting the mean in the definition of $J(\alpha)$ typically ensures differentiability.

Assumption $A2$ is a simple continuity assumption on the construction of more complicated estimators, $g_i$, from the more fundamental estimators $\bar{Y}^m$. This assumption is easily verified for the gradient estimators of interest here [15, 30], as well those used in other recursions [31], and appears not to be a restrictive assumption.

Assumption $A3$, which is a condition on the convergence behavior of the deterministic version, equation (5), reduces to a characterization of the function $H$. Consider what would happen if this condition were not satisfied for some deterministic algorithm. Let $x^{(1)},...,x^{(n)}$ be a sequence generated by the algorithm starting at $x^{(0)}$ and let $z^{(1)},...,z^{(n)}$ be the sequence generated when infinitesimally small perturbations are added to each $z^{(i)}$. In this case, violation of assumption $A3$ implies that arbitrarily small perturbations can cause $x^{(n)}$ and $z^{(n)}$ to be radically different in the sense that, while $x^{(n)}$ generates 'good' values of the objective function, $z^{(n)}$ results in 'poor' values. We argue that such algorithms are likely to be rejected in practice since their convergence behavior can be very different for arbitrarily small measurement errors.

In the case of algorithms used in practice, we note that if the function $H$ is continuous then $A3$ is automatically satisfied and this includes several numerical procedures [26, 33]. Furthermore, when $H$ is not continuous [10], proofs of deterministic convergence usually consist of showing a strict improvement of the objective function and, in these cases, $A3$ would need to be proved separately, perhaps using ideas from the proof of deterministic convergence. We also mention that, when multiple optima are present, assumption $A3$ should be rephrased to imply that the perturbed version satisfies the property of producing arbitrarily close values of the objective function (for $N \leq n \leq 2N$). Note also that an algorithm that satisfies $A3$ may produce original and perturbed paths characterized by close $x$-values for large values of $n$, but that are very different for smaller values of $n$.

Finally, some comments about the growth rate of $L(n)$. Conditions on $L(n)$ arise naturally from our method of proof and the growth rate needed here is far slower than the linear growth required in [34]. In [42], decreasing stepsizes were used in addition to increased sampling and convergence was shown according to a nonstandard definition of convergence. We have demonstrated that a fixed stepsize is sufficient for strong convergence in the usual sense. We believe that $L(n) = c_n \log n$ is the slowest growth rate possible since with a slower growth rate such as $L(n) = \log n$ the reverse Borel-Cantelli lemma together with results on large deviation lower bounds can be applied to show lack of convergence. The reader is referred to [19] for an example in the case of stepsize-controlled iteration.

# 4 Applications

In this section we present two applications of our sampling-controlled methods to established algorithms. The first application, solely for illustrative purposes, is a sampling-controlled version of the classic Robbins-Munro algorithm [31] and the second application is to to a well-known optimization algorithm [10] that has been used in routing problems.

## 4.1 The Robbins-Munro Algorithm

In this section we examine a sampling-controlled analog of the Robbins-Munro algorithm [31] for cases in which the deterministic version converges with a fixed stepsize. In particular, we consider the following *deterministic* recursion on a single variable in a compact set $D \subset \mathbb{R}$:

$$x^{(n+1)} = x^{(n)} - a\left(f(x^{(n)}) - \theta\right) \tag{10}$$

where $f(x^*) = \theta$ and $x^*$ is the desired point of convergence, i.e., $\lim_{n\to\infty} x^{(n)} = x^*$ for all $x^{(0)} \in D$. In this case if we define the operator $Tx = x - a(f(x) - \theta)$, then in several instances [33] $T$ can be shown to be a contraction mapping [33] with a unique fixed point at $x^*$ and, therefore, $\lim_{n\to\infty} x^{(n)} = x^*$ in equation (10).

Next, let $\bar{Y}^m(x)$ be an estimator based on $m$ i.i.d. random samples, $Y^1(x), .., Y^m(x)$, such that

$$\bar{Y}^m(x) = \frac{1}{m}\sum_{j=1}^{m} Y^j(x)$$

and

$$\lim_{m\to\infty} \bar{Y}^m(x) = f(x)$$

Further, by letting $\bar{g}^{(n)} = \bar{Y}^{L(n)}(X^{(n)})$ we have the following *sampling-controlled stochastic* version:

$$X^{(n+1)} = X^{(n)} - a\left(\bar{g}^{(n)} - \theta\right) \tag{11}$$

Here, according to the framework described in the previous section, $Y(x) = f(x)$ and $g(y) = y$ so that $h(x) = f(x) - \theta$ and $H(h, x) = -h$. In order to ensure that $A3$ holds, it might be necessary to enforce compactness by restricting $X^{(n)}$ to an interval $[C_1, C_2]$ where $C_1 < x^* < C_2$. Then, if $A1$ is satisfied, we have, by Theorem 3, $\lim_{n\to\infty} X^{(n)} = x^*$.

### 4.2 A Gradient-based Routing Algorithm

We examine the application of our sampling-control technique to a well-known gradient-based hill-climbing algorithm, Gallager's algorithm [10], that has found applications in routing in computer networks [4, 6, 10], load balancing [23, 29, 35] as well as in the area of learning automata [36]. We focus on augmenting this algorithm into a stochastic approximation procedure using sampling-control. We note that the algorithm is gradient-based and thus, the *deterministic* version uses analytic formulas for the gradient at each iterative step. Furthermore, we observe that several methods for direct gradient estimation in queueing systems have recently received a great deal of attention in the literature [15, 30] and therefore, a *stochastic* version of Gallager's algorithm, using these gradient estimates, is of general interest in the above application areas.

We concern ourselves with applications in which direct gradient estimation is possible, including those which have already received some attention. These examples include the optimization of parameters in a queueing network [9, 15, 24], load balancing [20, 29, 34] and routing [6], some of which, we note, have considered the use of stepsize-control [9, 15, 20, 24]. In order to demonstrate the use of sampling-control, we examine a particular application - the optimization of arrival rates in a queueing network - and develop a sampling-controlled extension of Gallager's algorithm that uses estimates of the derivatives of queueing delays (with respect to these arrival rates) in order to minimize the overall mean delay. We will employ the *likelihood-ratio* estimation method in [14, 30] and describe how derivative estimates may be obtained and used in Gallager's gradient-based optimization algorithm.

Let us define the following notation associated with a single queue in a queueing network [30]:

$N_j$ = the number of customers served in the $j^{th}$ busy period [17].

$W_{ij}$ = the waiting time of the $i^{th}$ customer in the $j^{th}$ busy period.

$W_j = \sum_{i=1}^{N_j} W_{ij}$

$\lambda$ = the arrival rate to the queue.

$T_j$ = the duration of the $j^{th}$ busy period.

$D$ = the expected steady-state waiting time for the queue.

We assume that the arrival process is Poisson. For a fixed arrival rate, $\lambda$, we describe the estimation scheme of [30] and formally define the following likelihood-ratio estimators, each of which will be used to construct a more complex, derivative estimator:

$$\begin{aligned}
\bar{Y}_1^m(\lambda) &= \frac{1}{m}\sum_{j=1}^{m} W_j \\
\bar{Y}_2^m(\lambda) &= \frac{1}{m}\sum_{j=1}^{m} \left(\frac{N_j}{\lambda} - T_j\right) W_j \\
\bar{Y}_3^m(\lambda) &= \frac{1}{m}\sum_{j=1}^{m} \left(\frac{N_j}{\lambda} - T_j\right) N_j \\
\bar{Y}_4^m(\lambda) &= \frac{1}{m}\sum_{j=1}^{m} N_j
\end{aligned}$$

The steady-state waiting time [17] can be written as a function of several variables identified with a regeneration period and when the likelihood-ratio technique is applied to these relevant variables, we obtain the above formulas. The reader is referred to [30] for an intuitive explanation and thorough discussion of the these estimators, too lengthy to be provided here.

We assume that the probability structure governing the queues is such that samples obtained from different busy periods of the queue are independent [18]. Then, from this independence property and the strong law [30], we have,

$$\lim_{m\to\infty} \bar{Y}_k^m(\lambda) = E\left[\bar{Y}_k^1(\lambda)\right], \qquad 1 \le k \le 4$$

Next, let

$$\breve{g}\left(\bar{Y}^m(\lambda)\right) = \breve{g}\left(\bar{Y}_1^m(\lambda), ..., \bar{Y}_4^m(\lambda)\right) = \frac{\bar{Y}_2^m(\lambda)}{\bar{Y}_4^m(\lambda)} - \frac{\bar{Y}_3^m(\lambda)\bar{Y}_1^m(\lambda)}{\left(\bar{Y}_4^m(\lambda)\right)^2} \tag{12}$$

In [30], it is shown that $\breve{g}$, as defined above, is a strongly consistent estimator of the derivative of the mean delay of the queue with respect to the arrival rate, i.e.,

$$\lim_{m\to\infty} \breve{g}\left(\bar{Y}^m(\lambda)\right) = \frac{dD}{d\lambda} \; a.s. \tag{13}$$

However, this estimator is biased due to the nonlinearity of $\breve{g}$ [14]. Note also, that we have assumed knowledge of the arrival rate $\lambda$, in the construction of the estimator. In the case that the arrival rate is unknown, an estimate of $\lambda$ may be used in place [14, 29] of $\lambda$. The resulting estimator will still be consistent, i.e. equation (13) will be satisfied.

Now, in a queueing network composed of $K$ queues, an overall delay function can be defined [20] as a differentiable function, $C(D_1, ..., D_K)$ of the delays in each of the $K$ individual queues. In that case, derivative estimators $g_i$ for each derivative $\frac{\partial C}{\partial x_i}$, $1 \le i \le K$, may be expressed in terms of the components of the vector function $\breve{g}$ such that when the arrival rate, $\lambda(x)$, is a function of the control variable $x$:

$$\lim_{n\to\infty} g_i\left(\bar{Y}_{i1}^m(\lambda), ..., \bar{Y}_{i4}^m(\lambda)\right) = \frac{\partial C}{\partial x_i}$$

We now define the gradient algorithm in terms of $x^{(n)} = (x_1^{(n)}, ..., x_K^{(n)})$ where $x_i^{(n)}$ is the $i^{th}$ parameter or $i^{th}$ component of the multidimensional control variable $x^{(n)}$ and where $n$ denotes the iteration number. In keeping with the notation defined in the previous section we let each component $h_i(x)$ of the vector $h$ denote $\frac{\partial C}{\partial x_i}$. Then, the *deterministic* version of the algorithm will be completely specified by listing expressions for the components $H_i\left(h(x^{(n)}), x^{(n)}\right)$ of $H$ in the recursion given in equation (5). Let $M = \arg\min_{1\leq i\leq K} h_i(x^{(n)})$, i.e., $h_M(x^{(n)}) = \min_{1\leq i\leq K} h_i(x^{(n)})$, and define the following component functions, $H_i$, which characterize the deterministic algorithm $H$:

$$H_i\left(h_1(x^{(n)}), ..., h_K(x^{(n)}), x^{(n)}\right) = h_M(x^{(n)}) - h_i(x^{(n)}) \qquad \forall i \neq M$$

$$H_M\left(h_1(x^{(n)}), ..., h_K(x^{(n)}), x^{(n)}\right) = \sum_{1\leq i\leq K, i\neq M} h_i(x^{(n)}) - h_M(x^{(n)}) \tag{14}$$

It can be shown that [4, 10], under certain smoothness conditions on $C$, if $x^*$ is the unique optimum solution that minimizes the delay function $C$, then

$$\lim_{n\to\infty} x^{(n)} = x^*$$

when $x^{(n)}$ is defined by the recursion, $x^{(n+1)} = x^{(n)} + aH\left((h(x^{(n)}), x^{(n)}\right)$ for small enough $a$.

For the applications in which such gradient-based optimization algorithms are employed, there are usually constraints on the control variables $x^{(n)}$. In the routing [10] and load balancing [35] problems, for example, the control variables represent probabilities and hence $x^{(n)}$ is restricted to the simplex defined by: $\sum_{i=1}^K x_i^{(n)} = 1$ and $0 \leq x_i^{(n)} \leq 1$, $1 \leq i \leq K$. The algorithm defined in equation (14) satisfies the first (equality) constraint and is easily modified [10] in order to handle the latter (inequality) constraint. Below, we directly present the *stochastic* version of the algorithm modified to handle both these types of constraints. In order to simplify the notation needed we define

$$\bar{g}_i^{(n)} = g_i\left(\bar{Y}_{i1}^{L(n)}(X_i^{(n)}), ..., \bar{Y}_{i4}^{L(n)}(X_i^{(n)})\right)$$

and obtain the *stochastic* version (in terms of the vector components of $X^{(n)}$) as:

$$X_i^{(n+1)} = X_i^{(n)} + a\left(\bar{g}_M^{(n)} - \bar{g}_i^{(n)}\right) \qquad \forall i \in A^{(n)}$$

$$X_M^{(n+1)} = X_M^{(n)} + a\left(\sum_{i\in A^{(n)}} \bar{g}_i^{(n)} - \bar{g}_M^{(n)}\right) \tag{15}$$

$$X_i^{(n+1)} = X_i^{(n)} \qquad \forall i \notin A^{(n)}$$

where $A^{(n)} = \left\{i \neq M \,\middle|\, X_i^{(n)} + a\left(\bar{g}_M^{(n)} - \bar{g}_i^{(n)}\right) > 0\right\}$, $M = \arg\min_i \bar{g}_i^{(n)}$.

From equation (12), we observe that assumption $A2$ of section 3 is satisfied, i.e., $g$ is a continuous function. Therefore, if we assume the existence of the moment generating function for the estimators $\bar{Y}_k^m(x)$ then Theorems 1 and 2 are easily proven. Assuming that $A3$ is also satisfied, we may apply Theorem 3 to conclude that

$$\lim_{n\to\infty} X^{(n)} = x^*$$

Since the functions $H_i$ are discontinuous, the verification of $A3$ for this algorithm depends on the application and, in particular, on the function $C$.

## 5 Conclusions

In this paper, we have presented a new stochastic approximation scheme, *sampling-controlled iteration*, that has the important capability of handling bias in estimators. Estimator bias is found in several problem areas and, consequently, sampling-controlled stochastic approximation finds application in related optimization, numerical and control algorithms. Since other research efforts [35] have also experimentally determined the usefulness of this method, we conclude that sampling-control offers an attractive alternative to the standard stochastic approximation method.

Our main contribution was a general convergence theorem which explored the bounds on growth rates of sampling required for convergence. Our proof methods, which are simple and accessible, use elementary large deviation bounds and easily determine the conditions on the rate of sampling. In addition to the convergence result, we apply our methods to an optimization problem in a queueing system. Furthermore, this being a relatively new procedure for stochastic approximation, we illustrate our method by presenting a sampling-controlled version of the classic Robbins-Munro algorithm. For future work, it would be interesting to compare the response to changing system statistics of modified stepsize- and sampling-controlled algorithms. Finally, we observe that although our focus has been on theoretical issues of convergence, much needs to be understood in terms of the application and empirical study of stochastic approximation methods.

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## Appendix:

**PROOF OF THEOREM 1:** Let

$$J(\alpha) = \sup_{x \in D} \log E\left[\exp\left\langle \alpha, Y^j(x) - Y(x)\right\rangle\right].$$

The function $J$ is convex [8] and by assumption, $\frac{\partial J(\alpha)}{\partial \alpha}$ exists at $\alpha = 0$; this can be computed and found to be $\frac{\partial J(\alpha)}{\partial \alpha} = 0$ . Define $L_Y$ as the Legendre-Fenchel transform [8] of $J$:

$$L_Y(\beta) = \sup_{\alpha} [< \alpha, \beta > -J(\alpha)].$$

Now, $J(0) = 0$ and $\frac{\partial J(\alpha)}{\partial \alpha} = 0$ imply $L_Y(\beta) = 0$ iff $\beta = 0$. The fact that $J$ is finite in a neighborhood of $\alpha = 0$ implies $L_Y(\beta) \to \infty$ as $|\beta| \to \infty$ [8]. We have therefore shown

(i) $L_Y(\beta) = 0$ iff $\beta = 0$.
(ii) The sets $\{\beta : L_Y(\beta) \leq s\}$ are compact for all $s \in [0, \infty)$.

Next we need to show part (iii) of the theorem using an adaptation of Cramer's large deviation result for i.i.d. random vectors [41].

Let $A$ be a closed subset of $\mathbb{R}^p$ with $0 < L_Y(A) < \infty$ (where $L_Y(A) = \inf_{x \in A} L_Y(x)$). Since $L_Y$ is an action functional, the covering lemma [8] implies that for every $0 < \epsilon < L_Y(A)$, there exist non-zero points $\alpha_1, ..., \alpha_r$ in $\mathbb{R}^p$ such that

$$A \subset \cup_{i=1}^{r} H_+(\alpha_i, L_Y(A) - \epsilon)$$

(using the notation in [8]). Consequently, for each $x$ in $D$ we have

$$\begin{aligned} P\left[\bar{Y}^m(x) - Y(x) \in A\right] &\leq \sum_{i=1}^{r} P\left[< \alpha_i, \bar{Y}^m(x) - Y(x) > -J(\alpha_i) \geq L_Y(A) - \epsilon\right] \qquad (16) \\ &\leq \sum_{i=1}^{r} E\left[\exp < \alpha_i, m(\bar{Y}^m(x) - Y(x)) >\right] \exp[-m(J(\alpha_i) + L_Y(A) - \epsilon)] \\ &= \sum_{i=1}^{r} \exp[mJ(\alpha_i, x)] \exp[-m(J(\alpha_i) + L_Y(A) - \epsilon)] \quad , m = 1, 2, ... \end{aligned}$$

whence

$$\sup_{x \in D} P\left[\bar{Y}^m(x) - Y(x) \in A\right] \leq r \exp[-m(L_Y(A) - \epsilon)] \qquad m = 1, 2, ... \qquad (17)$$

where the last inequality uses the fact that $J(\alpha, x) \leq J(\alpha)$ for all $\alpha$ and $x$. Since the integer $r$ and the points $\alpha_1, ..., \alpha_r$ all depend on $A$ and on $\epsilon$, additional arguments are needed in order to validate Part (iii) from (17). First note that (17) implies

$$\overline{\lim}_m \frac{1}{m} \log \left[ \sup_{x \in D} P\left[\bar{Y}^m(x) - Y(x) \in A\right] \right] \leq -L_Y(A) \qquad (18)$$

since $\epsilon > 0$ is arbitrary. As a result, for every closed set $A$ and every $\nu > 0$, there exists $m^* = m^*(\nu, A)$ such that for all $m \geq m^*$,

$$\sup_{x \in D} P\left[\bar{Y}(x) - Y(x) \in A\right] \leq \exp[-m(L_Y(A) - \nu)]. \qquad (19)$$

This yields a large deviation upper bound which is uniform in $x$.

To assert the uniformity in the closed sets $A$, we define the set $A_s = \{\beta : L_Y(\beta) \geq s\}$ and $A_s^*$ to be the closure of $A_s$. If $L_Y$ is finite for all $\beta$, the continuity (via convexity) of $L_Y$ implies that $A_s$ is closed, and the resulting uniformity follows since $\inf_{\beta \in A} L_Y(\beta) \geq s$ implies $A \subseteq A_s$.

Next consider the alternative case. If $L_Y(\beta) = +\infty$ for some values of $\beta$ it is possible to show that the probability that $\bar{Y}^m(x) - Y(x)$ is in $\{\beta : L_Y(\beta) = \infty\}$ is identically equal to zero for all $m$ and $x$. Therefore, when estimating $P\left[\bar{Y}^m(x) - Y(x) \in A\right]$ we obtain the same upper bound if we replace $A$ by its intersection with $\{\beta : L_Y(\beta) < \infty\}$, and we may then use the argument for the case when $L_Y(\beta)$ is finite for all values of $\beta$. □

**PROOF OF THEOREM 2:** Theorem 2 follows from Theorem 1 via the upper bound part of the 'contraction principle' (see pages 5 and 6 in [41]). For the reader's convenience we will sketch the proof of part (iii). Given $x \in D$ and closed $A$ satisfying $\inf_{\gamma \in A} L_g(x, \gamma) \geq s$, let

$$B_x = \{\theta : g(\theta) = \gamma + g(Y(x)) \text{ for some } \gamma \in A\}$$

Then $g\left(\bar{Y}^m(x)\right) - g\left(Y(x)\right) \in A$ implies $\bar{Y}^m(x) \in B_x$. Therefore

$$P\left[g(\bar{Y}^m(x)) - g(Y(x)) \in A\right] \leq P\left[\bar{Y}^m(x) - Y(x) \in \{\beta : \beta = \theta - Y(x), \theta \in B_x\}\right]$$

We also have

$$\begin{aligned}\inf_{\{\beta=\theta-Y(x):\theta\in B_x\}} L_Y(\beta) &= \inf_{\theta\in B_x} L_Y(\theta-Y(x))\\ &= \inf\{L_Y(\theta-Y(x)):g(\theta)=\gamma+g(Y(x)),\gamma\in A\}\\ &= \inf\{L_g(x,\gamma):\gamma\in A\}\end{aligned}$$

The upper bound together with the asserted uniformity in $x$ and in $A$ now follow from the conclusion of Theorem 1 $\square$

**PROOF OF THEOREM 3:** We will prove that $P\left[\left|X^{(n)}-x^*\right|>\delta\ i.o.\right]=0$ for all $\delta>0$ (here, the notation $i.o.$ refers to 'infinitely often' [32]). Fix $\delta>0$. Define

$$E_i=\left\{\left|X^{(n)}-x^*\right|>\delta \text{ for some } n,\ iN\le n\le iN+N\right\}$$

where $N$ is from assumption $A3$ for our chosen value of $\delta$. Clearly,

$$\left\{\left|X^{(n)}-x^*\right|>\delta\ i.o.\right\}=\{E_i\ i.o.\}\quad a.s. \tag{20}$$

where the first event in equation (20) is $i.o.$ in $n$ whereas the second is $i.o.$ in $i$. Next, in the manner of assumption $A3$, we may write the stochastic version as

$$X^{(n+1)}=X^{(n)}+aH\left(h(X^{(n)})+\epsilon_n,X^{(n)}\right),\qquad X^{(0)}\in D \text{ and } n=1,2,...$$

with

$$\epsilon_n=g\left(\bar{Y}^{L(n)}(X^{(n)})\right)-h(X^{(n)})=g\left(\bar{Y}^{L(n)}(X^{(n)})\right)-g\left(Y(X^{(n)})\right)$$

Therefore, the stability assumption, $A3$, implies that uniformly in $x\in D$, with $\epsilon$ from $A3$,

$$\left[\left|g\left(\bar{Y}^{L(n)}(X^{(n)})\right)-h(X^{(n)})\right|<\epsilon(\delta),\ iN-N\le n\le iN+N\right]\subseteq\left[\left|X^{(n)}-x^*\right|\le\delta,\ N\le n\le iN+N\right].$$

We now condition on the value of $X^{(n)}$ at iteration $iN-N$ in order to obtain a bound on the probability of $E_i$.

$$\begin{aligned}P\left[E_i\left|X^{(iN-N)}=x\right.\right] \le\ & P\left[\left|g\left(\bar{Y}^{L(n)}(X^{(n)})\right)-h\left(X^{(n)}\right)\right|>\epsilon\right.\\ &\qquad\left.\text{for some } n,\ iN-N\le n\le iN+N\left|X^{(iN-N)}=x\right.\right]\\ \le\ & \sum_{n=iN-N+1}^{iN+N}P\left[\left|g\left(\bar{Y}^{L(n)}(X^{(n)})\right)-h\left(X^{(n)}\right)\right|>\epsilon\left|X^{(iN-N)}=x\right.\right]\end{aligned}$$

But Theorem 2, via a simple conditioning argument, provides an upper bound on each term in the summation that is uniform in $x$. Hence

$$\begin{aligned}P\left[E_i\right] &\le \sum_{n=iN-N+1}^{iN+N}\exp\left(-c\left[L(n)\right]\right)\\ &\le \sum_{n=iN-N+1}^{iN+N}\exp\left(-c\left[\inf_{iN-N\le n\le iN+N}L(n)\right]\right)\\ &\le 2N\exp\left(-c\left[\inf_{iN-N\le n\le iN+N}L(n)\right]\right)\end{aligned}$$

for $i$ sufficiently large, where

$$c = \frac{1}{2} \inf_{|\gamma|>\epsilon, x \in D} L_g(x, \gamma) > 0.$$

To see that $c$ is in fact strictly positive, we note that $c = 0$ implies the existence of $x_i \to x \in D$ as $i \to \infty$ and $\gamma_i$ satisfying $|\gamma_i| > \epsilon$, such that $L_g(x_i, \gamma_i) \to 0$. Since $L_y(\beta) = 0$ only at $\beta = 0$, this implies the existence of $\theta_i$ such that $(\theta_i - Y(x_i)) \to 0$ and $|g(\theta_i) - g(Y(x_i))| \geq \epsilon$. This contradicts the assumed continuity of $g$.

We have broken up the sequence of iterations into blocks of size $N$ and placed a bound on the probability of $E_i$ via the Large Deviation result, Theorem 2. Now, we return to equation (20) in order to study the probability of $E_i$ occurring infinitely often. We have taken the sum below from $n = 0$ for simplicity in notation, with the understanding that $c_0 \log 0$ is defined to be zero. If $\sum_{i=0}^{\infty} P[E_i] < \infty$ then, by the Borel-Cantelli lemma [32], this implies that $P[E_i \; i.o.] = 0$. Now,

$$\begin{aligned}
\sum_{i=0}^{\infty} P[E_i] &\leq \sum_{i=0}^{\infty} 2N \exp\left(-c\left[\inf_{iN-N \leq n \leq iN+N} L(n)\right]\right) \\
&\leq 2N \sum_{n=0}^{\infty} \exp\left(-c\left[L(n)\right]\right) \\
&= 2N \sum_{n=0}^{\infty} n^{-c[c_n]}
\end{aligned}$$

which is a finite sum since the product $c[c_n] > 2$ for large enough $n$ and hence convergence follows. $\square$