Some Theorems on Conditional Pasta: A Stochastic Integral Approach

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Abstract

In this note, the "conditional pasta theorem" of van Doorn and Regterschot is proved under more general conditions and via a stochastic integral approach of independent interest. The limit theorems here require less restrictive assumptions and are shown for a wider class of arrival processes. Applications are also presented for particular cases of continuous and discrete-time arrivals.

1 Introduction

The purpose of this paper is to extend the recent work of van Doorn and Regterschot on "Conditional Pasta" [5] to a larger class of arrival processes and for a larger class of functionals. For recent related work on PASTA we refer the reader to the following articles: [2, 3, 8, 9, 7]. Our methods, however, are quite different from these authors and are based on a stochastic integral representation of the difference between two time averages which is of independent interest, cf., equation 2. One immediate consequence of this representation, in addition to the other extensions and generalisations in this paper, is our inequality (5) which sharpens and generalises inequality (7) of Wolff–see [12].

Notational conventions:

(i) $N(t)$ denotes the arrival process to some queueing system the state of which is denoted by $Y(t)$.
(ii) $\mathcal{F}^Y(t) := \sigma(Y(s), 0 \leq s \leq t)$
(iii)$N(t)$ is adapted to a sigma–field denoted by $\mathcal{F}(t)$.
(iv) We shall use the terms "arrival process" and "counting process" interchangeably. For all unexplained terminology we refer the reader to [1, 6].

In a typical application $N(t)$ denotes the (Poisson) arrival process to an M/G/1 queue and $Y(t)$ represents either the number in queue or the virtual waiting time process. More generally, one could assume that $N(t)$ is a Markov Modulated Poisson Process(MMPP) which is characterized by the requirement that its instantaneous arrival rate $\lambda(t)$ is determined by the state of a continuous time Markov chain $X(t)$ which is also called the *environment process.* More precisely, we assume $\lambda(t) = f(X(t))$ (or even $\lambda(t) = f(t, X(t))$, one of the non-stationary generalizations in this paper) where $X(t)$ denotes a Markov process with n states and state space denoted by $\mathcal{S} = (1, 2 \ldots n)$ and f is a function defined on $\mathcal{S}$ with $f(i) = \lambda_i > 0, i = 1, \ldots, n$. It is interesting to note that in deriving the PASTA and conditional PASTA properties we need only assume that the arrival process is what Liptser–Shiryaev ([6], Vol.II, p.249) call a *Poisson–type* process, i.e.,

$$M(t) = N(t) - \int_0^t \lambda(s)\, ds \text{ is an } \mathcal{F}(t) \text{ martingale.} \tag{1}$$

Definition 1 *The function $\Lambda(t) = \int_0^t \lambda(s)\, ds$ is called the "compensator" of the counting process $N(t)-$ see [6], Definition 2, p.239, vol.II.*

Remark 1 *Notice that $\Lambda(t)$ is (i)absolutely continuous, (ii) strictly increasing with $\lim_{t\to\infty} \Lambda(t) = +\infty$, a.s. (iii) bounded on finite intervals with bound $\Lambda(t) \leq L \cdot t$ where $L = \max\{\lambda_i : 1 \leq i \leq n\}$. For these properties to hold it is only necessary to assume that $\lambda(t)$ is bounded, measurable and bounded away from zero, i.e., $L \geq \lambda(t) \geq \delta > 0$.*

We want to compare the proportion of time that the process $Y(t) \in B$ with the corresponding proportion of customers who, upon arrival, see $Y(t) \in B$. The key observation is that the difference between these two quantities can be expressed as a stochastic integral with respect to the square integrable martingale $M(t)$ defined in equation (1). More precisely, let $U(t) = I_B(Y(t-))$, where

$I_B(x) = 1, x \in B, I_B(x) = 0$, otherwise. Thus,

$$W(t) = \int_0^t U(s)\, d\Lambda(s)$$

is a random weighted average of the amount of time during $[0,t]$ that $Y(t) \in B$. In the special case of the Poisson process $W(t) = \lambda \times$ the amount of time during $[0,t]$ that $Y(t) \in B$. Similarly ,

$$S(t) = \int_0^t U(s)\, dN(s)$$

counts the number of times that an arrival sees $Y(t) \in B$ during the interval of time $[0,t]$. Next observe that $R(t) = S(t) - W(t)$ can be written as a stochastic integral with respect to the square integrable martingale $M(t)$. More precisely, it is easy to verify that:

$$\begin{aligned} R(t) &= \int_0^t U(s)\, dM(s) && (2) \\ &= \int_0^t U(s)\, dN(s) - \int_0^t U(s)\, d\Lambda(s) \\ &= S(t) - W(t) \end{aligned}$$

Notice that the random function $U(t)$ is left continuous and is therefore *predictable* with respect to the $\sigma-$ field $\mathcal{F}(t)$. See [1], [6] for the definition of "predictable process."

Lemma 1 *Let $U(t)$ be a predictable process satisfying the condition:*

$$E\left(\int_0^T |U(s)|^2\, d\Lambda(s)\right) < \infty, \text{ for every } T > 0.$$

Then the process $\{R(t), \mathcal{F}(t), t \geq 0\}$, defined by the stochastic integral above, is a martingale.

Proof: This is an immediate consequence of the general theory of stochastic integration with respect to square integrable martingales [6], chap.5, section 5.4.

Our main result is that Wolff's lemma 2 is still valid for the much larger class of arrival processes considered here. More precisely, we have the following result:

Theorem 1 *Assume $N(t)$ is an arrival process with intensity function $\lambda(t)$ satisfying conditions (i), (ii), (iii) of remark (1). Then*

$$\lim_{t\to\infty} \frac{R(t)}{t} = 0, \text{ with probability one.} \quad (3)$$

2 Proof of Theorem

Theorem 1 is a special case of the following strong law of large numbers for martingales of the form:

$$R(t) = \int_0^t v(s)\, dM(s),$$

where $M(t)$ is a right-continuous square integrable martingale whose "compensator", denoted by $< M > (t)$, has the representation

$$< M > (t) = A(t) = \int_0^t a(s)\, ds \text{ with } a(t) \geq 0.$$

In addition we assume that $v(t)$ and $a(t)$ are both bounded i.e.,

$$\|v\| = \sup_{t \geq 0} \mid v(t) \mid < \infty \text{ and } \|a\| = \sup_{t \geq 0} \mid a(t) \mid < \infty.$$

Theorem 2 *Suppose $M(t)$ is a right-continuous square integrable martingale with compensator of the form*

$$< M > (t) = A(t) = \int_0^t a(s)\, ds \text{ with } a(t) \geq 0, \|a(t)\| \text{ bounded.}$$

Let $v(s)$ be a bounded predictable process.Then $R(t)$ defined above is a square integrable martingale and

$$\lim_{t \to \infty} \frac{R(t)}{t} = 0, \text{ with probability one.}$$

Remark 2 *This is a simple extension of the strong law of large numbers for discrete-time square integrable martingales to the continuous time case – see [11],p.138. It is most likely a special case of a more general result already available in the literature but we found it easier to sketch a direct proof.*

Proof: It is well known, see e.g. [6], vol.I, p.175, equation (5.72) and 5.4.6 on p.181, that $R(t)$ is a martingale such that:

$$\begin{aligned} ER(t)^2 &= E\left(\int_0^t v(s)\, dM(s)\right)^2 \\ &= E\int_0^t v(s)^2 a(s)\, ds \leq \|v\|^2 \times \|a\| \times t \end{aligned} \tag{4}$$

This proves that $R(t)$ is a square integrable martingale. Next observe that

$$E(R(t+h) - R(t))^2 = E\int_t^{t+h} v(s)^2 a(s)\, ds \leq \|v\|^2 \times \|a\| \times h \tag{5}$$

Note that the inequalities (4) and (5) sharpen and generalize inequalities (7) and (8) in [12]. From here on the proof proceeds in the same manner as outlined in [12] so we shall content ourselves with a sketch. Pick $h > 0$ and set $X_k = R(kh) - R((k-1)h), k = 1, 2, \ldots$ and put $R(0) = 0$, thus

$R(nh) = \sum_{k=0}^{n} X_k$. In addition inequality 5 implies that $EX_k^2 \le Ch$,where C is independent of h. Consequently, as in [12]

$$\sum_{n=1}^{\infty} \frac{E(X_n^2)}{n^2} < \infty \tag{6}$$

This implies, see e.g. Neveu [11], Prop.IV-6-1, p.138, that

$$\lim_{n\to\infty} \frac{R(nh)}{n} = 0 \tag{7}$$

The proof of Theorem 2 is now completed by using the same argument as in Wolff's paper [12] and the observation that there exists a constant C, independent of h, such that

$$R(nh) - Ch \le R(t) \le R((n+1)h) + Ch, \text{ for } nh \le t \le (n+1)h$$

3 Examples

3.1 Conditional Pasta

In this section of the paper we strengthen Theorem 1 of van Doorn and Regterschot ([5]) as a special case of our Theorem 1. To make it easier for the reader to follow the computations we follow their notation. Thus, $x(t)$, $y(t)$ and $N(t)$ denote respectively the environment process, the state of the system and the arrival process and $1(y(u) \in A) = I_A(y(u))$. Define

$$\begin{aligned} V_s(t) &= \frac{\int_0^t 1(x(u)=s)1(y(u)\in A)\,du}{\int_0^t 1(x(u)=s)\,du} \\ W_s(t) &= \frac{\int_0^t 1(x(u-)=s)1(y(u-)\in A)\,dN(u)}{\int_0^t 1(x(u-)=s)\,dN(u)} \end{aligned}$$

Theorem 3 *(van Doorn and Regterschot [5])*

$$\lim_{t\to\infty} V_s(t) = \lim_{t\to\infty} W_s(t), \ a.s.$$

in the sense that if either one of these limits exist then both do and they are equal.

Proof: Define $B = \{s\}\times A$ and $U(t) = 1(x(t-), y(t-)) \in B) = 1(x(t-) = s)1(y(t-) \in A)$. Clearly, $U(t)$ is a bounded predictable process; so Theorem 1 together with the fact that $\lambda(u) = \lambda_s$ on the set $\{x(u) = s\}$, can be applied to obtain:

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t 1(x(u-)=s)1(y(u-)\in A)\,dN(u) = \lambda_s \lim_{t\to\infty}\frac{1}{t}\int_0^t 1(x(u)=s)1(y(u)\in A)\,du.$$

and

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t 1(x(u-)=s)\,dN(u) = \lambda_s \lim_{t\to\infty}\frac{1}{t}\int_0^t 1(x(u)=s)\,du.$$

Consequently, by taking ratios the proof of Theorem 3 is completed.

Remark 3 *Our methods and results extend those of van Doorn and Regterschot in several ways: (i) Theorem (1) assumes only that the intensity function of the arrival process satisfies the conditions of remark (1). It is not necessary to assume that it is Poisson conditioned on each state* $x(t)$*. (ii) To derive the conditional pasta results of ([5]) we used the functional* $U(t) = 1((x(t-), y(t-)) \in \{s\} \times A)$*; but not all predictable processes are of this form. For example, if* $y(t)$ *is a vector* $y(t) = (y_1(t), ..., y_k(t))$ *and* f *is the vector function* $f = (f_1, ..., f_k)$*, then* $U(t) = \min_{1 \le i \le k} f_i(y_i(t-))$ *is an example of a functional that is not expressible as a simple product of indicator functions over* B*. Thus our results apply to the much larger class of bounded predictable processes.*

3.2 Geometric Arrivals

We may apply Theorem 1 very easily to show that, in the case of a discrete-time system, geometric arrivals (i.e., the interarrival times are geometric random variables) see time averages [7]. Without loss of generality, let arrivals occur with probability p (the parameter of the geometric distribution) and define the step function $[t]$ as $[t] = n, n \le t < n+1$. Next, for $t \in R$, let $B(t)$ denote the number of arrivals in $[0, t]$. Then, it is easily shown that $M(t) = B(t) - p[t]$ is a right-continuous martingale. Now,

$$E\left[(R(t+h) - R(t))^2\right] \le \sup_{s \in [t, t+h]} U^2(s) E\left[\left(\int_t^{t+h} dM(s)\right)^2\right] \le p(1-p)\lceil h \rceil$$

where the last inequality arises from the variance of the Binomial distribution. We have proved inequality (5) for geometric arrivals and hence, through the compensator $p[t]$, we obtain the discrete-time version of Pasta without requiring a separate proof.

3.3 Markov-Modulated Poisson Processes

As another example of directly applying Theorem 1, we compute the probability of a system state as seen by arrivals from a K-state Markov Modulated Poisson Process ($MMPP$) which has been stated elsewhere without proof [10]. Let $X(t), 1 \le X(t) \le K$ denote the state of a K-state $MMPP$ and $Y(t) \in \{0, 1, 2, ...\}$ the system state at time t respectively. We assume that $\{X(t), Y(t)\}$ is an ergodic Markov process. The arrival rate when the $MMPP$ is in state $X(t)$ is, as mentioned earlier, $f(X(t)) = \lambda_{X(t)}$. Let $\pi(i, j)$, $1 \le i \le K, j \ge 0$, be the limiting distribution of the Markov process $\{X(t), Y(t)\}$ and for $1 \le i \le K$ define $I_i(t) = 1$ if $X(t) = i$, $I_i(t) = 0$ otherwise. $U_j(t)$ is an indicator function for system state j: $U_j(t) = 1$ if $Y(t-) = j$, $U_j(t) = 0$ otherwise. The long term probability of an arrival seeing the event B is

$$P\,[\text{arrival sees } B] = \lim_{t \to \infty} \frac{1}{N(t)} \int_0^t \sum_{j \in B} U_j(s) dN(s)$$

Now, from Theorem 1

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \sum_{j \in B} U_j(s) dN(s) \quad = \quad \lim_{t \to \infty} \frac{1}{t} \int_0^t \sum_{j \in B} U_j(s) d\Lambda(s)$$

$$
\begin{aligned}
&= \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{j\in B} U_j(s) \left(\sum_{i=1}^{K} \lambda_i I_i(s) \right) ds \\
&= \sum_{j\in B} \sum_{i=1}^{K} \lim_{t\to\infty} \frac{1}{t} \int_0^t U_j(s) I_i(s) \lambda_i ds \\
&= \sum_{j\in B} \sum_{i=1}^{K} \lambda_i \pi(i,j)
\end{aligned}
$$

The last step follows from ergodic theorems for Markov processes (see Doob [4], Theorem 2.1, page 515). Thus the above limit and $\lim_{t\to\infty} \frac{N(t)}{t}$ may be combined to obtain expressions for the averages seen by arrivals [10].

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