Measure of goodness of a set of color-scanning filters

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Accurate scanning of a color image, which is absolutely essential for good color reproduction, can ensure that all relevant information about the color stimulus of a signal is obtained. The set of scanning filters is hence an important component of a color reproduction system. In this paper we introduce a measure of the goodness of a set of color-scanning filters. This measure relates the space spanned by the scanning filters to the human visual subspace. The q factor of a single color-scanning filter is shown to be a particular case of the measure. Experimental results are presented to justify the appropriateness of the measure.

1. INTRODUCTION

The purpose of color-scanning measurements is to record accurately some physical properties of a signal. This characterization is often used to reproduce a color image on a display or printer. Since the reproduction is viewed by a human observer, the measurements can be limited to those properties that permit the creation of an image that will appear the same as the original. The CIE has tabulated color-matching functions for this purpose. These functions are used as filters for the measurement of radiant sources. For reflective or transmissive sources, the functions together with an illuminant are used to take the necessary data.

It is well known that the scanning filters need not be exact duplicates of the color-matching functions but need be only a nonsingular transformation of them. This fact is used in determining the filters used in television and other optical applications. While the CIE color-matching functions are well defined, it is not always possible to duplicate the filters or a linear combination of them with various filter materials. The inclusion of an illumination spectrum in the light path makes the fabrication of scanning filters more difficult. In this paper we address the problem of determining how well a set of filters permits accurate measurement of color. The basis of this measure is the relation between the vector space defined by the human visual system that is determined from the CIE color-matching functions and the vector space defined by the scanning filters. The vector-space approach has recently found many applications.[1–5]

Most current research in color systems assumes that the visual frequency spectrum can be represented adequately by samples taken approximately 10 nm apart over the range 400–700 nm. Integrals are approximated by summations, and the infinite-dimensional Hilbert space of visible spectra with the usual 2 norm is reduced to an N-dimensional Hilbert space, where N is the number of samples (in this case $N = 31$). A continuous function of wavelength is represented by an N vector of its sampled values. Hence in this paper visual spectra will be treated as vectors in an N-dimensional Hilbert space. Thus the CIE color-matching functions are represented by N vectors, $\mathbf{a}_i$.

The characterization of color must include the effect of the illuminating source. The CIE tristimulus values are measured with respect to some illuminant $\mathbf{l}$. This illuminant is incorporated into the color-matching functions by the use of a diagonal matrix, $\mathbf{L}$, such that $\mathbf{L}_{ii} = \mathbf{l}_i$. The subspace spanned by the set of vectors $\{\mathbf{La}_i\}_{i=1}^{3}$ is defined as the human visual subspace (HVSS) for the illuminant $\mathbf{l}$. It is noted that accurate calculation of the tristimulus values is possible only if the space spanned by the color-scanning filters includes the HVSS for the illuminant $\mathbf{l}$[1]. The proposed measure of goodness is related to the projection of the space defined by the filters onto the HVSS.

The paper is organized as follows. Section 2 deals with the preliminaries of the vector-space approach to color systems[1] and the limitations of Neugebauer's q factor as a measure. The requirements of a measure of goodness are defined in Section 3. The analysis of an error measure leads to the measure of goodness in Section 4. It is possible that a set of three scanning filters that spans the HVSS is not realizable because of some practical limitations of the fabrication process. In such a situation, it is necessary to look for a set of four or more scanning filters that does span the HVSS. A perfect set of four scanning filters is presented in Section 5. It is demonstrated that the q factor of a single filter is not a good indicator of the appropriateness of the filter as a member of a set of more than three color-scanning filters. Experimental results comparing actual filter-set performance with the measure of goodness are presented in Section 6. The experiments are simulated on color ensembles from a thermal printer, an ink-jet printer, a color copier, and a 64-sample set of Munsell chips. The results demonstrate the appropriateness of the introduced measure. The measure is seen to be a good indicator of average mean-square error and average error in $L^*a^*b^*$ space for the signal ensembles used.

2. PRELIMINARIES

The notation in this paper follows that of Trussell.[1] Let $\mathbf{S} = [\mathbf{s}_1 \ \ \mathbf{s}_2 \ \ \mathbf{s}_3]$, where $\mathbf{s}_1, \mathbf{s}_2$ and $\mathbf{s}_3$ are N vectors that represent the color-sensitivity functions of the three types of cone in the eye. Let $\mathbf{A} = [\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3]$ and $\mathbf{P} = [\mathbf{p}_1 \ \ \mathbf{p}_2 \ \ \mathbf{p}_3]$, where $\mathbf{a}_i$ and $\mathbf{p}_i (i = 1, 2, 3)$ are N vectors

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representing the CIE matching functions (often referred to as $[\bar{x} \quad \bar{y} \quad \bar{z}]$ in the literature) and the corresponding CIE primaries (often referred to as $[\overline{X} \quad \overline{Y} \quad \overline{Z}]$ in the literature), respectively. The matrix $\mathbf{P}$ is defined by the equation $\mathbf{A}^T\mathbf{P} = \mathbf{I}$[1] and is not unique. The set $\{\mathbf{a}_i\}_{i=1}^3$ denotes the set of matching functions; $\{\mathbf{m}_i\}_{i=1}^r$ denotes any set of r scanning filters, and $\mathbf{M}$ denotes the matrix of scanning filters, $\mathbf{M} = [\mathbf{m}_1 \quad \mathbf{m}_2 \ldots \mathbf{m}_r]$. The range space of a matrix $\mathbf{X}$ is the span (set of linear combinations) of its column vectors and is denoted $R(\mathbf{X})$. Hence $R(\mathbf{M})$ denotes the set of linear combinations of the scanning filters. Note that since the CIE color-matching functions represent the same vector space as the sensitivity functions of the cones, $R(\mathbf{A}) = R(\mathbf{S})$.

Consider a reflectance spectrum $\mathbf{f}$, viewed under an illuminant $\mathbf{l}$. The reflectance spectrum $\mathbf{f}$ is seen as $\mathbf{Lf}$, where $\mathbf{L}$ is a diagonal matrix such that $\mathbf{L}_{ii} = \mathbf{l}_i$. For a perfect color match between signals $\mathbf{f}$ and $\mathbf{g}$, as viewed under illuminant $\mathbf{l}$, $\mathbf{S}^T\mathbf{Lf} = \mathbf{S}^T\mathbf{Lg}$, which is equivalent to[4] $\mathbf{A}^T\mathbf{Lf} = \mathbf{A}^T\mathbf{Lg}$, or $(\mathbf{LA})^T\mathbf{f} = (\mathbf{LA})^T\mathbf{g}$. Let the matrix $\mathbf{A}_L$ denote the matrix product $\mathbf{LA}$. Then $\mathbf{A}_L^T\mathbf{f} = \mathbf{A}_L^T\mathbf{g}$, and the spectrum $\mathbf{g}$ is known as a metamer of $\mathbf{f}$ under illuminant $\mathbf{l}$. The visual stimulus of a signal is determined uniquely by its projection onto the subspace that is spanned by the set of vectors $\{\mathbf{La}_i\}_{i=1}^3$. This subspace is defined as the HVSS for the illuminant $\mathbf{l}$ and can be denoted $R(\mathbf{LA})$. The projection of $\mathbf{f}$ onto the HVSS for $\mathbf{l}$ is

$$P_V\mathbf{f} = \mathbf{A}_L(\mathbf{A}_L^T\mathbf{A}_L)^{-1}\mathbf{A}_L^T\mathbf{f} \tag{1}$$

and is also called the fundamental of $\mathbf{f}$ with respect to the illuminant $\mathbf{l}$[2]. It can be shown that the fundamental with respect to a particular illuminant represents completely the visual stimulus of the signal with respect to that illuminant[1,6] and that

$$P_V\mathbf{f} = P_v\mathbf{g} \Leftrightarrow \mathbf{A}_L^T\mathbf{f} = A_L^T\mathbf{g}. \tag{2}$$

Color reproduction begins with correctly determining the fundamental or the projection of a given spectrum onto the HVSS. A perfect set of scanning filters is one whose measurements give the tristimulus values of the signal under a linear transformation. Equivalently, a perfect set of scanning filters determines the fundamental of the signal, and, also equivalently, a perfect set of scanning filters is one whose span includes the HVSS. The set $\{\mathbf{La}_i\}_{i=1}^3$ is a perfect set of scanning filters and a basis for the HVSS, but it is not the only one.

A number of problems arise in the implementation of a scanning system that obtains the required projection of $\mathbf{f}$. In particular, it is difficult to construct a designed scanning filter exactly, and any errors in filter construction will change the space spanned by the filters, resulting in an error in the measurement of the required projection. This error will lead to an error in the reproduction. Notice that this error will occur even if the measurements are noise free in all other respects. In an attempt to measure the goodness of a color-filter with respect to such an error, Neugebauer[7] defined the quality factor, or q factor, of a color filter. If $\mathbf{m}$ represents a color filter and $P_V(\mathbf{m})$ its orthogonal projection onto the HVSS, the q factor of $\mathbf{m}$ is defined as

$$q(\mathbf{m}) = \frac{\|P_V(\mathbf{m})\|^2}{\|\mathbf{m}\|^2}, \tag{3}$$

where $\|\cdot\|$ is the 2 norm in N-dimensional vector space. Notice that $0 \le q(\mathbf{m}) \le 1$, and the closer the value of $q(\mathbf{m})$ to unity, the better the color-scanning filter $\mathbf{m}$. If the value of $q(\mathbf{m})$ is small compared with unity, the filter measurement does not give much information about the fundamental of the measured signal, and hence the filter is not appropriate for color scanning. The q factor seems to be a reasonable quality measure for filters not in the HVSS, because $\|\mathbf{m}\|^2[1 - q(\mathbf{m})]$ is the square of the Euclidean distance of $\mathbf{m}$ from the HVSS. If any one of a set of three scanning filters $\{\mathbf{m}_i\}_{i=1}^3$ is not in the three-dimensional HVSS [i.e., $q(\mathbf{m}_i) \neq 1$ for some i], then $R(\mathbf{LA}) \neq R(\mathbf{M})$, and $\{\mathbf{m}_i\}_{i=1}^3$ is inaccurate for color sensing. A major disadvantage of the q factor is that it is designed to be used with only a single filter. A measure that extends the idea of the q factor to judgment of the effectiveness of a set of color-scanning filters would be useful.

In most existing scanning systems, only three scanning filters are used. Three linearly independent scanning filters span the three-dimensional HVSS if all three have unit q factors. Hence the q factor indicates a perfect set of filters if $q(\mathbf{m}_i) = 1$ for every i and the $\mathbf{m}_i$ are linearly independent vectors. It does not indicate whether the three filters are linearly independent, nor can it assist in differentiating among imperfect sets of filters. Hence the q-factor cannot be used by itself to indicate a "better" imperfect set of filters. Another disadvantage of the q factor is that it may be used to judge at most a set of three filters.

There are at least two reasons that more than three filters may be used to improve the quality of the color reproduction. First, in many cases, three parameters are not enough to define sufficiently the visual stimulus of an N-dimensional signal for color correction. Typically, such a situation arises when the color reproduction is to be viewed under two different illuminants. In such a case, as many as six parameters (representing the projections of the signal onto the two different three-dimensional HVSS's defined by the two different illuminants) may be required for accurate representation of the signal.[1,5] Second, the constraint of constructability on the filters might imply that no set of three constructable filters can span the HVSS, although a set of four filters could be constructed so that the required projection would be obtained. When more than three parameters (four scanning filters, for example) are necessary, the q factor is not an effective measure of the goodness of even each single filter as part of the set of more than three filters. For example, suppose that $\{\mathbf{m}_i\}_{i=1}^4$ is a set of scanning filters. It is possible that the HVSS is contained in the span of the set of four filters, i.e., that

$$R(\mathbf{M}) \supseteq R(\mathbf{A}_L), \tag{4}$$

but $q(\mathbf{m}_i) < 1$ for $i = 1, 2, 3, 4$. Such a set could provide perfect color scanning, although the individual q factors would not indicate this. An example of such a set is presented in Section 5.

The performance of a set of filters can be judged by the reproduction quality of a set of signals. Usually this set of signals is chosen so as to represent the ensemble of signals on which the set of scanning filters is to be used. The average error in the reproduction is often used as an indicator of the goodness of the set of filters. The q factor of a color-scanning filter has the disadvantage of not being a good indicator of the perceptual error in color reproduction.

As discussed above, the q factor has three major disadvantages. First, it measures a single filter independently of other filters in the scanning set. A measure that extends the idea of the q factor to include judging the effectiveness of a set of color-scanning filters would certainly be useful. Second, the q factor can be used to judge the merit of a single filter as part of a set of three filters in a limited sense, because it indicates a perfect set of three filters only when they are linearly independent. The q factor itself does not indicate linear independence for a perfect set of filters, nor does it differentiate among imperfect filter sets. Third, the q factor is not useful for judging the merit of a single filter as part of a set of more than three filters. It is possible, however, to develop a measure that overcomes these disadvantages. In Section 3 we discuss the requirements of an effective measure, and in Section 4 we present such a measure. We also discuss the relationship of the proposed measure to the q factor.

3. REQUIREMENTS OF AN EFFECTIVE MEASURE

If a set of scanning filters is linearly independent, the measurements produced (the values of $\mathbf{m}_i^T\mathbf{f}$) will be nonredundant and can be used for projection of the measured spectrum onto the space spanned by the scanning filters. If these filters are not linearly independent, not all measurements are necessary for finding the projection onto the space spanned by the filters. In this case the extra measurements may help to eliminate noise in the scanning process. In either case, a measure of goodness of the filter set is related to the space spanned by the scanning filters and to the relation of this space with the HVSS. The measure should not be related to the noise performance of the filters, as it should measure performance of the filters with respect to an error that occurs independently of any additional measurement noise. An effective measure of the goodness of a set of scanning filters should satisfy the following conditions: First, the measure should depend only on the space spanned by the scanning filters and not on particular, individual filters. For perfect scanning, the HVSS should be contained in the space spanned by the scanning filters, as indicated in expression (4). Hence a second requirement is that the measure indicate a perfect set of scanning filters. When the scanning is not perfect, the measure should distinguish among filter sets according to the goodness of the approximation to projections in the HVSS. Third, the measure should be generalizable to an arbitrary number of filters and an arbitrary reproduction space.

Let $\{\mathbf{v}_i\}_{i=1}^s$ denote a set of vectors that define the space to be spanned (typically the HVSS). This space is denoted $R(V)$, where $V = [\mathbf{v}_1 \quad \mathbf{v}_2 \ldots \mathbf{v}_s]$. We must note here that $R(\mathbf{V})$ need not be three dimensional; for example, effective color correction requires the projection onto a space of dimension greater than three.[1,5] The dimensions of $R(\mathbf{V})$ and $R(\mathbf{M})$ may not be equal; a large set of filters $\{\mathbf{m}_i\}_{i=1}^r$ may be used to ensure the spanning of $R(\mathbf{V})$.

A measure that immediately comes to mind is the dimension of the intersection of $R(\mathbf{M})$ and $R(\mathbf{V})$. On closer examination this measure is found to be too coarse. It takes on only integer values and does not distinguish well enough between good and not-so-good sets of scanning filters. This problem is illustrated in Fig. 1 for a hypothetical three-dimensional spectral space with a two-dimensional HVSS. The spaces spanned by two different sets of scanning filters are shown. It can be seen that the projection onto $R(\mathbf{M}_2)$, denoted $P_{M1}\mathbf{f}$, is a better approximation to the fundamental, denoted $P_V\mathbf{f}$, than is the projection onto $R(\mathbf{M}_1)$, denoted $P_{M1}\mathbf{f}$. This is not indicated by the dimension of the intersection of the spaces with the HVSS; in both cases this dimension is 1.

For an example of this problem in an N-dimensional space, consider the space spanned by the N vectors

$$\begin{aligned}\mathbf{v}_1 &= [1, 0, 0, 0, \ldots, 0]^T,\\ \mathbf{v}_2 &= [0, 1, 0, 0, \ldots, 0]^T,\\ \mathbf{v}_3 &= [0, 0, 1, 0, \ldots, 0]^T\end{aligned} \tag{5}$$

as the space to be spanned. Let the N vectors

$$\begin{aligned}\mathbf{m}_1 &= [1, 0, 0, 0, \ldots, 0]^T,\\ \mathbf{m}_2 &= [0, 0, 1, 0, \ldots, 0]^T,\\ \mathbf{m}_3 &= [0, 0, 0, 1, \ldots, 0]^T\end{aligned} \tag{6}$$

be the scanning filters. The dimension of the intersection, $R(\mathbf{V}) \cap R(\mathbf{M})$, is 2. Now suppose that

$$\begin{aligned}\mathbf{m}_1' &= [1, 0, 0, 0, \ldots, 0]^T,\\ \mathbf{m}_2' &= [0, 0.95, 0, 0.05, \ldots, 0]^T,\\ \mathbf{m}_3' &= [0, 0, 1.0, 0, \ldots, 0]^T\end{aligned}$$

is another set of scanning filters. The dimension of the intersection is also 2, but the second set will provide much more information about the required projection (for ex-

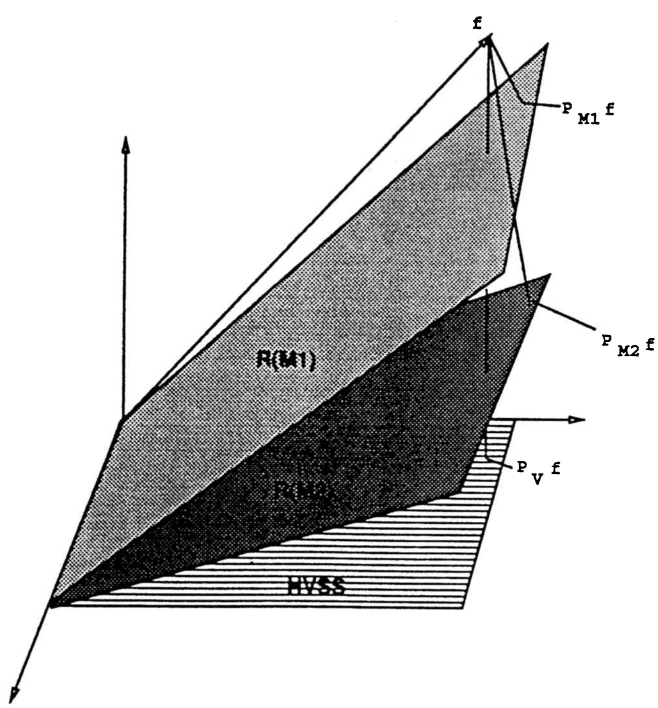

Fig. 1. $R(\mathbf{M}_2)$ is closer to the HVSS than is $R(\mathbf{M}_1)$.

ample, information about the second coordinate of the signal can be obtained more reliably from the second set of scanning filters than from the first). A measure is needed that will distinguish between such sets of filters.

The problem of finding a measure of the goodness of a set of scanning filters is directly related to the error associated with such a set in the absence of any additional noise. The error is defined as the difference between the required signal and the signal obtained with the scanning-filter set. This error depends on the particular reflectance vector **f** that is being measured. As the human eye is sensitive only to errors in the HVSS, the error may be defined as the difference between the actual and the reconstructed fundamentals. It is common to consider an average error over some well-defined set of reflectance spectra, $\{\mathbf{f}_k\}$. Among the many error measures that one may use to judge the performance of a filter set are the mean-square error in N space, the mean-square tristimulus error, and the mean-square error in a uniform color space such as the CIE $L^*a^*b^*$ space. The mean-square error and the mean error in CIE $L^*a^*b^*$ are studied in this paper. The mean-square error in N space will now be related to an error measure that has the desired properties.

4. AN ERROR MEASURE AND A RELATED MEASURE OF GOODNESS

Two major error measures are often used in the evaluation of color reproductions: mean-square error, and mean $L^*a^*b^*$ error. In either case, the mean error indicates an average over a particular data set and is hence dependent on the data set. The problems of correlating mean-square error with perceptual error are well known. The mean-square error is addressed because it is easy to manipulate and because its analysis provides valuable insight into the problem of reproduction errors that are due to filter-construction errors. The mean ΔE_{Lab} error measure is far more difficult to analyze and manipulate but is a valid measure of perceptual error. While there are cases in which color estimates may have low mean-square errors and high ΔE_{Lab} errors and vice versa, the average of the errors over a data set are generally in qualitative agreement, as is demonstrated in Section 6.

Before an error expression can be obtained, some notation needs to be established. Let an orthonormal basis for $R(\mathbf{V})$ be defined by $\mathbf{N} = [\mathbf{n}_1 \quad \mathbf{n}_2 \ldots \mathbf{n}_\alpha]$ such that $R(\mathbf{N}) = R(\mathbf{V})$ and

$$\mathbf{N}^T\mathbf{N} = \mathbf{I}. \tag{7}$$

Such a basis may be obtained by the Gram–Schmidt orthogonalization procedure.[8] The number of orthonormal vectors, α, is the rank of $\{\mathbf{v}_i\}_{i=1}^s$, and α equals s if $\{\mathbf{v}_i\}_{i=1}^s$ is a linearly independent set. Similarly, we define an orthonormal basis for $R(\mathbf{M})$ by $\mathbf{O} = [\mathbf{o}_1 \quad \mathbf{o}_2 \ldots \mathbf{o}_\beta]$ such that $R(\mathbf{O}) = R(\mathbf{M})$ and

$$\mathbf{O}^T\mathbf{O} = \mathbf{I}. \tag{8}$$

Again, notice that β is the rank of $\{\mathbf{m}_i\}_{i=1}^r$ and that β equals r if $\{\mathbf{m}_i\}_{i=1}^r$ is a linearly independent set. The orthonormal bases **N** and **O** need not represent realizable filters. Let $P_M(\cdot)$ represent the orthogonal projection operator onto $R(M)$ in N space. Similarly, let $P_V(\cdot)$ represent the orthogonal projection operator onto $R(\mathbf{V})$ in N space. Define $P_V(\mathbf{O}) = [P_V(\mathbf{o}_1)P_V(\mathbf{o}_2)\ldots P_V(\mathbf{o}_\beta)]$ and likewise $P_M(\mathbf{N}) = [P_M(\mathbf{n}_1)P_M(\mathbf{n}_2)\ldots P_M(\mathbf{n}_\alpha)]$. The projection of a visual spectrum **f** onto $R(\mathbf{V})$ is the information required for accurate color reproduction as defined by the designers of the reproduction system. From Eqs. (1) and (7) this required projection of a visual spectrum **f** onto $R(\mathbf{V})$ is

$$P_V(\mathbf{f}) = \mathbf{N}\mathbf{N}^T\mathbf{f}. \tag{9}$$

In the special case in which $R(\mathbf{V})$ is the HVSS, $P_V(\mathbf{f})$ is the fundamental of the spectrum **f**. The projection obtained by using the scanning filters $\{\mathbf{m}_i\}_{i=1}^s$ is the projection onto $R(\mathbf{M})$:

$$P_M(\mathbf{f}) = \mathbf{O}\mathbf{O}^T\mathbf{f}. \tag{10}$$

The projection of this onto the HVSS is

$$P_V P_M(\mathbf{f}) = \mathbf{N}\mathbf{N}^T\mathbf{O}\mathbf{O}^T\mathbf{f}. \tag{11}$$

In the special case in which $R(\mathbf{V})$ is the HVSS, $P_V P_M(\mathbf{f})$ is the estimated fundamental of the spectrum **f** and is the relevant information about **f** that can be obtained from the scanned data. Expressions (9) and (11) give the required projection and the obtained projection, respectively.

A. Mean-Square Error

The difference between the two fundamentals $P_V(\mathbf{f})$ and $P_V P_M(\mathbf{f})$ is the difference between expressions (9) and (11):

$$\mathbf{e} = \mathbf{N}\mathbf{N}^T(\mathbf{I} - \mathbf{O}\mathbf{O}^T)\mathbf{f}. \tag{12}$$

It can be shown that

$$\mathbf{e} = \mathbf{0} \Leftrightarrow R(\mathbf{M}) \supseteq R(\mathbf{V}), \tag{13}$$

and perfect reproduction is possible if and only if the space spanned by the scanning filters includes the space to be spanned. See Theorem 1 of Appendix A for details of the proof of expression (13). If **f** is part of a known ensemble of signals, then it is reasonable to look at the mean-square value of **e**. Assume that $E[\mathbf{f}\mathbf{f}^T]$ is known, and let $E[\mathbf{f}\mathbf{f}^T] = \mathbf{R}$, the correlation matrix. Under these assumptions the expression for the mean-square value of the error is

$$E[\|\mathbf{e}\|^2] = \mathrm{Trace}[\mathbf{N}^T(\mathbf{I} - \mathbf{O}\mathbf{O}^T)\mathbf{R}(\mathbf{I} - \mathbf{O}\mathbf{O}^T)\mathbf{N}]. \tag{14}$$

For details, see Theorem 2 of Appendix A. This expression may not be simplified further unless assumptions are made about the nature of **R**. For a given sample set, however, this error is easily computed and will be useful for many who now use sample sets, anyway.

B. Bounds on the Error Expression

Bounds on error expression (14) may be obtained in terms of the structure of the matrix **R**. It can be shown that

$$0 \le E[\|\mathbf{e}\|^2] \le \mathrm{Trace}\,\mathbf{R}. \tag{15}$$

For details, see Theorem 3 of Appendix A.

To obtain some heuristic understanding of the meaning of these bounds, consider a simple example demonstrating the extreme cases of zero error and of maximum error. Let the space to be spanned be defined by the set $\{\mathbf{v}_i\}_{i=1}^3$ defined in Eq. (5) of Section 3. Let the scanning filter set be the set $\{\mathbf{m}_i\}_{i=1}^{3}$ defined in Eq. (6) of Section 3. Notice that in this particular case the scanning filter set is orthonormal. The set $\{\mathbf{v}_i\}_{i=1}^{3}$ is also orthonormal. Hence in this case $\mathbf{M} = \mathbf{O}$ and $\mathbf{N} = \mathbf{V}$. Simple matrix multiplication indicates that $E[\|\mathbf{e}\|^2] = R_{22}$, the value of the second element on the diagonal of $\mathbf{R}$. Let $\mathbf{R}$ be a diagonal matrix with only one nonzero diagonal element. Let this element be γ. Then

$$\text{Trace } \mathbf{R} = \gamma .$$

The case of zero error occurs when the nonzero element of the diagonal of $\mathbf{R}$ is any element other than the second. The maximum error occurs when the nonzero element of the diagonal of $\mathbf{R}$ is the second. For example, this happens when the ensemble of spectra for this synthetic example is defined by

$$\{\mathbf{f} \,|\, \mathbf{f}_i = u \quad \text{if } i = 2, \quad \mathbf{f}_i = 0 \text{ else}\},$$

where u is a Gaussian random variable with variance σ^2 and mean m. The value of the maximum error in this case is $\sigma^2 + m^2$.

From the above example and the proof of inequality (15), it can be seen that the zero error occurs when the energy in the signal that is in the space to be spanned is also in space spanned by the scanning filters. In the special case when the space that needs to be spanned is the HVSS, the zero-error case will occur when all the energy in the signal that is in the HVSS is also in the space spanned by the scanning filters. Similarly, the maximum error occurs when all the energy of the signal escapes the scanning filters but lies in the space to be spanned.

From the above observations it is clear that the correlation structure of the signal determines the error to a large extent. When nothing is known about the correlation structure, it is common to assume that $\mathbf{R}$ is a scalar multiple of the identity matrix and that the signal power is equally distributed in all directions. This assumption implies that concentration of signal power in a particular direction will not bias the error to make it maximum or minimum, as in the cases just discussed. This leads to an error measure that is demonstrated to be particularly useful and is independent of the data set.

C. An Error Measure

In the particular case when $\mathbf{R}$ is a scalar multiple of the identity matrix, i.e., when the spectrum $\mathbf{f}$ can be expressed as a sequence of independent, identically distributed random variables, the error expression is considerably simplified. With this assumption, the error expression can be related to the q-factor measure. This assumption is often made when no information is available about the signal statistics and the random variables of the signal are assumed independent of one another. It is a maximum-ignorance assumption for signal estimation. Error expression (14) is then

$$E[\|\mathbf{e}\|^2] = \sigma^2 \left\{ \sum_{i=1}^{\alpha} [1 - \lambda_i^2(\mathbf{O}^T\mathbf{N})] \right\}, \tag{16}$$

where σ^2 is the variance of a single component of $\mathbf{f}$ and $\lambda_i(\mathbf{O}^T\mathbf{N})$ denotes the ith singular value of $\mathbf{O}^T\mathbf{N}$. For details, see Theorem 2 of Appendix A. It is appropriate to mention that

$$\lambda_i^2(\mathbf{O}^T\mathbf{N}) = \cos^2(\theta_i) \qquad i = 1, \ldots \alpha, \tag{17}$$

where θ_i is the ith principal angle between $R(\mathbf{V})$ and $R(\mathbf{M})$.[8] Note that

$$1 - \lambda_i^2(\mathbf{O}^T\mathbf{N}) \geq 0 \qquad i = 1, \ldots \alpha, \tag{18}$$

because $1 - \lambda_i^2(\mathbf{O}^T\mathbf{N})$ $i = 1, \ldots \alpha$ are the eigenvalues of the quadratic form $N^T(\mathbf{I} - \mathbf{O}\mathbf{O}^T)\mathbf{N}$. Hence

$$0 \leq \lambda_i^2(\mathbf{O}^T\mathbf{N}) \leq 1 \qquad i = 1, \ldots \alpha, \tag{19}$$

which implies that

$$0 \leq E[\|\mathbf{e}\|^2] = \sigma^2 \left[\alpha - \sum_{i=1}^{\alpha} \lambda_i^2(\mathbf{O}^T\mathbf{N}) \right] \leq \sigma^2 \alpha . \tag{20}$$

Perfect reproduction is possible when

$$\alpha = \sum_{i=1}^{\alpha} \lambda_i^2(\mathbf{O}^T\mathbf{N}) \tag{21}$$

or

$$\lambda_i(\mathbf{O}^T\mathbf{N}) = 1 \qquad i = 1, \ldots \alpha . \tag{22}$$

Note that the conditions for perfect reproduction are independent of the variance of the components of $\mathbf{f}$. The term α in expression (20) is the dimension of the space to be spanned and is assumed invariate. The summation in expression (20) may be used as a measure of the goodness of the filters, because a high value of the summation indicates a low error. A normalized measure of the goodness of the filter set is

$$\nu(\mathbf{V}, \mathbf{M}) = \left[\sum_{i=1}^{\alpha} \lambda_i^2(\mathbf{O}^T\mathbf{N}) \right] \Big/ \alpha . \tag{23}$$

The error is zero if and only if the set of filters is perfect and Eq. (22) holds. Perfect reproduction implies and is implied by

$$\nu(\mathbf{V}, \mathbf{M}) = 1 . \tag{24}$$

The error measure can be shown to be related to the principal angles between the two subspaces, $R(\mathbf{V})$ and $R(\mathbf{M})$. Principal angles are well known in numerical linear algebra; this relationship is discussed in detail in Subsection 4.E.

D. Relationship with the q Factor

It can be shown that the measure of goodness defined in Eq. (23) is related to Neugebauer's q-factor measure. Specifically, this measure of goodness is the sum of the q factors of the vectors $\mathbf{o}_i$ divided by the dimension of $R(\mathbf{V})$. For details, see Theorem 4 of Appendix A. Thus the sum of q factors of the filters represented by $\mathbf{O}$, an orthonormal basis for the space spanned by the scanning filters, is a valid measure of goodness for the set of filters when the signal $\mathbf{f}$ is from an ensemble of independent, identically distributed random variables. The sum may

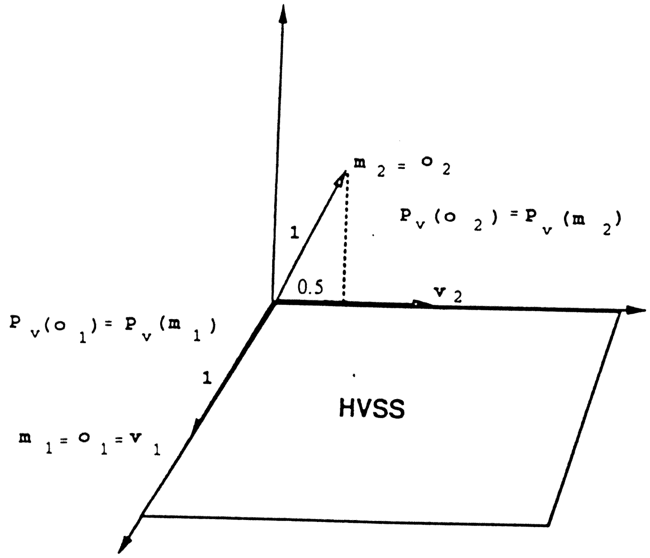

Fig. 2. Measure is the sum of q factors of orthogonal scanning filters.

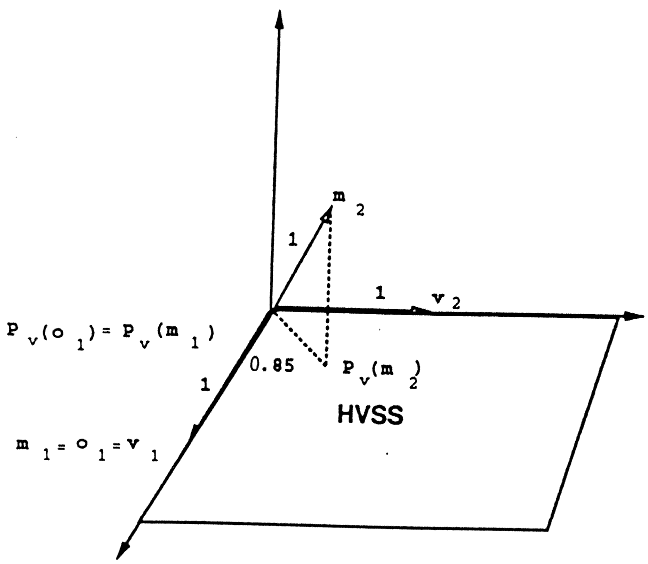

Fig. 3. Measure is not the sum of q factors of nonorthogonal scanning filters.

be normalized so that a maximum value of unity indicates perfect reproduction. This normalized form is

$$\nu(\mathbf{V},\mathbf{M}) = \left[\sum_{i=1}^{\beta} q(\mathbf{o}_i)\right] \Big/ \alpha . \tag{25}$$

Here the term q factor is used in a general sense to mean the norm of the projection of the normalized vectors $\mathbf{o}_i$ onto the α-dimensional $R(\mathbf{V})$. Neugebauer's original definition of the q factor implies the specific case where this space is the three-dimensional HVSS.

For the measure to be independent of the particular set of filters used to span the respective spaces, a necessary condition is that the filters $\mathbf{O}$ be orthonormal. So, for example, $\Sigma_{i=1}^{r} q(\mathbf{m}_i)$ cannot be used instead of $\Sigma_{i=1}^{\beta} q(\mathbf{o}_i)$ as a measure because filters with high values of $|\mathbf{m}_i^T\mathbf{m}_j|$ for $i \neq j$ may have high q factors but poor joint performance. Ensuring that the filters $\mathbf{O}$ are orthogonal removes the effect of the correlation ($|\mathbf{m}_i^T\mathbf{m}_j|$ for $i \neq j$). To see that the normalized sum of the q factors of the scanning filters themselves is not a valid measure, consider the following example (refer to Figs. 2 and 3 for illustration).

Let $\mathbf{v}_1 = [1,0,0]^T$ and $\mathbf{v}_2 = [0,1,0]^T$ be the vectors representing a simple two-dimensional HVSS. Let $\mathbf{m}_1 = [1,0,0]^T$ and $\mathbf{m}_2 = [0, \cos(\pi/3), \sin(\pi/3)]^T$ be vectors representing the scanning filters. The normalized sum of the q factors is 0.75. Now, suppose that another set of scanning filters is chosen that spans the same space as the first set. Suppose that the set is $\mathbf{m}_1 = [1,0,0]$ and that $\mathbf{m}_2 = [2/\sqrt{5}, [\cos(\pi/3)]/\sqrt{5}, [\sin(\pi/3)]/\sqrt{5}]$. Notice that this set of scanning filters is not orthogonal. The normalized sum of q factors for this set is 0.925. Let

$$\phi(\mathbf{V},\mathbf{M}) = \sum_{i=1}^{2} q(\mathbf{m}_i). \tag{26}$$

Then $\phi(\mathbf{V},\mathbf{M})$ can be seen to be dependent on the particular filters used and is not just a function of the space spanned by the filters, unless the orthogonality condition is imposed on the filters. Hence it is not a valid measure of goodness. Note that if the filters $\mathbf{m}_i$ are narrow band with little overlap, they are close to orthogonal. In this case, $\phi(\mathbf{V},\mathbf{M})$ may be a good approximation to $\nu(\mathbf{V},\mathbf{M})$. The measure $\nu(\mathbf{V},\mathbf{M})$ avoids this problem entirely.

If each scanning filter is to be evaluated by itself and not as part of a larger set of filters, the proposed measure can be used to evaluate the set consisting of a single scanning filter. In this case, the proposed measure is exactly the same as the q factor of Neugebauer, and Neugebauer's definition of the q factor may be seen as a specific instance of the measure defined in Eq. (23).

E. Relation to Principal Angles

As mentioned in Subsection 4.C, the idea of the principal angles between two subspaces is well known in numerical linear algebra. It is important to see the connection between the measure developed in this paper and this well-established mathematical concept. The principal angles[8] $\theta_1, \theta_2, \ldots, \theta_\alpha \in [0, \pi/2]$ between $R(\mathbf{M})$ and $R(\mathbf{V})$ are defined recursively by

$$\cos(\theta_k) = \max_{\mathbf{u}\in R(\mathbf{V})} \max_{\mathbf{v}\in R(\mathbf{M})} \mathbf{u}^T\mathbf{v} = \mathbf{u}_k{}^T\mathbf{v}_k , \tag{27}$$

subject to

$$\|\mathbf{u}\| = \|\mathbf{v}\| = 1,$$
$$\mathbf{u}^T\mathbf{u}_i = 0 \qquad i = 1, \ldots, k-1,$$
$$\mathbf{v}^T\mathbf{v}_i = 0 \qquad i = 1, \ldots, k-1.$$

It can be shown that[8]

$$\cos(\theta_k) = \lambda_k(\mathbf{O}^T\mathbf{N}). \tag{28}$$

Hence

$$\nu(\mathbf{V},\mathbf{M}) = \left[\sum_{i=1}^{\alpha} \cos^2(\theta_i)\right] \Big/ \alpha . \tag{29}$$

Notice that the measure is large when the angles between the subspaces are small, and it is interesting that a measure developed by using the mean-square error is consistent with the observation that the error in the scanning process is larger if the angle between the subspaces is larger.

F. Relation to $L^*a^*b^*$ Error

The mean-square error is not directly related to the $L^*a^*b^*$ error, and it is possible that reconstructions with low mean-square errors have high $L^*a^*b^*$ errors, and vice versa. However, the average $L^*a^*b^*$ error over an ensemble of signals used in colometric experiments is usually highly correlated with the mean-square error. Heuristically this is reasonable. The $L^*a^*b^*$ transformation is based on the tristimulus values; therefore small errors in the tristimulus values usually will imply small $L^*a^*b^*$ errors. Experience with the proposed measure has indicated that small differences do not always yield errors in the same order as the measure would indicate but that larger differences always do. The results of the experiments with several different ensembles demonstrate this when data correction is used.

G. Data Correction

The error measure given in Subsection 4.C, with its relations to the q factor and principal angles, can be considered a measure of the difference in the subspaces spanned by $\mathbf{A}$ and $\mathbf{M}$. As it is based on the assumption of independent, identically distributed random variables, it may not give a good estimate of the error that can be obtained for a particular data set. However, it is still a good estimate of the quality of the data obtained, because this quality depends on the closeness of the spaces.

Given a set of color measurements for an ensemble with known tristimulus values, it is common to derive the 3×3 matrix that premultiplies the measurements to give a minimum mean-square estimate of the tristimulus values. This correction is dependent on the particular ensemble. It is commonly performed in colorimetry when the scanning filters are to be used on a well-characterized data set. In the simulation experiments the fundamental is estimated from the corrected data, where the corrected data set consists of the linear minimum mean-square-error approximations of the actual tristimulus values. Such a fundamental will represent a signal that has the corrected data as its tristimulus values. As demonstrated in Section 6 below, the correction reduces both mean-square and $L^*a^*b^*$ errors considerably. The corrected scanning filter data set is

$$\mathbf{h} = (\mathbf{A}^T\mathbf{RM}(\mathbf{M}^T\mathbf{RM})^{-1})(\mathbf{M}^T\mathbf{f}), \tag{30}$$

where $\mathbf{R} = E[\mathbf{ff}^T]$ is the sample correlation matrix of the ensemble. The estimated fundamental is

$$\hat{P}_V\mathbf{f} = [\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}]\mathbf{h}, \tag{31}$$

which gives

$$\hat{P}_V\mathbf{f} = [\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{RM}(M^T\mathbf{RM})^{-1}](\mathbf{M}^T\mathbf{f}). \tag{32}$$

Since the transformation from fundamental to tristimulus values is linear, the same expression is obtained by minimizing the error between fundamentals. The corrected data set always provides a lower mean-square error than the uncorrected data because the corrected fundamental is *the* minimum mean-square-error linear estimate of the true fundamental. Hence the mean-square error between the corrected fundamental estimate and the true fundamental will be lower than that between the true fundamental and any other linear estimate, including the uncorrected fundamental. It should be noted that data correction is a way of making the best use of the data obtained from a set of scanning filters if the correlation of the data set is known. Data correction does not change the data obtained in any fundamental manner, and the color estimate can be only as good as the data obtained with the set of scanning filters.

5. PERFECT FILTER SET

In Section 2 it was mentioned that it is possible that a four-filter set can ensure perfect color scanning. An example is presented here of an imperfect three-filter set with fairly high individual q factors. It is demonstrated that the addition of a fourth filter to the set makes the set perfect, although the q factor of the fourth filter is less than 0.25.

Consider the set of three all-positive filters shown in Figs. 4–6. This set of filters has a measure, $\nu(\mathbf{A}, \mathbf{M})$, of 0.953. The filters have q factors of 0.873, 0.891, and 0.939. Clearly this set does not span the HVSS and is not a perfect set of scanning filters. The mean-square error between the fundamental and the estimated uncorrected fundamental for a 64-signal subset of the Munsell chip set is 0.188, and the average ΔE_{Lab} error for the same set is 3.4. The addition of a fourth filter to the set, shown in Fig. 7, increases the measure to 1.0, and the resulting four-filter set spans the HVSS. The q factor of the fourth filter is 0.246. This is an example of a case in which the q factor of a filter is not indicative of its appropriateness as a scanning filter.

The fact that a perfect scanning-filter set can be obtained by the addition of a filter with such a low q factor indicates a problem for the design of such filter sets. Consider the problem of selecting or designing scanning filters from a specified set of filters, for example, the Kodak Wratten set of gelatin filters. If only three filters are to be used in an attempt to span the HVSS, then all three must have high q factors. This means that filter candidates can be limited. If more than three filters are to be used, then individual q factors are useless, and the possible number of filters that must be considered is limited only by the total number in the set to be used for construction. The optimal design of such filter sets is an open problem.

6. EXPERIMENTAL RESULTS

Several ensembles were used to study the appropriateness of the proposed measure, $\nu(\mathbf{V}, \mathbf{M})$. The mean-square error and the ΔE_{Lab} error were calculated on sets of signals from a lithographic printer, a thermal printer, an ink-jet printer, a color copier, and a 64-sample set of Munsell chips. The first four ensembles are representative of the range of printed materials that would be scanned in a publishing or a copying application. The Munsell chip set is the only one generated not by a three- or four-color process but by a more diverse set of pigments. The light sources were assumed uniform.

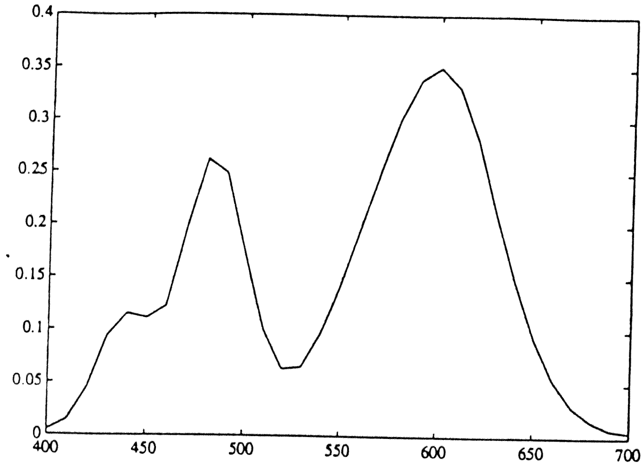

Fig. 4. Filter 1.

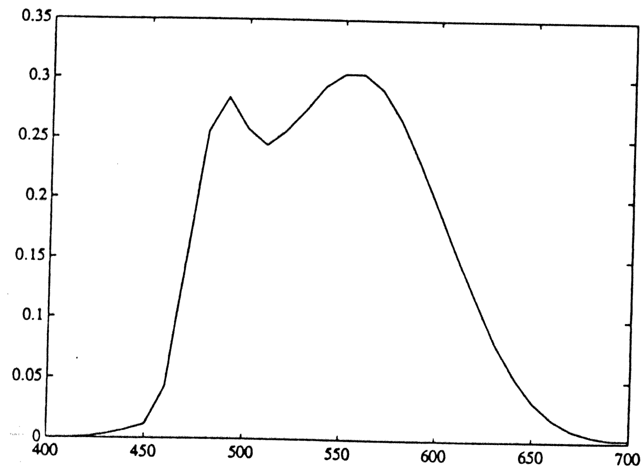

Fig. 5. Filter 2.

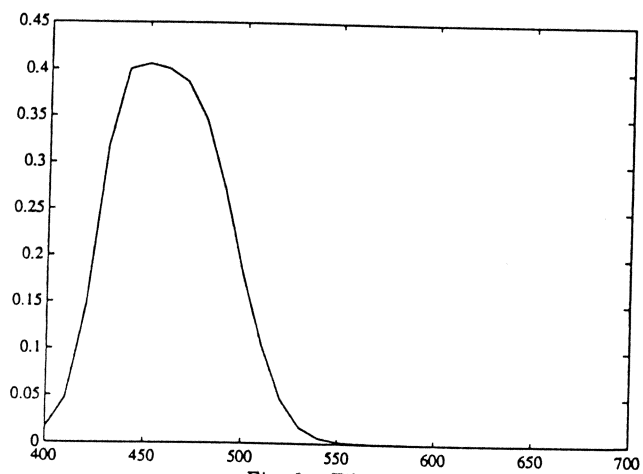

Fig. 6. Filter 3.

The filter set that is shown here is representative of several that were used in experiments during this study. Filter set 1 is the set of Kodak Wratten gelatin filters Nos. 52, 49, and 72B. Filter set 2 is the above set with a fourth filter added, the Kodak Wratten gelatin filter No. 57. Filter set 3 consists of five filters, the four in set 2 and Kodak Wratten gelatin filter No. 30.

In the first experiment, the fundamental was reconstructed directly from the uncorrected scanning-filter data. Hence the estimated projection was calculated as

$$\hat{P}_V\mathbf{f} = P_V P_M \mathbf{f} = [\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{M}(\mathbf{M}^T\mathbf{M})^{-1}](\mathbf{M}^T\mathbf{f}), \quad (33)$$

where $\mathbf{M}^T\mathbf{f}$ is the scanning-filter data. The mean-square error between this reconstructed fundamental and the true fundamental, $P_V\mathbf{f}$, is represented by e_1 in column 3 in Tables 1–5. The ΔE_{Lab} error calculated on the basis of tristimulus values estimated from $P_M\mathbf{f}$,

$$\mathbf{A}^T P_M \mathbf{f} = [\mathbf{A}^T\mathbf{M}(\mathbf{M}^T\mathbf{M})^{-1}](\mathbf{M}^T\mathbf{f}), \quad (34)$$

is represented by E_1 in column 4 in Tables 1–5. The data manipulations in the first experiment hence are independent of the particular data set.

In the second experiment, the scanning-filter data were corrected for a best fit of the actual tristimulus values, assuming knowledge of second-order signal statistics. The ΔE_{Lab} error with the use of Eq. (30) as the estimated tristimulus values is represented by E_2 in column 6 of Tables 1–5. The mean-square error between the reconstructed fundamental in Eq. (32) and the actual fundamental is represented by e_2 in column 5 in Tables 1–5. The mean-square errors e_1 and e_2, the ΔE_{Lab} errors E_1 and E_2, and the measure $\nu(\mathbf{A},\mathbf{M})$ are tabulated in Tables 1–5 for the five sets of signals used.

Notice that the mean-square error and the ΔE_{Lab} error behave similarly in the two experiments. A conclusion is that the average mean-square error is a reasonable indicator of the average perceptual error for the ensembles studied here.

As the proposed measure is an accurate theoretical indicator of mean-square error when the signal is composed of independent random variables, it is possible that it may not be so accurate an indicator of mean-square error when the signal is correlated. In fact, this may be observed in the uncorrected experimental results for filter set 2 for the color-copier data set, the lithographic-printer data set, and the ink-jet-printer data set. Here, the errors do not monotonically decrease as the measure increases. It can be seen that an increase in the number of filters in these cases increases the error; this is because the data set is correlated and the estimates used are not the best estimates for the particular data set. Hence an increase in the number of filters does not ensure that the additional information is used appropriately for the particular data

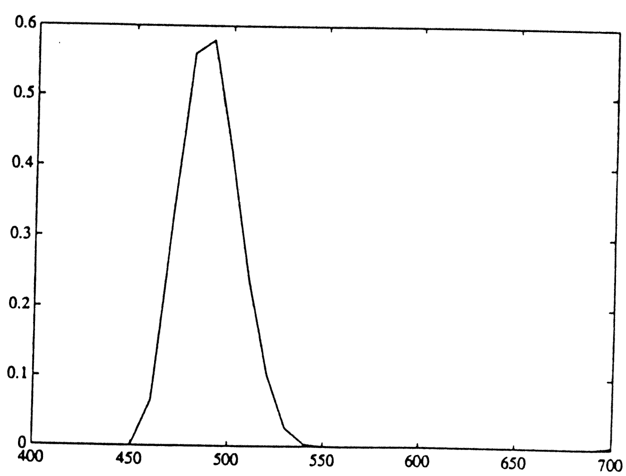

Fig. 7. Filter 4.

Table 1. Munsell-Chip Set

Filter Set	Measure	Mean-Square Error, e_1, Uncorrected	ΔE_{Lab} E_1 Uncorrected	Mean-Square Error, e_2, Corrected	ΔE_{Lab} E_2 Corrected
1	0.858	0.188	16.41	0.0022	2.10
2	0.913	0.152	15.73	0.0011	1.04
3	0.943	0.004	2.43	0.0002	0.49

Table 2. Color-Copier Data Set

Filter Set	Measure	Mean-Square Error, e_1, Uncorrected	ΔE_{Lab} E_1 Uncorrected	Mean-Square Error, e_2, Corrected	ΔE_{Lab} E_2 Corrected
1	0.858	0.050	12.96	0.00120	2.94
2	0.913	0.054	14.62	0.00005	0.49
3	0.943	0.002	2.72	0.00002	0.41

Table 3. Lithographic-Printer Data Set

Filter Set	Measure	Mean-Square Error, e_1, Uncorrected	ΔE_{Lab} E_1 Uncorrected	Mean-Square Error, e_2, Corrected	ΔE_{Lab} E_2 Corrected
1	0.858	0.070	14.32	0.00125	2.56
2	0.913	0.073	15.38	0.00007	0.54
3	0.943	0.002	2.40	0.00001	0.31

Table 4. Thermal-Printer Data Set

Filter Set	Measure	Mean-Square Error, e_1, Uncorrected	ΔE_{Lab} E_1 Uncorrected	Mean-Square Error, e_2, Corrected	ΔE_{Lab} E_2 Corrected
1	0.858	0.061	12.58	0.00086	2.65
2	0.913	0.049	11.74	0.00066	1.81
3	0.943	0.003	2.68	0.00007	0.87

Table 5. Inkjet-Printer Data Set

Filter Set	Measure	Mean-Square Error, e_1, Uncorrected	ΔE_{Lab} E_1 Uncorrected	Mean-Square Error, e_2, Corrected	ΔE_{Lab} E_2 Corrected
1	0.858	0.096	15.89	0.0012	2.30
2	0.913	0.097	16.48	0.0005	1.21
3	0.943	0.011	6.84	0.0002	0.70

set. In contrast, it can be seen that the goodness measure is a reasonable indicator of the error when the data set is corrected.

The observations indicate that the measure should not be used for fine tuning a set of filters but can give a good indication of performance for larger differences in measure when knowledge of the signal statistics is not available.

7. CONCLUSIONS

The qualities of a good measure were stated, and a measure was developed that satisfies most of these requirements. The proposed measure is a valid measure of the goodness of a filter set with respect to the mean-square error between the fundamental and its estimate when the signal components are independent identically distributed. If the signal components are not independent identically distributed, the proposed measure is at least as good as the q factor and does eliminate some of the disadvantages of the q factor. The proposed measure is a better indicator of filter performance when the filter measurements are corrected for a specific data set.

APPENDIX A

Theorem 1. $\mathbf{e} = \mathbf{0}$ *if and only if* $R(\mathbf{M}) \supseteq R(\mathbf{V})$.

Proof: To prove the "if" portion of the theorem, note that

$$R(\mathbf{M}) \supseteq R(\mathbf{V})$$

implies that[9]

$$P_V P_M \mathbf{f} = P_V \mathbf{f},$$

$$\mathbf{e} = P_V \mathbf{f} - P_V P_M \mathbf{f} = \mathbf{0}.$$

To prove the "only if" portion, note that it is equivalent to showing that

$$R(\mathbf{O}) \supseteq R(\mathbf{N}),$$

which is in turn equivalent to

$$\mathrm{Null}(\mathbf{N}) \supseteq \mathrm{Null}(\mathbf{O}).$$

In the special case when $R(\mathbf{V})$ is the HVSS, this means that the black space of the human visual system includes the black space of the scanning filters. Let $\mathbf{f}$ lie in the black space of the scanning filters, $\mathbf{f} \in \mathrm{Null}(\mathbf{O})$, i.e.,

$$\mathbf{O}^T\mathbf{f} = \mathbf{0}.$$

As $\mathbf{e} = \mathbf{0}$, this implies that $\mathbf{N}\mathbf{N}^T\mathbf{f} = \mathbf{0}$. This implies that

$$\mathbf{f}^T\mathbf{N}\mathbf{N}^T\mathbf{f} = (\mathbf{N}^T\mathbf{f})^T\mathbf{N}^T\mathbf{f} = \mathbf{0},$$

which implies that[10]

$$\mathbf{N}^T\mathbf{f} = \mathbf{0}$$

and $\mathbf{f} \in \mathrm{Null}(\mathbf{N})$. This completes the proof.

Theorem 2. *$E[\|\mathbf{e}\|^2] = \sigma^2\{\Sigma_{i=1}^{\alpha}[1 - \lambda_i^2(\mathbf{O}^T\mathbf{N})]\}$ when $E[\mathbf{f}\mathbf{f}^T] = \mathbf{R} = \sigma^2\mathbf{I}$.*

Proof: The sum of the squares of the error components, or the square error, is

$$\|\mathbf{e}\|^2 = [\mathbf{N}\mathbf{N}^T(\mathbf{I} - \mathbf{O}\mathbf{O}^T)\mathbf{f}]^T\mathbf{N}\mathbf{N}^T(\mathbf{I} - \mathbf{O}\mathbf{O}^T)\mathbf{f}. \tag{A1}$$

From Eq. (7) and the symmetric nature of the matrices $\mathbf{N}\mathbf{N}^T$ and $\mathbf{O}\mathbf{O}^T$, Eq. (A1) reduces to

$$\|\mathbf{e}\|^2 = \mathbf{f}^T(\mathbf{I} - \mathbf{O}\mathbf{O}^T)\mathbf{N}\mathbf{N}^T(\mathbf{I} - \mathbf{O}\mathbf{O}^T)\mathbf{f},$$

which can be rewritten as

$$\|\mathbf{e}\|^2 = \mathrm{Trace}[\mathbf{N}^T(\mathbf{I} - \mathbf{O}\mathbf{O}^T)\mathbf{f}\mathbf{f}^T(\mathbf{I} - \mathbf{O}\mathbf{O}^T)^T(\mathbf{N}^T)^T].$$

If $E[\mathbf{f}\mathbf{f}^T] = \mathbf{R}$ and $\mathbf{R} = \sigma^2\mathbf{I}$, then

$$E[\|\mathbf{e}\|^2] = \sigma^2\,\mathrm{Trace}[\mathbf{N}^T(\mathbf{I} - \mathbf{O}\mathbf{O}^T)(\mathbf{I} - \mathbf{O}\mathbf{O}^T)\mathbf{N}].$$

This may be rewritten as

$$E[\|\mathbf{e}\|^2] = \sigma^2\,\mathrm{Trace}(\mathbf{I}_\alpha - \mathbf{N}^T\mathbf{O}\mathbf{O}^T\mathbf{N}).$$

This simplifies to

$$E[\|\mathbf{e}\|^2] = \sigma^2\left\{\sum_{i=1}^{\alpha}[1 - \lambda_i^2(\mathbf{O}^T\mathbf{N})]\right\},$$

where $\lambda_i(\mathbf{O}^T\mathbf{N})$ denotes the ith singular value of $\mathbf{O}^T\mathbf{N}$.

Theorem 3. *$0 \le E[\|\mathbf{e}\|^2] \le \mathrm{Trace}\ \mathbf{R}$.*

Proof: The following inequality[11] is needed for the proof:

$$0 \le \mathrm{Trace}\ \mathbf{X}^T\mathbf{Q}\mathbf{X} \le \mathrm{Trace}\ \mathbf{Q} \tag{A2}$$

for positive semidefinite $\mathbf{Q}$ and $\mathbf{X}$ such that $\mathbf{X}^T\mathbf{X} = \mathbf{I}$. The maximum value is achieved when all the energy in $\mathbf{Q}$ lies inside the space spanned by the column vectors of $\mathbf{X}$. When all the energy in $\mathbf{X}$ lies outside the subspace spanned by the columns of $\mathbf{N}$,

$$\mathrm{Trace}\ \mathbf{X}^T\mathbf{Q}\mathbf{X} = 0.$$

Application of the inequality to error expression (14) gives

$$0 \le E[\|\mathbf{e}\|^2] \le \mathrm{Trace}(\mathbf{I} - \mathbf{O}\mathbf{O}^T)\mathbf{R}(\mathbf{I} - \mathbf{O}\mathbf{O}^T), \tag{A3}$$

because $(\mathbf{I} - \mathbf{O}\mathbf{O}^T)\mathbf{R}(\mathbf{I} - \mathbf{O}\mathbf{O}^T)$ is positive semidefinite and $\mathbf{N}^T\mathbf{N} = \mathbf{I}$. Application of inequality (A2) to inequality (A3) gives the result.

Theorem 4. *$\Sigma_{i=1}^{\alpha}\lambda_i^2(\mathbf{O}^T\mathbf{N}) = \Sigma_{i=1}^{\beta}q(\mathbf{o}_i)$.*

Proof:

$$\sum_{i=1}^{\beta}q(\mathbf{o}_i) = \sum_{i=1}^{\beta}\|P_V(\mathbf{o}_i)\|^2 = \mathrm{Trace}\{[P_V(\mathbf{O})]^T P_V(\mathbf{O})\}.$$

Now

$$\mathrm{Trace}[P_V(\mathbf{O})]^T P_V(\mathbf{O}) = \mathrm{Trace}(\mathbf{O}^T\mathbf{N}\mathbf{N}^T\mathbf{N}\mathbf{N}^T\mathbf{O}) = \mathrm{Trace}(\mathbf{O}^T\mathbf{N}\mathbf{N}^T\mathbf{O}).$$

Hence

$$\sum_{i=1}^{\beta}q(\mathbf{o}_i) = \mathrm{Trace}(\mathbf{O}^T\mathbf{N})(\mathbf{O}^T\mathbf{N})^T = \mathrm{Trace}(\mathbf{N}^T\mathbf{O})(\mathbf{N}^T\mathbf{O})^T,$$

as the eigenvalues of $\mathbf{X}\mathbf{X}^T$ are identical to those of $\mathbf{X}^T\mathbf{X}$. This gives

$$\sum_{i=1}^{\beta}q(\mathbf{o}_i) = \sum_{i=1}^{\alpha}\lambda_i^2(\mathbf{O}^T\mathbf{N}).$$

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