CHAPTER 5

Error Analysis in the Design of Colour Scanning Filters

Chapter 4 demonstrates the use of the measure developed in Chapter 3 as an optimization criterion in the problem of the design of colour scanning filters. It also demonstrates that it is often not possible to fabricate the designed 'optimal' filters exactly. The fabricated filters will not have the specified transmissivities, and this perturbation in filter transmissivities leads to a general degradation of filter performance demonstrated by larger average ΔE_{Lab} errors and smaller values of the measure. This chapter addresses the effect of the perturbation on the average square ΔE_{Lab} error of the scanner output and on the data-independent measure of Chapter 3. Quantitative estimates of the sensitivity of average square ΔE_{Lab} errors and of the data-independent measure to filter transmissivities as a function of wavelength are presented. Such estimates are very useful for the manufacturer. Experimental data demonstrates the accuracy of the sensitivity estimates. Error modelling provides a basis for calculating worst-case bounds on filter fabrication errors. Bounds on maximum allowable filter fabrication errors as a function of wavelength are presented.

In section 3.5.4, it is explained why the ΔE error is not an appropriate error measure for the definition of a measure of goodness of a set of scanning filters. The case of the sensitivity of the average ΔE error to small perturbations in filter characteristics is different, however. 'Optimal' scanning filters can be designed using the data-independent measure of Chapter 3. The filters may be 'trimmed' to obtain filter sets resulting in small mean square ΔE_{Lab} errors. Thereafter, required accuracy in filter fabrication at particular wavelengths may be determined by sensitivity curves of the mean square ΔE_{Lab} error.

The chapter is organized as follows. Section 1 presents the Taylor series for scalar functions of matrices. The Taylor series expansion is the basis for the expression developed for sensitivity of the average square ΔE_{Lab} error and of the data-independent measure to filter fabrication errors. Section 2 develops a quantitative measure of the effect of small perturbations in the colour scanning filters on the data-independent measure and section 3 develops the same for the average square ΔE_{Lab} error. Section 4 demonstrates how these quantitative estimates can be used to define bounds on the allowable error in filter fabrication, given a particular tolerance for maximum average square ΔE_{Lab} error or a minimum allowable value for the data-independent measure. Section 5 presents simulation results to support the claims in Sections 2, 3 and 4. Conclusions are presented in Section 6.

5.1 Taylor Series Expansion for Scalar Functions of Matrices

The Taylor series forms the basis of approximation techniques that use the derivatives of a function. The definitions and approximations associated with the Taylor series are presented in this section to provide the mathematical background for the approximations and the sensitivity analysis of the following sections. To begin, consider the Taylor formula for a scalar function of a single variable. For a scalar function y of a single variable x, the first differential at x_0 due to a change δx in the argument, is the linear part of the increment in the function. It may be expressed as [21, pg. 81]

$$dy(x_0; \delta x) = \left.\frac{\partial y}{\partial x}\right|_{x=x_0} \delta x$$

Notice that the first differential depends on both x_0 and δx, but that the term $\left.\frac{\partial y}{\partial x}\right|_{x=x_0}$ depends only on x_0. Hence the first differential is linear in δx. Similarly, the second differential is a quadratic expression in δx. It can be written as:

$$d^2y(x_0; \delta x) = \left.\frac{\partial^2 y}{\partial^2 x}\right|_{x=x_0} (\delta x)^2$$

and is a function of both x_0 and δx. The second-order Taylor series approximation for the value of y at a point $x + \delta x$ is:

$$y(x_0 + \delta x) \approx y(x_0) + dy(x_0; \delta x) + \frac{1}{2} d^2 y(x_0; \delta x)$$

The ideas of first and second differentials and their use in the approximation of the increment in a function due to a change in the argument can be extended to scalar functions of vector-valued arguments as follows.

The following second-order Taylor formula may be used to approximate scalar function g in the neighbourhood of a vector-valued argument $\mathbf{x}_0$ if g is twice-differentiable at $\mathbf{x}_0$ [21, pg. 108]:

$$g(\mathbf{x}_0 + \delta\mathbf{x}) = g(\mathbf{x}_0) + dg(\mathbf{x}_0; \delta\mathbf{x}) + \frac{1}{2} d^2 g(\mathbf{x}_0; \delta\mathbf{x}) + r_c(\delta\mathbf{x})$$

where

$$\lim_{||\delta\mathbf{x}|| \to 0} \frac{r_c(\delta\mathbf{x})}{||\delta\mathbf{x}||^2} = 0$$

and dg is the first differential of g, d^2g the second differential, and both are functions of $\mathbf{x}_0$ and $\delta\mathbf{x}$. Hence the change in scalar function g when $\mathbf{x}$ changes by a 'small' amount $\delta\mathbf{x}$ may be approximated as

$$\delta g = g(\mathbf{x}_0 + \delta\mathbf{x}) - g(\mathbf{x}_0) \approx dg(\mathbf{x}_0; \delta\mathbf{x}) + \frac{1}{2}d^2g(\mathbf{x}_0; \delta\mathbf{x}) \qquad (5.1)$$

Given a scalar function of a matrix $\mathbf{X}$, the function may be written as a scalar function g of the vector $vec(\mathbf{X})$. In the particular application treated here, it is sought to approximate the change in the average square ΔE_{Lab} error over a data set (or the change in the data-independent measure) if the designed filters $\mathbf{M}$ are perturbed to give fabricated filters $\mathbf{M} + \Delta\mathbf{M}$. Recall that the average square ΔE_{Lab} error over a data set is denoted $\frac{\sum_{\mathbf{f}} \Delta E_{Lab}^2(\mathbf{f})}{n}$, where $\sum_{\mathbf{f}}$ denotes the sum over all data points and n is the number of data points. The differential of the mean square ΔE_{Lab} was derived in section 4.5.3, and can be written as (see equation 4.12):

$$dg = d\frac{\sum_{\mathbf{f}} \Delta E_{Lab}^2(\mathbf{f})}{n} = \mathbf{D}\frac{\sum_{\mathbf{f}} \Delta E_{Lab}^2(\mathbf{f})}{n}(vec\; d\mathbf{M}) \approx \mathbf{D}\frac{\sum_{\mathbf{f}} \Delta E_{Lab}^2(\mathbf{f})}{n}(vec\boldsymbol{\Delta}\mathbf{M})$$

if $\boldsymbol{\Delta}\mathbf{M}$ is 'small'. Similarly, from equation (4.17), section 4.5.2, the differential of the data-independent measure can be written as:

$$dg = d\nu(\mathbf{M}) \approx \{vec(\frac{2\mathbf{H}(\mathbf{I} - P_{M_H})P_V\mathbf{M}_H(\mathbf{M}_H^T\mathbf{M}_H)^{-1}}{3})\}^T(vec\boldsymbol{\Delta}\mathbf{M})$$

The above formulae provide an explicit means of quantifying the first differential of the data-independent measure ν or the mean square ΔE_{Lab} error if the perturbation $\boldsymbol{\Delta}\mathbf{M}$ is known.

When the designed filters are trimmed with respect to the mean square ΔE_{Lab} error, the resulting trimmed filters will be close to optimal, and $d\frac{\sum_{\mathbf{f}} \Delta E_{Lab}^2(\mathbf{f})}{n}$ will be close to zero. Similarly, when the designed filters are trimmed with respect to ν, $d(\nu)(\mathbf{M})$ will be close to zero. This means that the second term in equation (5.1) will be dominant in the expression for δg in either case. Hence, it is necessary to obtain expressions for the second differential which are similar to the above expressions for the first differential, so that the second differential may be calculated if the perturbation $\mathbf{\Delta M}$ is known. Experimental results confirm that the second term, $\frac{1}{2}(d^2 g)$, is a fairly good approximation to the change, and that it contributes a much larger part of the change than does the first differential for the range of values of $\mathbf{\Delta M}$ under consideration. The next section deals with the problem of finding the second differential of the data-independent measure ν.

5.2 The Second Differential of the Data-Independent Measure ν

The following notation simplifies the expression for the second differential. Given an $m \times n$ matrix,

$$\mathbf{X} = \begin{bmatrix} x_{11} & \cdots\cdot & x_{1n} \\ \vdots & & \vdots \\ x_{m1} & \cdots\cdot & x_{mn} \end{bmatrix},$$

and any other matrix $\mathbf{Y}$, the symbol $\oplus$ represents the Kronecker product,

$$\mathbf{X} \oplus \mathbf{Y} = \begin{bmatrix} x_{11}\mathbf{Y} & \cdots\cdot & x_{1n}\mathbf{Y} \\ \vdots & & \vdots \\ x_{m1}\mathbf{Y} & \cdots\cdot & x_{mn}\mathbf{Y} \end{bmatrix}$$

Let $\mathbf{K}$ be such that [21]

$$vec\ d\mathbf{M} = vec\ d\mathbf{M}^T\mathbf{K}$$

Equation (4.16) gives an expression for the first differential, $d\nu$. The second differential, $d^2\nu$, is the first differential of the first differential, i.e., $d^2\nu = d(d\nu)$. The second differential of the data-independent measure is (see Appendix, Theorem 9)

$$d^2\nu = -(vecd\mathbf{M})^T \mathcal{H} vecd\mathbf{M}$$

where

$$\mathcal{H} = \frac{1}{2}(\mathcal{A} + \mathcal{A}^T)$$

and

$$\begin{aligned}\mathcal{A} = \frac{2}{3}\{\mathbf{H}[-(\mathbf{M}_H^T\mathbf{M}_H)^{-1} \oplus (\mathbf{I} - P_{M_H})P_V(\mathbf{I} - P_{M_H}) \\ + (\mathbf{M}_H^T\mathbf{M}_H)^{-1}\mathbf{M}_H^T P_V \mathbf{M}_H(\mathbf{M}_H^T\mathbf{M}_H)^{-1} \oplus (\mathbf{I} - P_{M_H}) \\ + 2\mathbf{K}(\mathbf{I} - P_{M_H})P_V\mathbf{M}_H(\mathbf{M}_H^T\mathbf{M}_H)^{-1} \oplus (\mathbf{M}_H^T\mathbf{M}_H)^{-1}\mathbf{M}_H^T]\mathbf{H}\}\end{aligned}$$

The matrix $\mathcal{H}$ is assumed positive semi-definite because it is the negative of the Hessian matrix of a function near a local maximum. If it were not positive semi-definite, there would be values of $\mathbf{\Delta M}$ for which the change in the data-independent measure ν would be positive, i.e. the measure ν would not be a local maximum. The next section deals with the problem of finding the second differential for the average square ΔE_{Lab} error.

5.3 The Second Differential of the Mean-Square ΔE_{Lab} Error

In this chapter, no explicit reference will be made to the dependence on the data point $\mathbf{f}$ of the matrices $\mathbf{\Omega}$ defined in equation (4.20) and $\mathbf{\Xi}$ defined in equation (4.24). Instead of $\mathbf{\Omega}(\mathbf{f})$ and $\mathbf{\Xi}(\mathbf{f})$, the matrices will be denoted as $\mathbf{\Omega}$ and $\mathbf{\Xi}$. The column vectors $\hat{\mathbf{t}}(\mathbf{f})$ and $\mathbf{c}(\mathbf{f})$ are also denoted $\hat{\mathbf{t}}$ and $\mathbf{c}$. All other matrices that are defined in this chapter will be denoted similarly. The following notation simplifies the expression for the second differential. Let

$$\mathbf{\Theta} = (\mathbf{M}_H^T \mathbf{R} \mathbf{M}_H)^{-1},$$

$$\mathbf{\Psi} = \mathbf{I} - \mathbf{M}_H (\mathbf{M}_H^T \mathbf{R} \mathbf{M}_H)^{-1} \mathbf{M}_H^T \mathbf{R},$$

$$\mathbf{\Lambda} = 2 \begin{bmatrix} \frac{500(a-a_f)}{9(x^5 x_n)^{1/3}} & 0 & 0 \\ 0 & \frac{116(L-L_f) - 500(a-a_f) + 200(b-b_f)}{9(y^5 y_n)^{1/3}} & 0 \\ 0 & 0 & \frac{-200(b-b_f)}{9(z^5 z_n)^{1/3}} \end{bmatrix},$$

and

$$\mathcal{G} = 2\mathbf{R}\mathbf{A}(\mathbf{\Omega}\mathbf{\Upsilon}^T\mathbf{\Upsilon}\mathbf{\Omega} - \mathbf{\Lambda})\mathbf{A}^T\mathbf{R}$$

where Υ is defined in equation(1.5) and Ω in equation (4.20). The second differential of the mean square ΔE_{Lab} error is (see Appendix, Theorem 10):

$$d^2 g = (vecd\mathbf{M})^T \mathcal{H} vecd\mathbf{M}$$

where

$$\mathcal{H} = \frac{1}{2}(\mathcal{A} + \mathcal{A}^T)$$

and

$$\mathcal{A} = (\mathbf{H}[(\mathbf{\Theta} \oplus \mathbf{\Psi}^T(\frac{\sum_{\mathbf{f}} \mathbf{\Xi}}{n})\mathbf{\Psi}) + \frac{2}{n}\sum_{\mathbf{f}}(\mathbf{K}\mathbf{\Psi}^T\mathcal{G}\mathbf{M}_H\mathbf{\Theta} \oplus \mathbf{\Theta}\mathbf{M}_H^T\mathbf{f}\mathbf{f}^T\mathbf{\Psi})$$

$$+ \frac{1}{n}\sum_{\mathbf{f}}(\mathbf{\Theta}\mathbf{M}_H^T\mathcal{G}\mathbf{M}_H\mathbf{\Theta} \oplus \mathbf{\Psi}^T\mathbf{f}\mathbf{f}^T\mathbf{\Psi}) + \frac{1}{n}\sum_{\mathbf{f}}(\mathbf{\Theta}\mathbf{M}_H^T\mathbf{f}\mathbf{f}^T\mathbf{M}_H\mathbf{\Theta} \oplus \mathbf{\Psi}^T\mathcal{G}\mathbf{\Psi})$$

$$-2\,(\mathbf{K}\mathbf{R}\mathbf{M}_H\mathbf{\Theta} \oplus \mathbf{\Theta}\mathbf{M}_H^T(\frac{\sum_{\mathbf{f}} \mathbf{\Xi}}{n})\mathbf{\Psi}) - (\mathbf{\Theta}\mathbf{M}_H^T(\frac{\sum_{\mathbf{f}} \mathbf{\Xi}}{n})\mathbf{M}_H\mathbf{\Theta} \oplus \mathbf{R}\mathbf{\Psi})]\mathbf{H})$$

The matrix $\mathcal{H}$ is assumed positive semi-definite because it is the Hessian matrix of a function near a local minimum.

5.4 Worst Case Bounds on Fabrication Errors

The expressions for the second differentials of the data-independent measure and the mean square ΔE_{Lab} error, derived in sections 5.2 and 5.3 respectively, may be used to study the effect of a small change $\mathbf{\Delta M}$ in the scanning filters. In both cases the absolute value of the change in the scalar function due to a small change of ΔM in close-to-optimal scanning filters $\mathbf{M}$, may be expressed as

$$|\delta g| \;\approx\; |\frac{1}{2}d^2 g(vec\ \mathbf{M}; vec\ \mathbf{\Delta M})| \;\approx\; \frac{1}{2}(vec\mathbf{\Delta M})^T \mathcal{H} vec d\mathbf{\Delta M} \qquad (5.2)$$

for relevant matrices $\mathcal{H}$ which have been derived in the previous sections. Given the matrix $\mathcal{H}$ it is possible to calculate the maximum allowable fabrication errors for the scanning filters, given maximum allowable change in g. Three methods of perturbing the trimmed filter designs will be investigated. The first one deals with perturbing the filter design at a single wavelength. The second deals with perturbing the filter design by the same amount at each wavelength. The third deals with perturbing the filter design by an error vector of fixed euclidean-norm.

Suppose that an error of $\pm\omega_k$ is made at the k^{th} wavelength and that all other fabrication errors are zero. Then,

$$|\delta g| \;\approx\; \frac{1}{2}\mathcal{H}_{kk}\omega_k^2$$

As the matrix $\mathcal{H}$ is defined so as to be positive semi-definite, $\mathcal{H}_{kk}$ is non-negative, and

$$|\delta g| \leq \epsilon$$

is ensured by

$$|\omega_k| \leq \sqrt{\frac{2\epsilon}{\mathcal{H}_{kk}}} \;\stackrel{\text{def}}{=}\; \omega_1(k) \qquad (5.3)$$

Thus, inequality 5.3 provides a measure of the maximum allowable fabrication error as a function of wavelength. The bound of inequality (5.3) will be referred to as the *single-wavelength bound.* Note that isolated fabrication errors are not likely to occur. Hence, the single-wavelength bound does not provide a bound that is likely to represent a physical situation. Further, the second differential is not a linear expression in the fabrication errors at individual wavelengths, hence the single-wavelength bound does not provide changes that can be added when fabrication errors occur at more than one wavelength. The advantage of the single-wavelength bound is that it provides a qualitative estimate of which wavelengths are most/least sensitive to fabrication errors.

A bound on the maximum change if the error of $\pm\omega$ occurs at each wavelength is given by

$$|\delta g| \leq \frac{1}{2}\omega^2 \sum_{i=1}^{rN} \sum_{j=1}^{rN} |\mathcal{H}_{ij}|$$

Hence, using the maximum variation for each element in the perturbation vector, ΔM, the change in the mean square ΔE_{Lab} error or the data-independent measure can be bounded by ρ:

$$|\delta g| \leq \rho$$

if

$$\omega \; = \; \overset{max}{i} \; (|vec\mathbf{\Delta M}_i|) \leq \sqrt{\frac{2\rho}{\sum_{i=1}^{rN} \sum_{j=1}^{rN} |\mathcal{H}_{ij}|}} \; \stackrel{\text{def}}{=} \; \omega_2 \qquad (5.4)$$

It is not easy to find the error vector for which the bound of inequality 5.4 is achieved, in fact it is not necessary for the bound to be achieved. Inequality 5.4 relates the infinity norm of the error vector $vec\ \mathbf{\Delta M}$ to the maximum acceptable change in the mean square ΔE_{Lab} error or the data-independent measure. This bound on the maximum change at all wavelengths will be referred to as the *all-wavelength bound.* This bound presents a realistic bound for errors that are likely to occur in real situations. On the other hand, it does not provide any information about wavelengths that are more or less sensitive to fabrication errors.

In order to relate the change in g to the 2-norm or euclidean norm of the error vector, observe the following [12]:

$$\frac{1}{2}(vec\mathbf{\Delta M})^T \mathcal{H} vecd\mathbf{\Delta M} \leq \frac{1}{2}||vec\mathbf{\Delta M}||^2 \lambda_{max}$$

where λ_{max} is the largest eigenvalue of $\mathcal{H}$, and $||.||$ denotes the 2-norm or the euclidean norm. This implies that

$$\delta g \leq \gamma$$

if the fabrication error $\mathbf{\Delta M}$ is such that

$$||vec\mathbf{\Delta M}|| \leq \sqrt{\frac{2\gamma}{\max_i \lambda_i}} \stackrel{\text{def}}{=} \omega_3 \tag{5.5}$$

where ω_3 is defined as the euclidean-norm bound. The bound γ is achieved when vec $\mathbf{\Delta M}$ is an eigenvector of $\mathcal{H}$ corresponding to the largest eigenvalue.

It is possible to relate the single-wavelength and the all-wavelength bounds as follows. Suppose δg is the allowed change in the mean square ΔE_{Lab} error or in the data-independent measure ν. Then, from equations (5.3) and (5.4), (using $\rho = \epsilon = \delta g$)

$$\omega_1(k) = \sqrt{\frac{\sum_{i=1}^{rN} \sum_{j=1}^{rN} |\mathcal{H}_{ij}|}{\mathcal{H}_{kk}}} \omega_2$$

which implies that

$$\omega_1(k) \geq \omega_2$$

as expected. The allowable error at a single wavelength (represented by the single-wavelength bound, ω_1) should be larger than the allowable error at all wavelengths (represented by the all-wavelength bound ω_2) for a fixed change in the measure or in the mean square ΔE_{Lab} error.

To relate the single-wavelength and all-wavelength bounds with the euclidean-norm bound, consider the following. Suppose δg is the allowed change in the mean square ΔE_{Lab} error or in the data-independent measure ν. Then, from equations (5.3) and (5.5) (where $\epsilon = \gamma = \delta g$),

$$\omega_1(k) = \sqrt{\frac{\lambda_{max}}{\mathcal{H}_{kk}}}\ \omega_3$$

It is not possible to assign an ordering to these bounds without knowing the values of λ_{max} and $\mathcal{H}_{kk}$, unless the matrix $\mathcal{H}$ is diagonal. If $\mathcal{H}$ is diagonal, the values of $\mathcal{H}_{kk}$ are the eigenvalues of $\mathcal{H}$, and $\omega_1(k) \geq \omega_3$. Further, from equations (5.4) and (5.5) (where $\rho = \gamma = \delta g$),

$$\omega_2 = \sqrt{\frac{\lambda_{max}}{\sum_{i=1}^{rN}\sum_{j=1}^{rN}|\mathcal{H}_{ij}|}}\ \omega_3$$

As $\mathcal{H}_{kk}$ is non-negative, and the trace of a matrix is the sum of its eigenvalues, $\omega_2 \leq \omega_3$. Thus, the all-wavelength bound is smaller than both the single-wavelength bound and the euclidean-norm bound. Whether the single-wavelength bound is larger than the euclidean-norm bound is case-specific and cannot be determined *a priori*.

5.5 Experimental Results

The bounds derived in the previous section are tested in this section. The tests are first performed on the bounds for the data-independent measure ν, and then on those for the mean square ΔE_{Lab} error.

5.5.1 Data-Independent Measure ν

The second-order approximation to the change in the measure ν is tested on the trimmed designed filters in Fig. 4.25 (referred to as Filter Set 1) and Fig 4.26 (referred to as Filter Set 2). The single-wavelength bounds are tested for a change of 0.005 and the all-wavelength bound is tested for a change of 0.05 in the measure ν.

Second Order Approximation

Consider the vector $vec\ \Delta\mathbf{M}$ defined by the difference between the trimmed, specified Filter Sets 1 and 2 and the manufactured filters indicated by dotted lines in Figs. 4.9 - 4.11 and Figs. 4.12 - 4.14 respectively. Using this vector $\Delta\mathbf{M}$, equation (5.2) is used to predict the change in the measure ν. This prediction is compared to the actual change in Table 5.1.

Table 5.1: Predicted and Actual Values of Measure ν-1

Set	Original Measure	Predicted Measure Equation (5.2)	Actual Measure	$\|\|vec\ \Delta\mathbf{M}\|\|^2$
1	0.9994	0.9849	0.9900	0.0777
2	0.9988	0.9593	0.9769	0.4623

The euclidean norm of $vec\ \Delta\mathbf{M}$ can be used to compute a bound on the change in the measure, calculated from inequality (5.5). The bound on the measure predicted by this change is -0.2006 for Filter Set 1 and -35.43 for Filter Set 2. It is clear that the euclidean-norm bound of equation (5.5) provides too large an estimate for the bound on the change in the measure, in general.

Table 5.2 provides a comparison similar to the one in Table 5.1 between predicted and actual errors for some randomly generated actual filters sets while using the trimmed filter sets 1 and 2. The observations indicate that equation (5.2) provides a fair estimate of the change in the measure. The reason for the slight discrepancy is that the second-order Taylor series approximation is not accurate. This is either because the error is too large or the higher-order derivatives too high for the approximation to be totally valid.

Table 5.2: Predicted and Actual Values of Measure ν-2

Set	Original Measure	Predicted Measure Equation (5.2)	Actual Measure	$\|\|vec\ \Delta\mathbf{M}\|\|^2$
1	0.9994	0.9623	0.9774	0.1615
1	0.9994	0.9812	0.9887	0.0628
1	0.9994	0.9885	0.9926	0.0280
2	0.9988	0.9898	0.9928	0.1232
2	0.9988	0.9975	0.9977	0.0378
2	0.9988	0.9943	0.9937	1.3492

Single-Wavelength Bounds

Single-wavelength bounds calculated from equation (5.3) for $\epsilon = 0.005$ are plotted as a function of wavelength in Figs. 5.1-5.6 for Filter Sets 1 and 2. The dotted lines indicate the bounds, and the solid line indicates the designed filter in all plots. It is clear from the figures that the red filter is the most sensitive to fabrication errors, because the single-wavelength bounds for the red filter are the smallest among all three filters for both sets 1 and 2. Further, the bounds appear smallest close to the slope of the red filter, and it is this slope that needs to be fabricated with accuracy.

Table 5.3 lists the maximum and minimum allowable fabrication errors at a single-wavelength for a change of 0.005 in the measure. The filters and the wavelengths at which the respective errors may occur are also listed. This allows comparison of the sensitivities of the two sets of filters. The values of the maximum and minimum allowable perturbations are similar for the two sets of filters, which implies similar sensitivities.

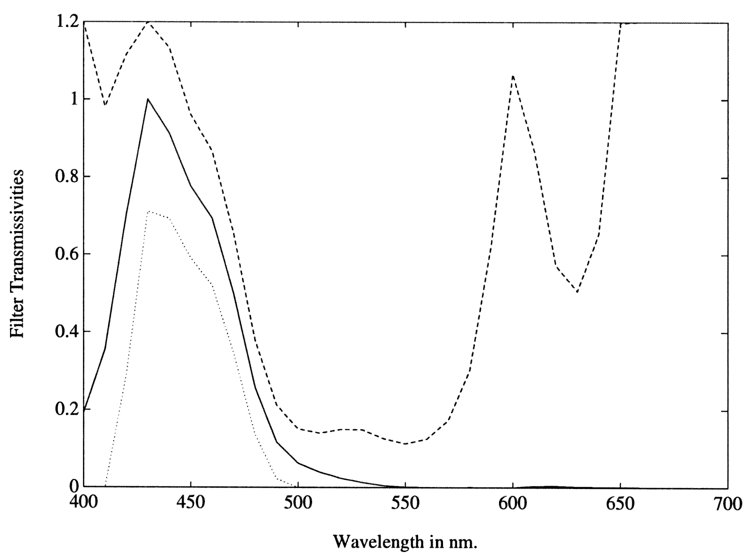

Figure 5.1: Single-Wavelength Bounds for Blue Filter of Filter Set 1 for a Predicted Change of 0.005 in Measure ν

The validity of the single-wavelength bounds was tested as follows. The trimmed filter sets were perturbed at exactly one wavelength by an amount equal to the single-

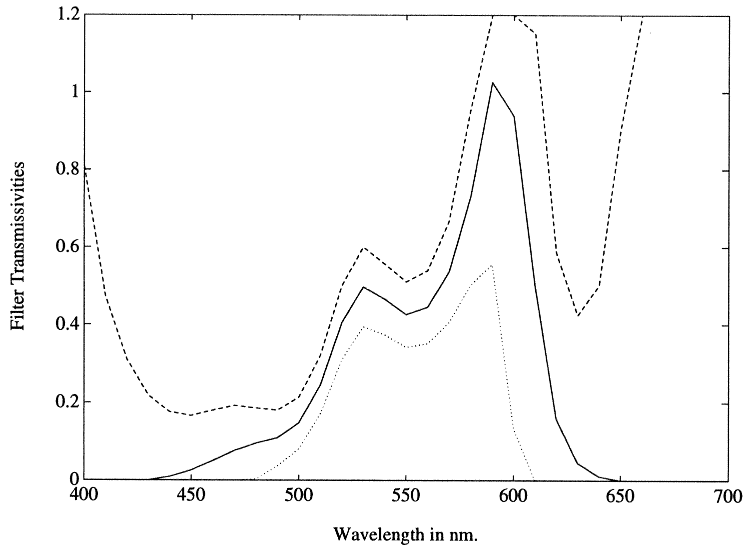

Figure 5.2: Single-Wavelength Bounds for Green Filter of Filter Set 1 for a Predicted Change of 0.005 in Measure ν

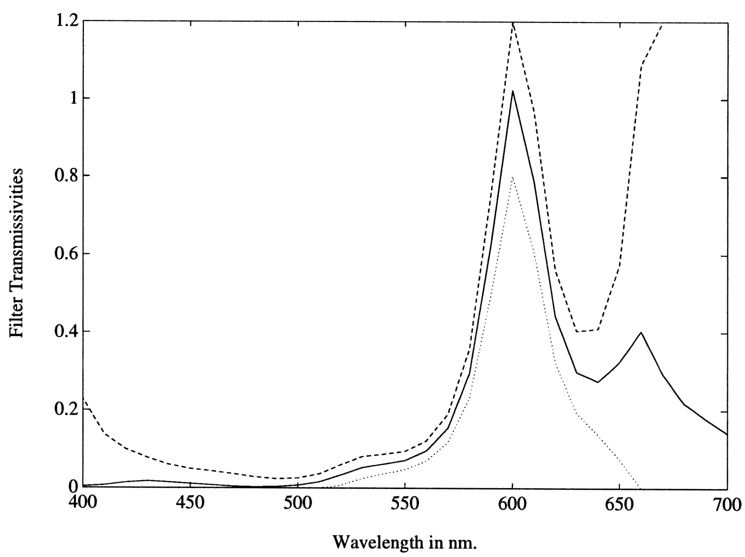

Figure 5.3: Single-Wavelength Bounds for Red Filter of Filter Set 1 for a Predicted Change of 0.005 in Measure ν

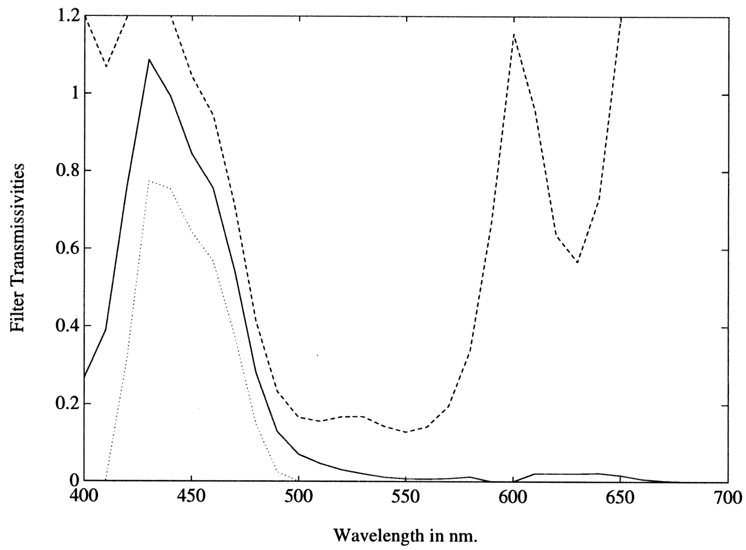

Figure 5.4: Single-Wavelength Bounds for Blue Filter of Filter Set 2 for a Predicted Change of 0.005 in Measure ν

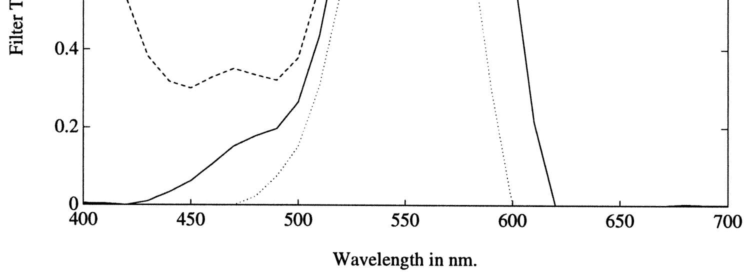

Figure 5.5: Single-Wavelength Bounds for Green Filter of Filter Set 2 for a Predicted Change of 0.005 in Measure ν

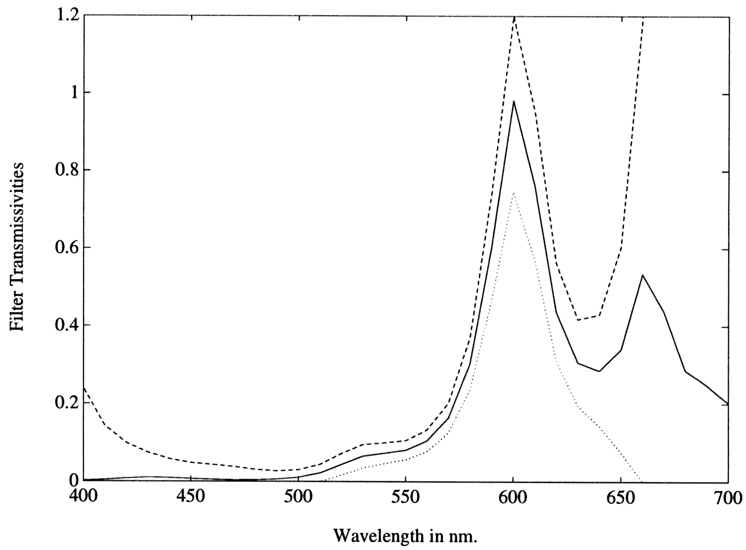

Figure 5.6: Single-Wavelength Bounds for Red Filter of Filter Set 2 for a Predicted Change of 0.005 in Measure ν

Table 5.3: Maximum and Minimum Values of Single-Wavelength Bound for a Predicted Change of 0.005 in Measure ν

Set		Perturbation	Filter	Wavelength in nm.
1	Minimum	0.0184	Red	500
	Maximum	113.4	Blue	680
2	Minimum	0.0195	Red	500
	Maximum	146.1	Green	680

wavelength bound at that wavelength, corresponding to a change of 0.005 in the measure. The measure of the perturbed filter set was calculated. This was done for each wavelength and each filter set. The lowest value of the measure obtained thus for Filter Set 1 was 0.9944 at the wavelength 670 nm for the red filter. The bound on the measure for the filter set is also 0.9944. Similarly, the lowest value of the measure obtained thus for Filter Set 2 was 0.9942 at the wavelength 680 nm for the red filter. The bound on the measure for this set is 0.9938. This implies that the second differential provides a reasonable bound for single-wavelength perturbations corresponding to a change of the order of 0.005 in the measure for the filter sets used.

All-Wavelength Bound

All-wavelength bounds corresponding to a change in measure $\rho = 0.05$ are 0.0203 and 0.0220 for Filter Sets 1 and 2 respectively. The bounds for the trimmed designs of Figs. 4.25 and 4.26 are plotted in Figs. 5.7-5.9 and Figs. 5.10-5.12. Note that the bounds plotted represent the same fabrication error at each wavelength, though at some points it appears as though the bounds are unequal. This is because of different slopes in the graphs, because of which the bounds appear differemt. Table 5.4 lists the error measures for the original designs and the corresponding perturbed designs indicated by the + symbol in the figures. In Table 5.4, the measure $\nu(\mathbf{A}, \mathbf{M}_H)$ is denoted ν, the predicted value of the measure Pν, the average ΔE_{Lab} error E, the maximum ΔE_{Lab} error E_{max}, the root mean square ΔE_{Lab} error RMS, and the mean square ΔE_{Lab} error MS. The calculated all-wavelength bound is denoted ω_2. The actual values of the measures are within the predicted bounds. Recall that the all-wavelength bound does not signify an error bound that is necessarily achievable.

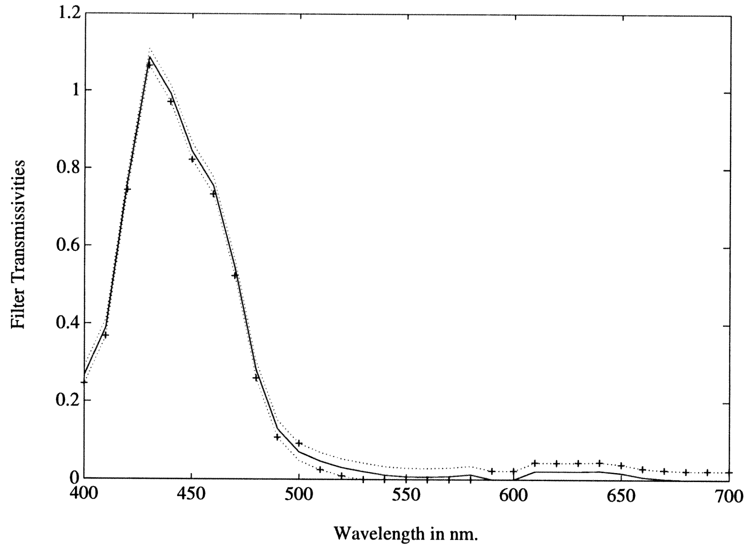

Figure 5.7: All-Wavelength Bound for Blue Filter of Filter Set 1 for a Predicted Change of 0.05 in Measure ν

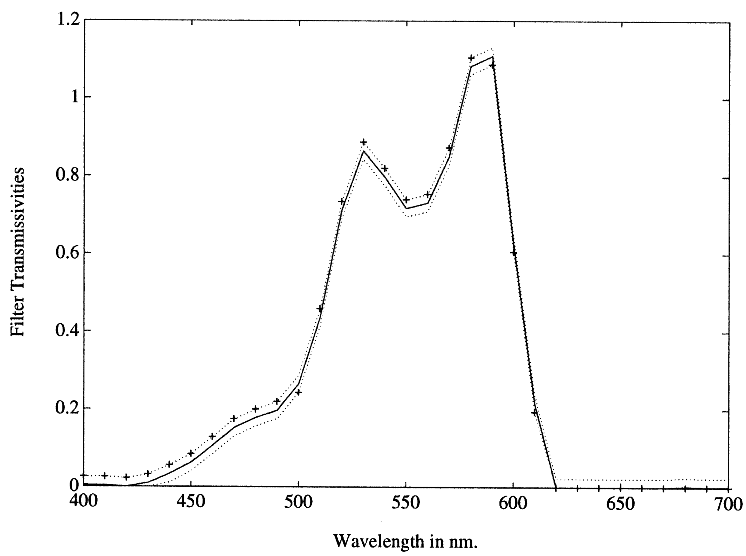

Figure 5.8: All-Wavelength Bound for Green Filter of Filter Set 1 for a Predicted Change of 0.05 in Measure ν

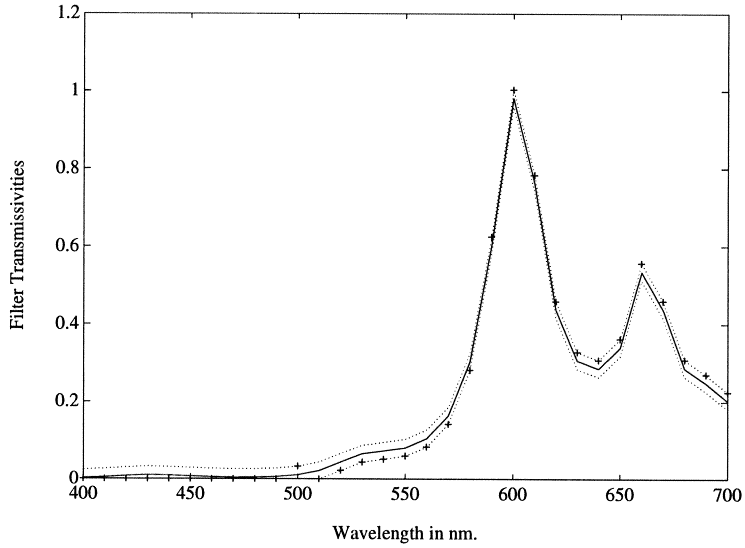

Figure 5.9: All-Wavelength Bound for Red Filter of Filter Set 1 for a Predicted Change of 0.05 in Measure ν

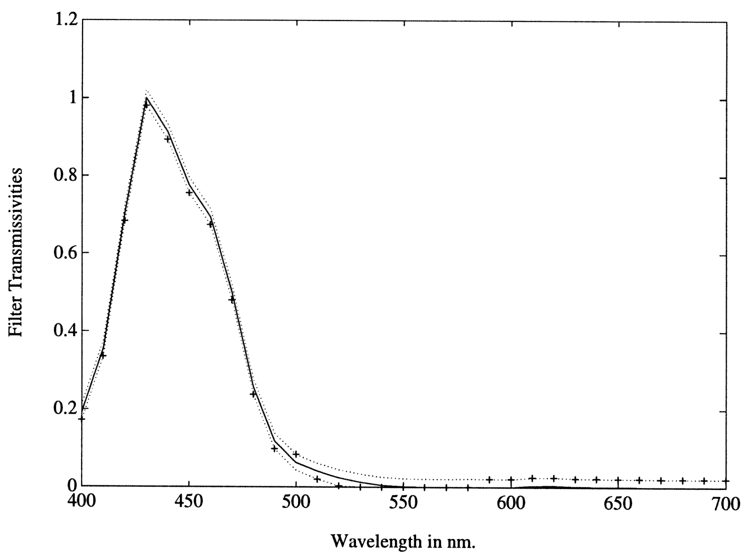

Figure 5.10: All-Wavelength Bound for Blue Filter of Filter Set 2 for a Predicted Change of 0.05 in Measure ν

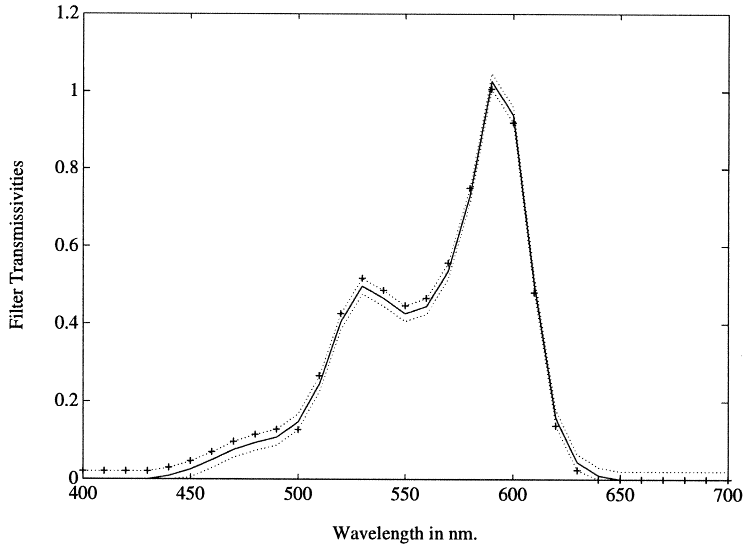

Figure 5.11: All-Wavelength Bound for Green Filter of Filter Set 2 for a Predicted Change of 0.05 in Measure ν

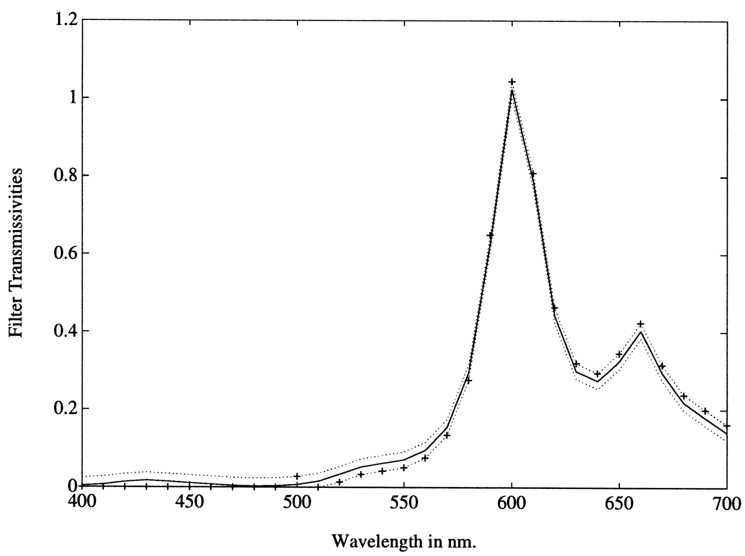

Figure 5.12: All-Wavelength Bound for Red Filter of Filter Set 2 for a Predicted Change of 0.05 in Measure ν

Table 5.4: Comparison Between Errors of Original Design and Perturbed Design for a Predicted Change of 0.05 in Measure ν

Set		ν	Pν	E	E_{max}	RMS	MS	ω_2	$\|\|vec\ \Delta\mathbf{M}\|\|^2$
1	Original	0.9994	0.9994	0.28	1.23	0.38	0.14	0	0
	Perturbed	0.9882	0.9494	0.39	1.64	0.54	0.29	0.0203	0.0383
2	Original	0.9988	0.9988	0.51	3.38	0.79	0.62	0	0
	Perturbed	0.9878	0.9488	0.54	3.31	0.81	0.65	0.0220	0.0450

5.5.2 Mean Square ΔE_{Lab} Error

The second-order approximation to the average square ΔE_{Lab} error is tested on the designed filters in Figs. 4.30 and 4.31. The single-wavelength bounds (inequality (5.3)) are tested for a change in mean square ΔE_{Lab} error of 0.005. The all-wavelength bound (inequality (5.4)) is calculated for a change in value of 1 in mean square ΔE_{Lab} error. All results are compared to corresponding results due to similar perturbations of the CIE matching functions under a uniform illuminant. As the CIE matching functions are the defining standards for the purpose of colour scanning, it is of interest to compare the sensitivity of the designed filters to that of the CIE matching functions. The comparison indicates that the designed filters are not more sensitive than the CIE functions themselves.

The trimmed single-gaussian model under Illuminant 2 of Fig. 4.30 is denoted Filter Set 1. The trimmed sum-of-gaussian model under Illuminant 1 (Set 1) of Fig. 4.31 is denoted Filter Set 2. The CIE matching functions under uniform illumination, normalized to a maximum value of unity to facilitate comparison with realizable filter transmissivities, are denoted Filter Set 3.

The Second-Order Approximation

Consider the value of $\Delta \mathbf{M}$ defined by the difference between the Filter Sets 1 and 2 and the manufactured filters indicated by dotted lines in Figs. 4.6-4.8 and 4.9-4.11 respectively. As in the previous section, this value of $\Delta \mathbf{M}$ is used to predict the change in mean square ΔE_{Lab} error. This prediction is compared to the actual change in Table 5.5.

The bound on the mean square ΔE_{Lab} error calculated from the euclidean norm

Table 5.5: Predicted and Actual Values of Mean Square ΔE_{Lab} Error-1

Set	Original Error	Predicted Error Equation (5.2)	Actual Error	$\|\|vec\ \Delta\mathbf{M}\|\|^2$
1	0.1001	1.0486	1.2366	0.0440
2	0.0322	0.4063	0.4088	0.0324

of $vec\ \Delta\mathbf{M}$ is 7.0 for Filter Set 1 and 28.28 for Filter Set 2. The error predicted using the euclidean-norm bound of equation (5.5) is much larger than the actual error, and, as in Table 5.1, this bound does not prove to be valuable is assessing changes in the ΔE_{Lab} error. Table 5.6 lists the predicted and actual errors for the trimmed filter sets 1 and 2, and randomly generated actual filter sets. The results tabulated in Tables 5.5 and 5.6 indicate that the second differential provides a fair estimate of the effect of fabrication error on mean square ΔE_{Lab} error. The estimate is slightly lower than the actual values. The reason for the discrepancy is, as in the previous section, that the higher-order derivatives contribute a non-negligible amount of the change, or that the norm of the error vector is not small enough.

Single-Wavelength Bounds

Single-wavelength bounds calculated from equation (5.3) for $\epsilon = 0.005$ are plotted as a function of wavelength in Figs. 5.13-5.21 for Filter Sets 1-3.

Table 5.7 lists the maximum and minimum value of the perturbations plotted for each of the designed filter sets, in the same manner as in the previous section. Notice that the minimum allowable error at a single wavelength is higher for the single-gaussian design than for the CIE matching functions. Further, the sum-of-gaussian

Table 5.6: Predicted and Actual Values of Mean Square ΔE_{Lab} Error-2

Set	Original Error	Predicted Error Equation (5.2)	Actual Error	$\|\|vec\ \Delta\mathbf{M}\|\|^2$
1	0.1001	0.2072	0.2717	0.0146
1	0.1001	0.3012	0.3398	0.0208
1	0.1001	0.1631	0.1936	0.0097
2	0.0322	0.3970	0.4201	0.0818
2	0.0322	0.2130	0.2249	0.0106
2	0.0322	0.1912	0.2024	0.0170

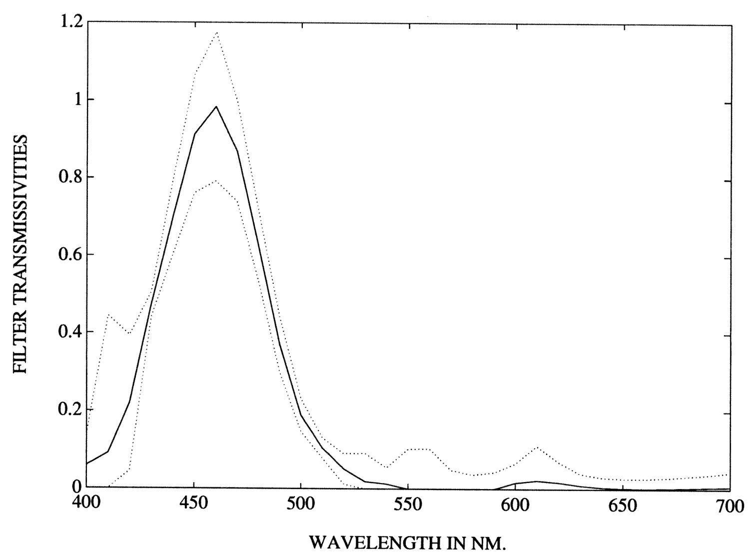

Figure 5.13: Single-Wavelength Bounds for Blue Filter of Trimmed Single Gaussian Model and Illuminant 2 for a Predicted Change of 0.005 in the Mean Square ΔE_{Lab} Error

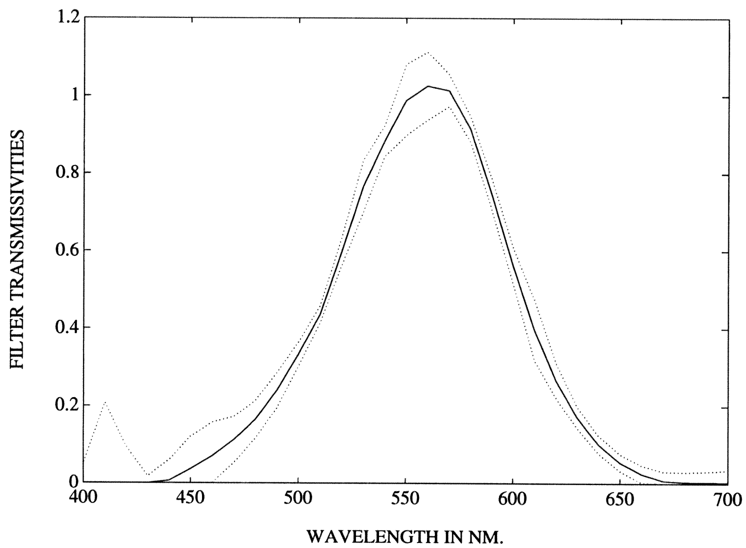

Figure 5.14: Single-Wavelength Bounds for Green Filter of Trimmed Single Gaussian Model and Illuminant 2 for a Predicted Change of 0.005 in the Mean Square ΔE_{Lab} Error

Table 5.7: Maximum and Minimum Values of the Single-Wavelength Bound for a Predicted Change of 0.005 in the Mean Square ΔE_{Lab} Error

Set		Perturbation	Filter	Wavelength in nm.
1	Minimum	0.0174	Green	430
	Maximum	0.3538	Blue	410
2	Minimum	0.0045	Red	500
	Maximum	7.4102	Green	680
3	Minimum	0.0123	Blue	700
	Maximum	0.2031	Blue	460

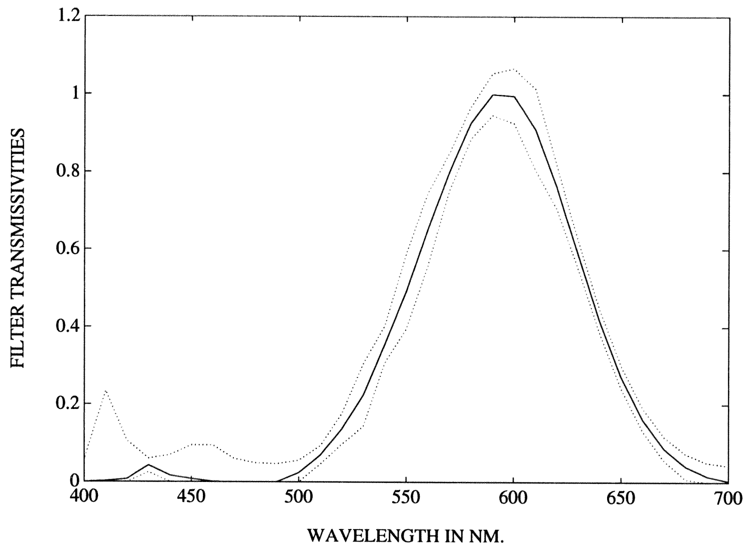

Figure 5.15: Single-Wavelength Bounds for Red Filter of Trimmed Single Gaussian Model and Illuminant 2 for a Predicted Change of 0.005 in the Mean Square ΔE_{Lab} Error

design allows a very large maximum value of fabrication error as compared to that allowed by the CIE matching functions. This indicates that the single gaussian and sum-of-gaussian designs are not unduly sensitive.

The single-wavelength bound was tested for each wavelength and each filter set, as in the previous section. The largest value of the mean square ΔE_{Lab} error on perturbation for Filter Set 1 was 0.1255, which is slightly larger than the bound for this set, which is 0.1051. This error occured at wavelength 400 nm in the blue filter. The largest value for Filter Set 2 was 0.0502 at 700 nm, for the red filter, for a bound of 0.0372. For Filter Set 3, the largest value was 0.0054 for a bound of 0.005, at 440 nm for the red filter.

There were many wavelengths in each filter where the mean square ΔE_{Lab} error

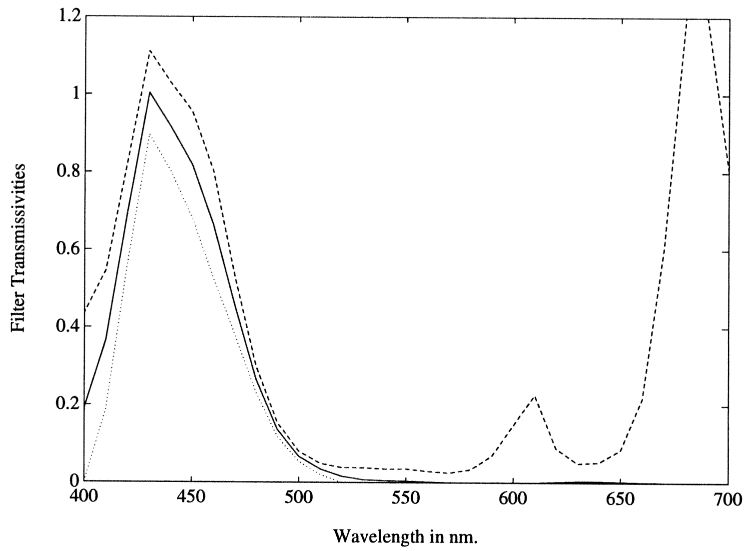

Figure 5.16: Single-Wavelength Bounds for Blue Filter of Trimmed Sum-of-Gaussian Model and Illuminant 1 for a Predicted Change of 0.005 in the Mean Square ΔE_{Lab} Error

was larger than the bound. In some of these cases the norm of the vector $vec\ \Delta\mathbf{M}$ was large enough to indicate that the Taylor series approximation of the second order was insufficient. In cases where the norm is not particularly high, it is not immediately clear what the reason for the inadequacy of the predicted bound is. A distinct possibility is that the values of the higher derivatives are large enough at these points to make a second-order Taylor approximation invalid.

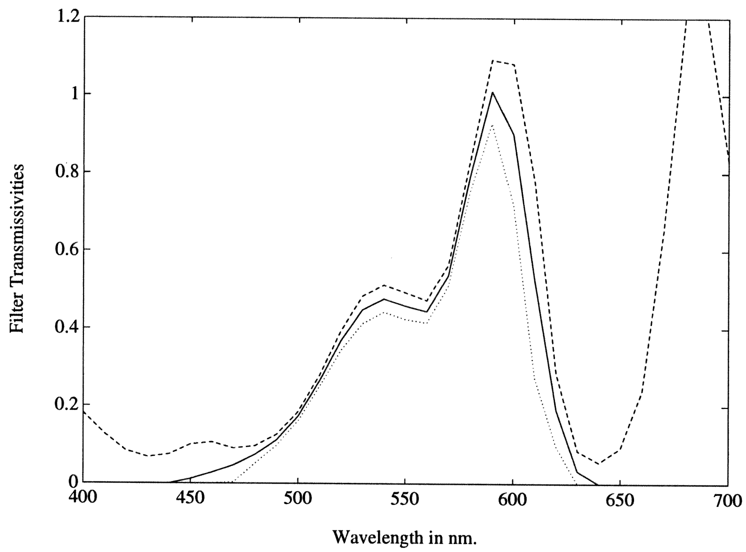

Figure 5.17: Single-Wavelength Bounds for Green Filter of Trimmed Sum-of-Gaussian Model and Illuminant 1 for a Predicted Change of 0.005 in the Mean Square ΔE_{Lab} Error

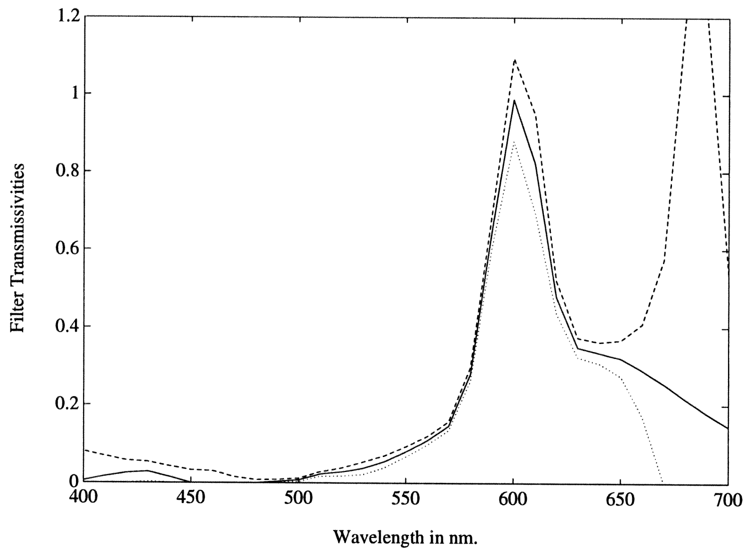

Figure 5.18: Single-Wavelength Bounds for Red Filter of Trimmed Sum-of-Gaussian Model and Illuminant 1 for a Predicted Change of 0.005 in the Mean Square ΔE_{Lab} Error

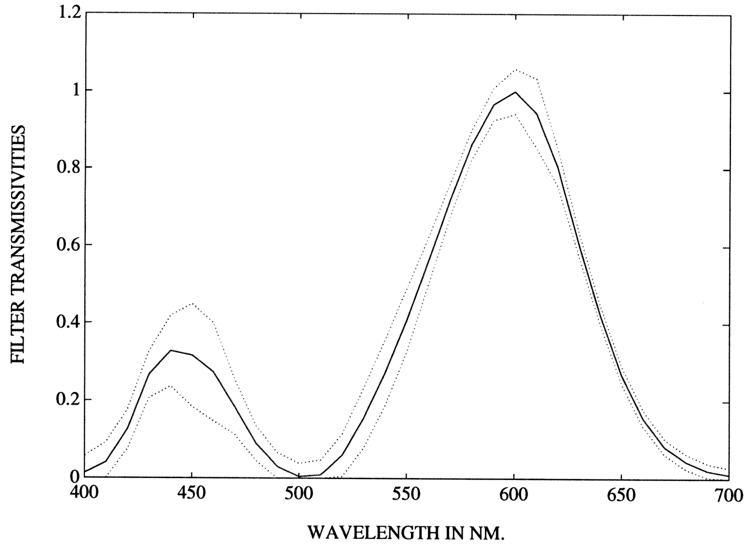

Figure 5.19: Single-Wavelength Bounds for Blue Filter of the CIE functions under a Uniform Illuminant for a Predicted Change of 0.005 in the Mean Square ΔE_{Lab} Error

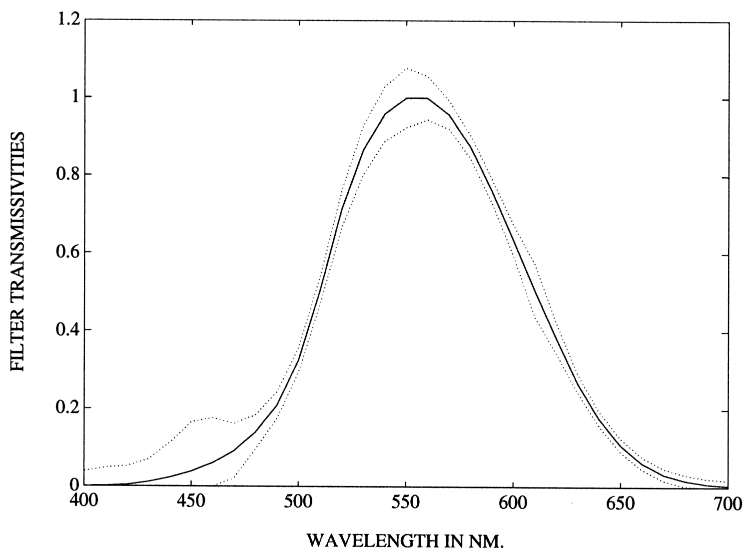

Figure 5.20: Single-Wavelength Bounds for Green Filter of the CIE functions under a Uniform Illuminant for a Predicted Change of 0.005 in the Mean Square ΔE_{Lab} Error

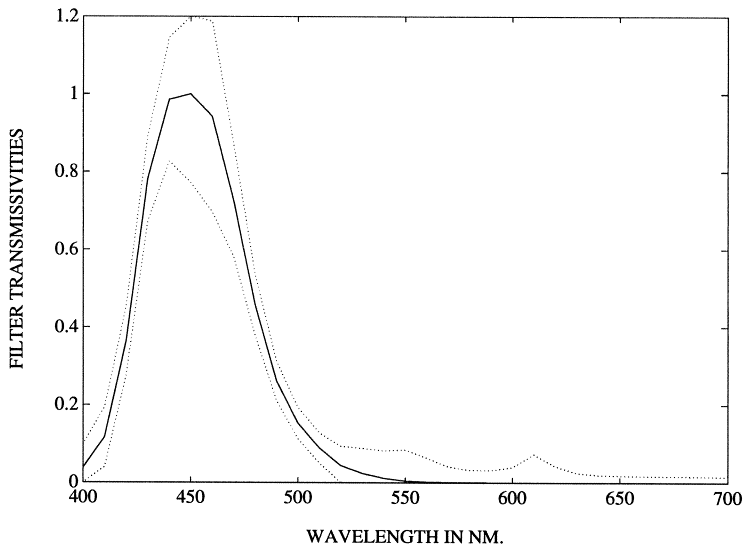

Figure 5.21: Single-Wavelength Bounds for Red Filter of CIE functions under a Uniform Illuminant for a Predicted Change of 0.005 in the Mean Square ΔE_{Lab} Error

All-Wavelength Bounds

The all-wavelength bounds indicated by equation (5.4) are calculated for a maximum allowable change of 1 in the average square ΔE_{Lab} error, and plotted in Figs. 5.22-5.30, the same manner as in the previous section.

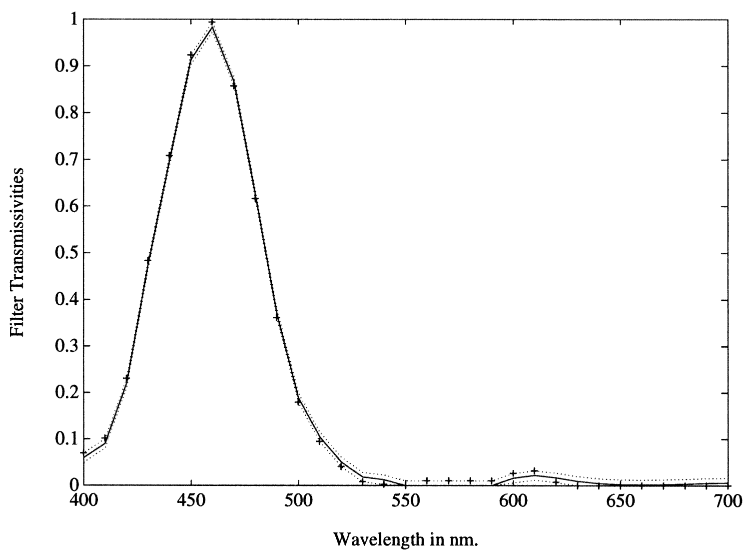

Figure 5.22: All-Wavelength Bound for Trimmed Blue Filter of Single Gaussian Model and Illuminant 2 for a Predicted Change of 1 in the Mean Square ΔE_{Lab} Error

Table 5.8 lists the error measures for the original designs and the corresponding perturbed designs indicated by the + symbol in the figures, as in the previous section. The mean square ΔE_{Lab} error is denoted MS. The predicted mean square ΔE_{Lab} error (the bound corresponding to the perturbation) is denoted PMS.

The filters that Barr Associates can manufacture to match Filter Set 1 indicate a largest manufacturing error of value 0.06 with respect to the trimmed filters. The filters that can be used to match Filter Set 2 also indicate a largest manufacturing

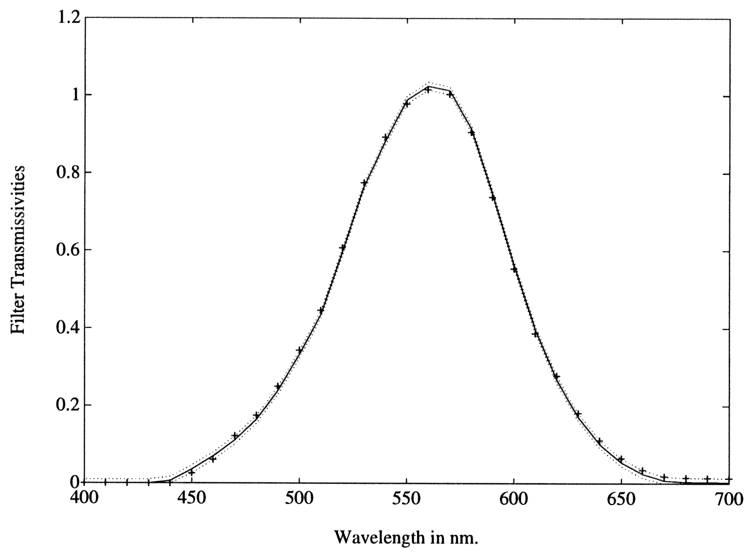

Figure 5.23: All-Wavelength Bound for Trimmed Green Filter of Single Gaussian Model and Illuminant 2 for a Predicted Change of 1 in the Mean Square ΔE_{Lab} Error

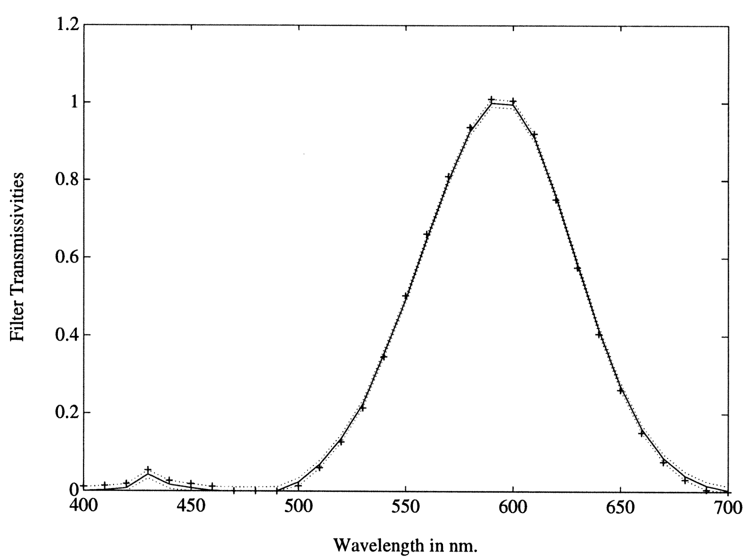

Figure 5.24: All-Wavelength Bound for Trimmed Red Filter of Single Gaussian Model and Illuminant 2 for a Predicted Change of 1 in the Mean Square ΔE_{Lab} Error

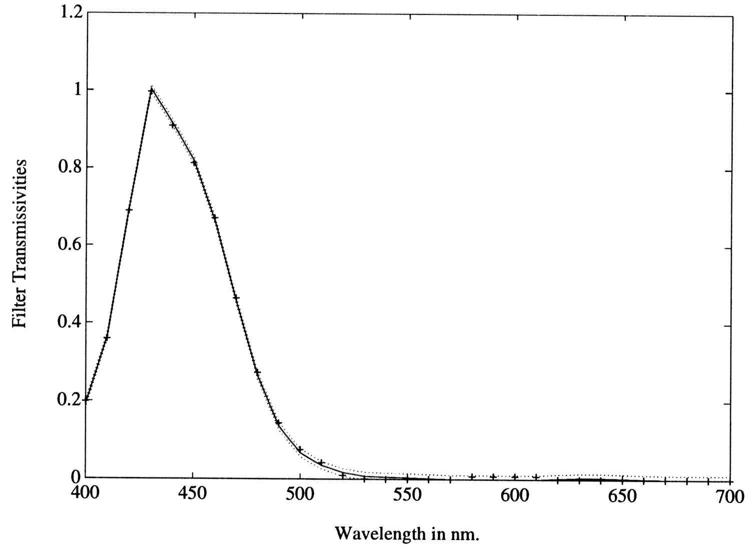

Figure 5.25: All-Wavelength Bound for Trimmed Blue Filter of Sum-of-Gaussian Model and Illuminant 1 for a Predicted Change of 1 in the Mean Square ΔE_{Lab} Error

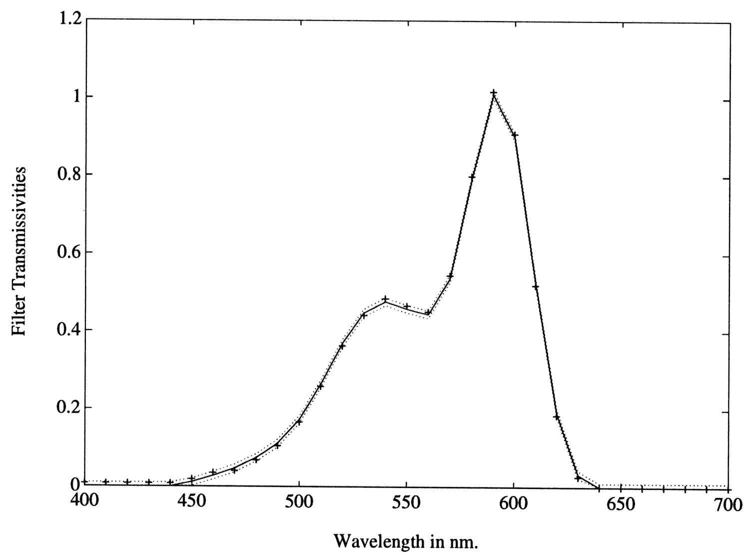

Figure 5.26: All-Wavelength Bound for Trimmed Green Filter of Sum-of-Gaussian Model and Illuminant 1 for a Predicted Change of 1 in the Mean Square ΔE_{Lab} Error

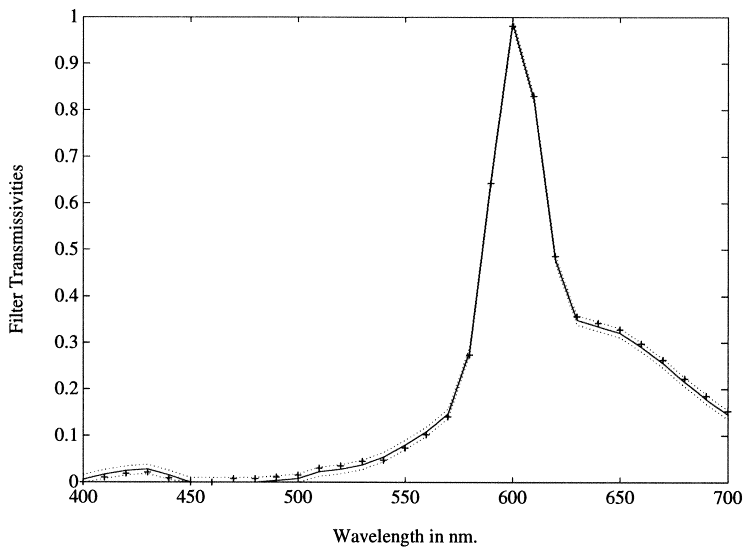

Figure 5.27: All-Wavelength Bound for Trimmed Red Filter of Sum-of-Gaussian Model and Illuminant 1 for a Predicted Change of 1 in the Mean Square ΔE_{Lab} Error

Table 5.8: Comparison Between Errors of Original Design and Perturbed Design for a Predicted Change of 1 in Mean Square ΔE_{Lab} Error

Set		ν	E	E_{max}	RMS	MS	PMS	ω_2	$\|\|vec\ \Delta\mathbf{M}\|\|^2$
1	Original	0.9508	0.27	0.67	0.32	0.10	0.10	0	0
	Perturbed	0.9484	0.71	2.63	0.91	0.83	1.10	0.010	0.0096
2	Original	0.9918	0.12	0.73	0.18	0.03	0.03	0	0
	Perturbed	0.9589	0.66	3.43	0.93	0.86	1.03	0.007	0.0046
3	Original	1.0	0	0	0	0	0	0	0
	Perturbed	0.9993	0.57	2.99	0.81	0.65	1.0	0.008	0.0059

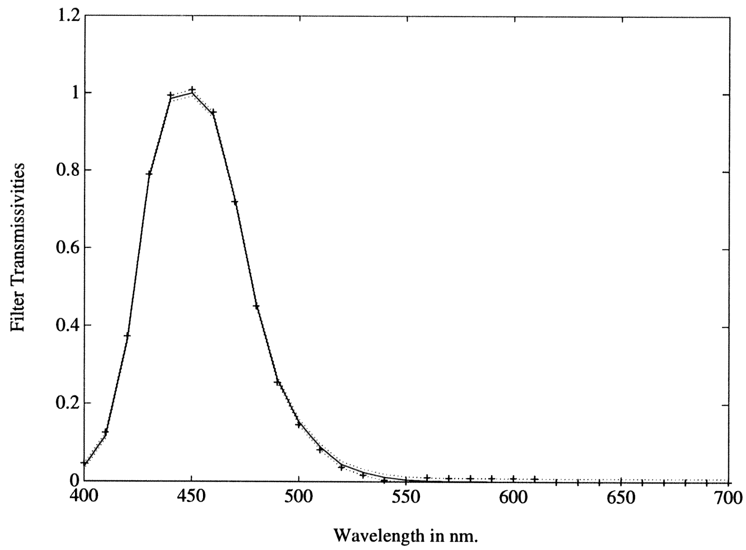

Figure 5.28: All-Wavelength Bound for Blue Filter of the CIE functions under a Uniform Illuminant for a Predicted Change of 1 in the Mean Square ΔE_{Lab} Error

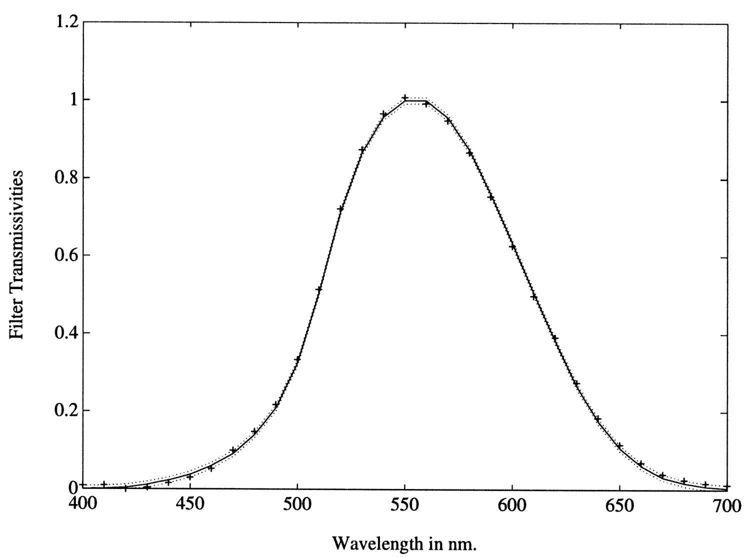

Figure 5.29: All-Wavelength Bound for Green Filter of the CIE functions under a Uniform Illuminant for a Predicted Change of 1 in the Mean Square ΔE_{Lab} Error

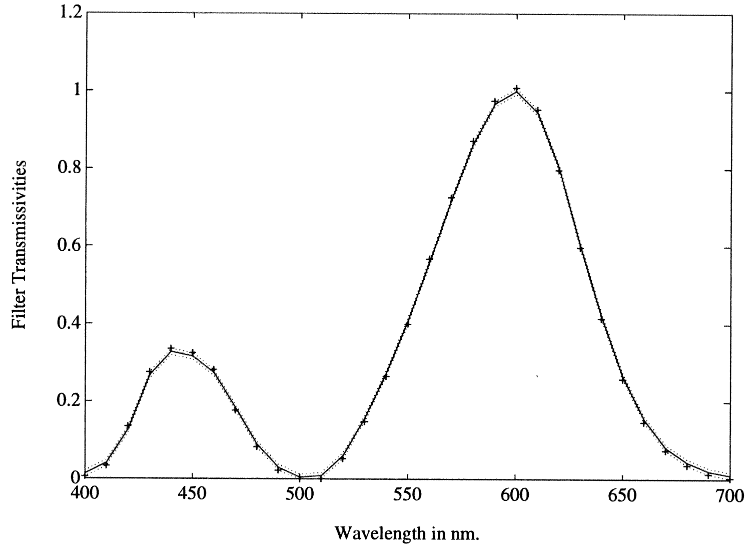

Figure 5.30: All-Wavelength Bound for Red Filter of CIE functions under a Uniform Illuminant for a Predicted Change of 1 in the Mean Square ΔE_{Lab} Error

error of 0.06. This value is much higher than the all-wavelength bound for a change of value 1 in the mean square ΔE_{Lab} error. Thus all the experiments indicate that the designed filters are highly sensitive given current industry capabilities with regard to accuracy in the filter manufacturing process. The experiments also indicate that the designed filters are not any more sensitive than the CIE matching functions. The fact that the cone sensitivities vary considerably among individuals has been mentioned in Chapter 1. Given this, small variations in scanning filters are not unacceptable if visual output is the final criterion, because the human colour sensors (cones) that the scanning filters are designed to imitate vary considerably themselves. On the other hand, if quality control is the final criterion, the small variations do make considerable difference.

5.6 Conclusions

An expression for the second differential is obtained in the chapter. The expression is used to approximate the change in mean square ΔE_{Lab} error over a particular data set due to filter perturbations, using a second-order Taylor series approximation. It is also used to approximate the change in the data-independent measure ν. Simulation results indicate that these approximations are valid. Further, the approximation provides estimates of error sensitivity as a function of wavelength. The approximation is used to obtain bounds on allowable filter fabrication errors given maximum acceptable changes in mean square ΔE_{Lab} error. The experiments demonstrate that the bounds thus obtained are valid. The experiments also indicate that the designed filters are highly sensitive to fabrication errors. The fact that the CIE matching functions display similar sensitivities to fabrication error implies that the sensitivity could be characteristic of the scanning process.