Hartenberg – Denavit Method

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Screw Connections

- Four parameters are needed to describe a screw connection between two links.
- Any revolute or prismatic connection is a limiting case of a screw connection.
  - A hinge is a screw with zero pitch
  - A prismatic (sliding) joint is a screw with infinite pitch
  - A ball joint can be modeled as 3 intersecting revolutes.

Symbolic Representation of Mechanisms

- A mechanism can be viewed as a sequence of connections.
- Each connection consists of a pair and is characterized by a *pair variable*.
- There are six lower pairs (with area contact)-
  - Revolute R
  - Prismatic P
  - Screw S_L (where L is the lead of the screw)
  - Cylindric C
  - Spheric G (For globular)
  - Planar F (For flat)

Symbolic Representation of Mechanisms

- As limiting cases of the screw pair we can consider revolutes and prismatic joints as special cases:
  - Screw S_L (where L is the lead of the screw)
  - Revolute $R = S_0$ (A screw with zero lead)
  - Prismatic $P = S_\infty$ (A screw with infinite lead)

Symbolic Representation of Mechanisms

- We can write the two halves of a pair as
  - R^+ and R^-
  - S_L^+ and S_L^-
  - P^+ and P^-

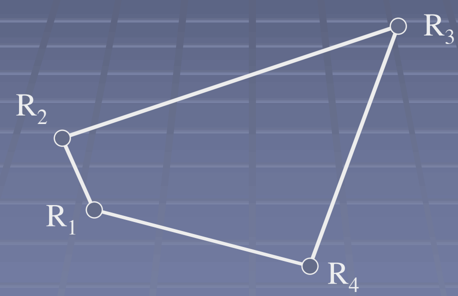
  and so forth.
- Lower pairs are invertible so it doesn't really matter which half is the plus or minus half.

Symbolic Representation of Mechanisms

- Relative motion between pair elements describes the relative motion between the links carrying the elements.
- That relative motion is described by the pair variables.
- These are variables such as □ for rotation and s for translation.

Symbolic Representation of Mechanisms

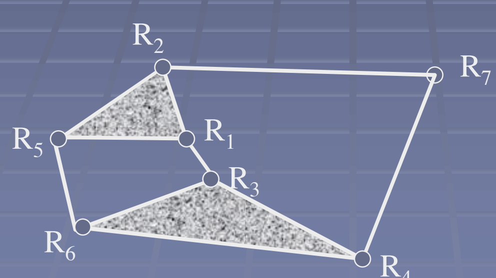
- We can give a symbolic description to a simple closed kinematic chain like a four-bar



as $R_1 R_2 R_3 R_4$ or $R_1 R_4 R_3 R_2$ and so forth.

Symbolic Representation of Mechanisms

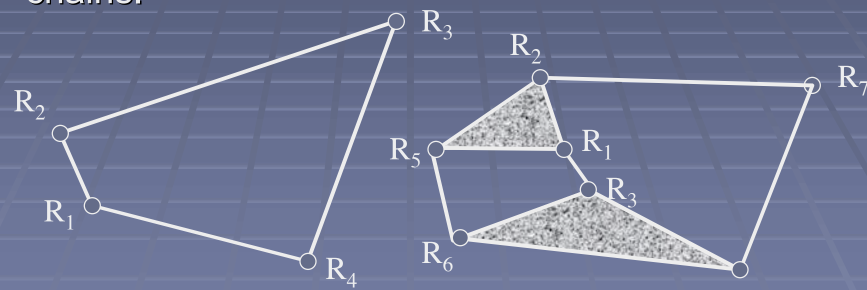
- A compound closed kinematic chain like the six-bar below could be described symbolically by giving two independent loops that include all the pairs. For instance,



as $R_6 R_3 R_1 R_5$ & $R_1 R_2 R_7 R_4 R_3$ for example.

Symbolic Representation of Mechanisms

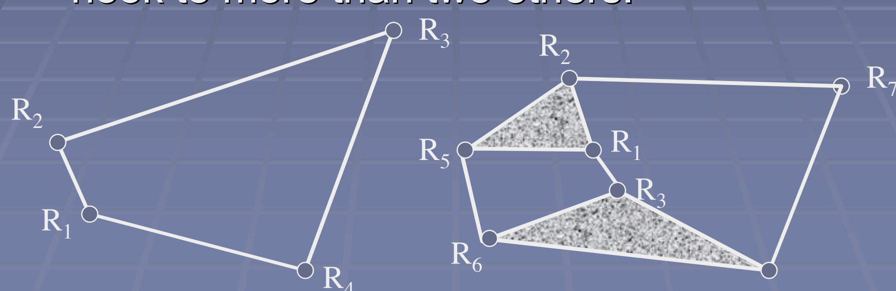
- These are both examples of closed kinematic chains.



- A typical robot would be an example of an open kinematic chain.

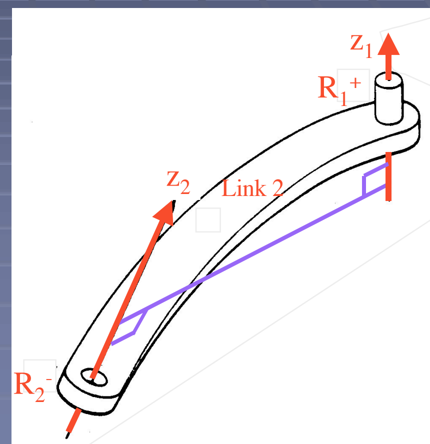
Symbolic Representation of Mechanisms

- In a simple closed kinematic chain each link connects to two and *only* two other links.
- In a compound closed chain some links hook to more than two others.



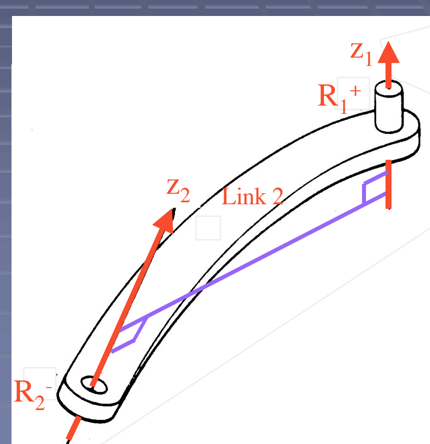
Description of a Simple Chain

- Relative positions of the successive pair axes on a link can be described by use of the *unique* common perpendicular between the pair axes.



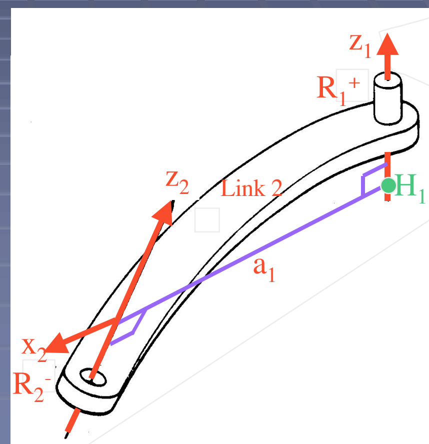
Description of a Simple Chain

- Coordinate systems are fixed in each joint using a simple convention.
- The z axes are chosen to define the orientations of the revolute, screw, or prismatic pairs



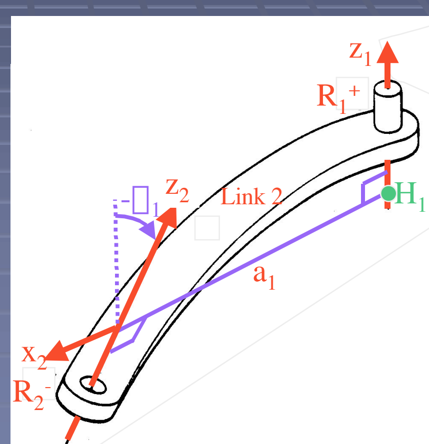
Description of a Simple Chain

- The x axis at a joint is chosen to lie along the common perpendicular from a point H on the previous z axis to the current one on the link.
- The length of that common perpendicular from z_k to z_{k+1} is called a_k.



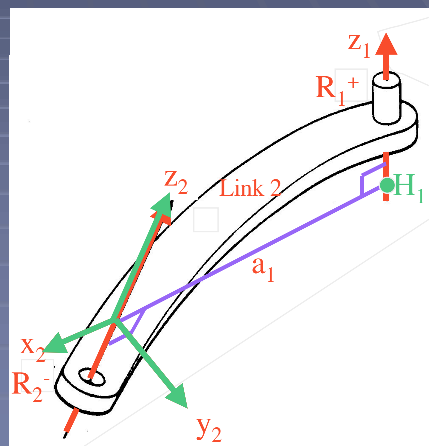
Description of a Simple Chain

- A general link has a "Dual Angle" □ (a_1, □$_1$) between the vectors z_1 and z_2.
- By that I mean that there could be both an offset a_1 and a twist angle □$_1$ (measured about the x axis in a plane perpendicular to the common normal).
- □ (Because of the right-hand rule the angle □$_1$ shown is negative.)



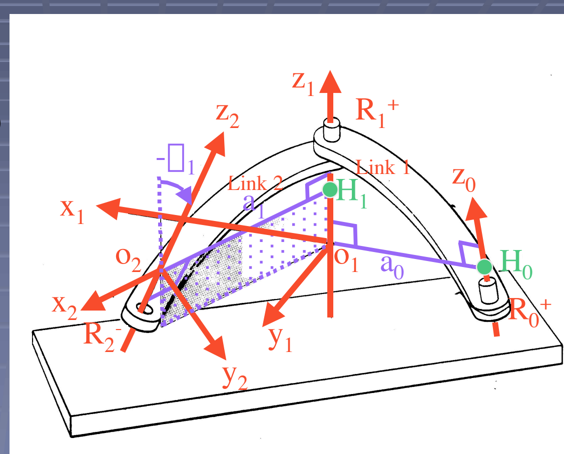
Description of a Simple Chain

- Finally, the y axis is chosen so as to give a right-handed rectangular coordinate system.



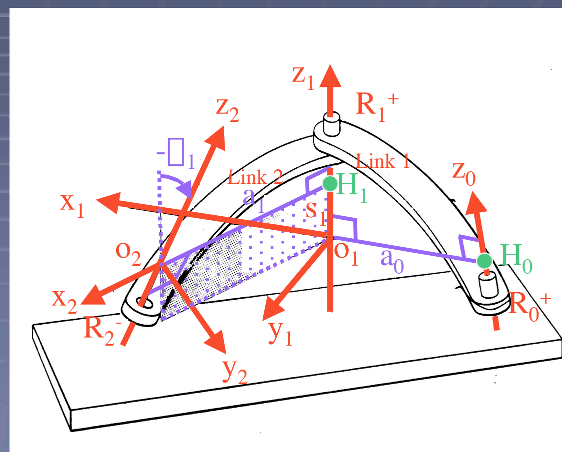
Description of a Simple Chain

- Here is what the situation looks like so far for two typical links of a spatial chain.
- To clarify the numbering conventions used, they were chosen as links #1 and #2.



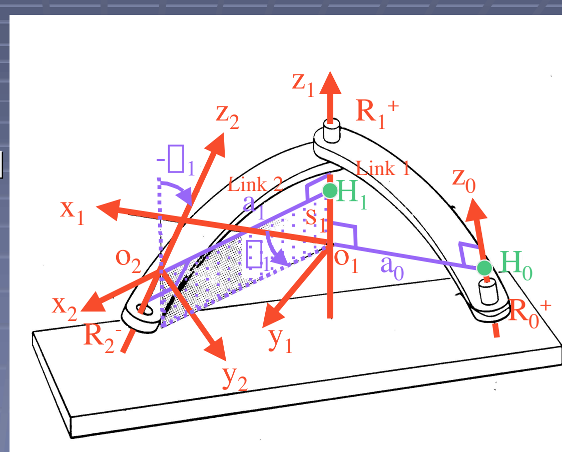
Description of a Simple Chain

- The offset along the z_1 axis between the origin o_1 and the point H_1 is called s_1.



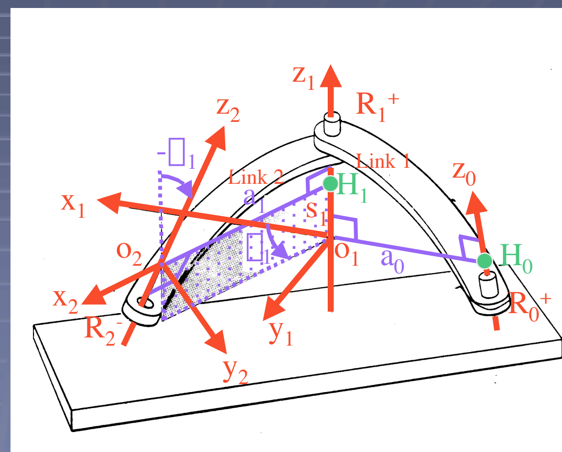
Description of a Simple Chain

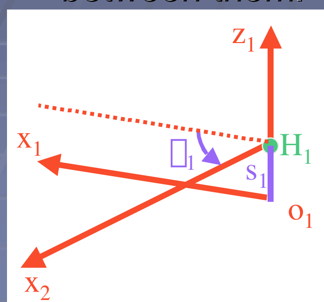
- Finally, the rotation angle between the x_1 and x_2 axes (and measured about the z_1 axis) is called □$_1$.



Description of a Simple Chain

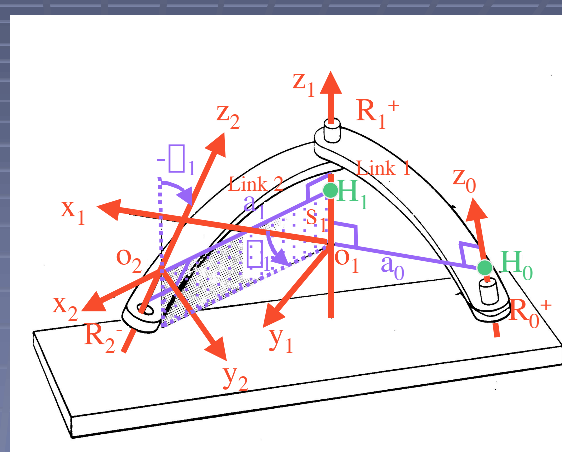
- Thus the x_1 and x_2 axes subtend a second “dual angle” □(s_1, □$_1$) between them.





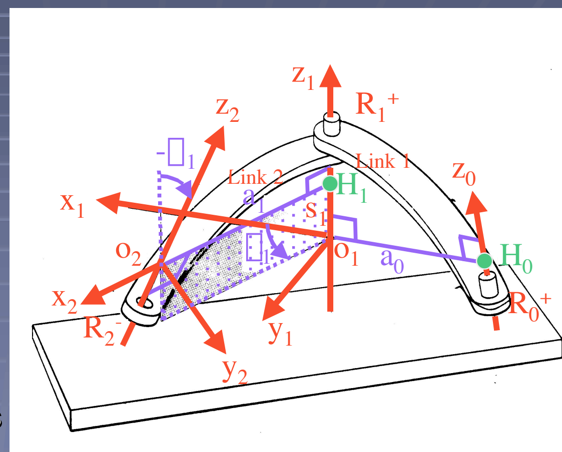
Description of a Simple Chain

- In this way, a rectangular cartesian coordinate system is uniquely specified in each link.



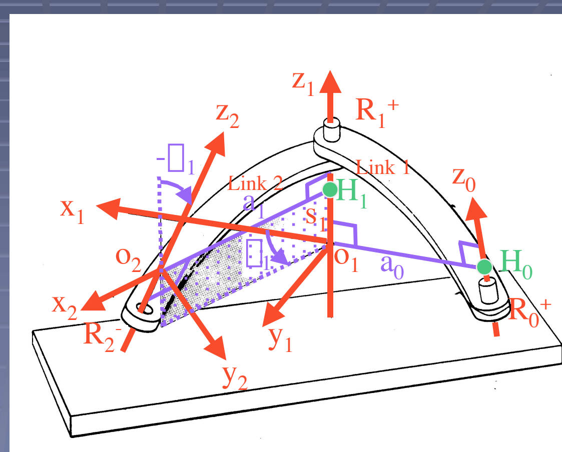
Description of a Simple Chain

- The relative positions of successive links is expressed in terms of the four parameters of the two dual angles $\square(s_1, \square_1)$, $\square(a_1, \square_1)$.
- These uniquely define the relative positions of successive systems of coordinates.;



Description of a Simple Chain

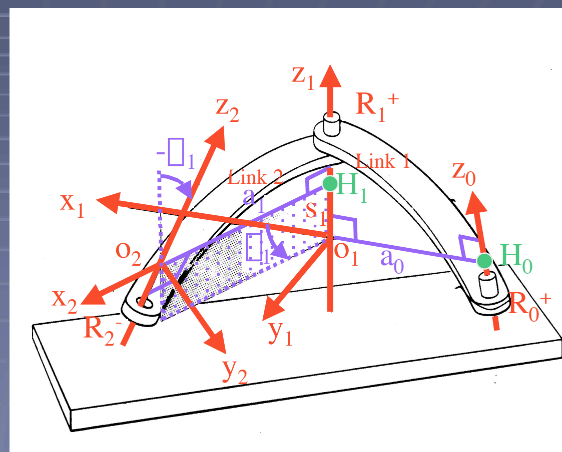
- For generality, we can assume the joints are all made up of screw pairs, (shown symbolicaly as $S_{L\ k)}$.



Description of a Simple Chain

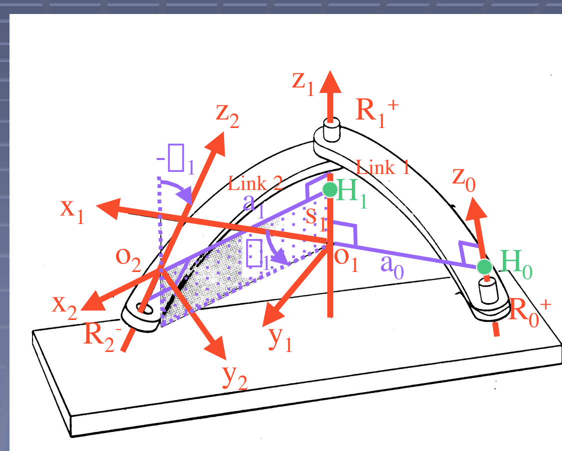
- In a screw the parameters θ_k and s_k would be related by the lead of the screw.

$$\frac{\Delta\theta_k}{2\pi} = \frac{\Delta s_k}{L_k}$$



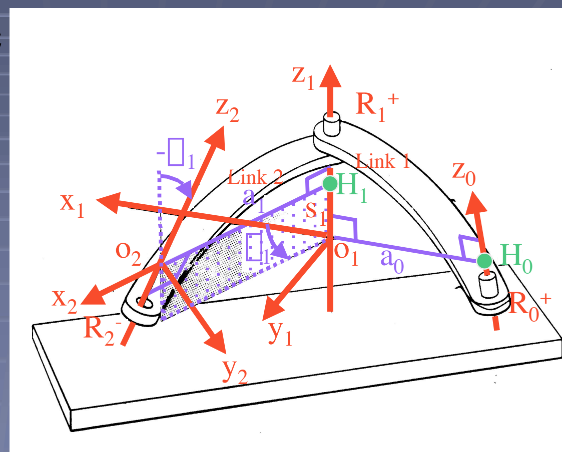
Description of a Simple Chain

- For a revolute pair ($L_k = 0$) only the parameter θ_k varies.



Description of a Simple Chain

- For a prismatic pair ($L_k = \infty$) only the parameter s_k varies.



Description of a Simple Chain

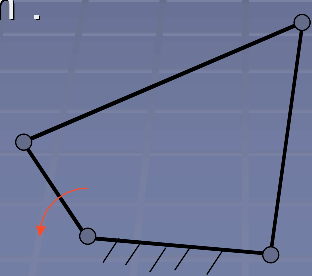
- Thus we can write a general equation for a simple closed chain of screws in the following symbolic form:

$$S_{L_1}\begin{vmatrix} a_1 \\ \alpha_1 \\ \theta_1 \\ s_1 \end{vmatrix} S_{L_2}\begin{vmatrix} a_2 \\ \alpha_2 \\ \theta_2 \\ s_2 \end{vmatrix} S_{L3}\begin{vmatrix} a_3 \\ \alpha_3 \\ \theta_3 \\ s_3 \end{vmatrix} . \quad . \quad . \quad .S_{L_k}\begin{vmatrix} a_k \\ \alpha_k \\ \theta_k \\ s_k \end{vmatrix} . \quad . \quad . \quad .S_{Ln}\begin{vmatrix} a_n \\ \alpha_n \\ \theta_n \\ s_n \end{vmatrix} = I$$

- Here, the identity symbol I is used to indicate that when you work your way around the closed chain you get back to the start.
- Later, when this is expressed with matrices, it will have the same function mathematically.

A brief digression…

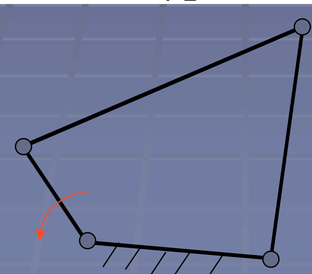
- How many degrees-of-freedom does a four-link four-revolute linkage have?
- If you think of it as a planar four-bar you automatically think “One degree-of-freedom”.



A brief digression…

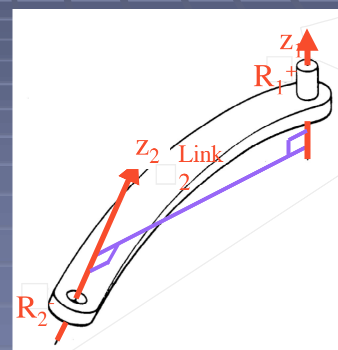
- Using the basic planar D.O.F. formula we get

$$
\begin{aligned}
D.O.F. &= 3(n - 1) - 2j \\
&= 3(4 - 1) - 2 * 4 \\
&= +1
\end{aligned}
$$



A brief digression…

- What if we think of these links as being general, spatial links connected by revolutes?
- In space, a revolute joint removes five relative degrees-of-freedom and leaves only one.

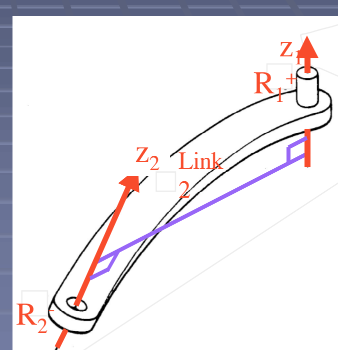


A brief digression…

- So if we have four bodies connected by revolutes the degrees-of-freedom should be:

$$
\begin{aligned}
D.O.F. &= 6(n-1)-5R \\
&= 6(4-1)-5*4 \\
&= 18-20 \\
&= -2
\end{aligned}
$$

- In other words, a general spatial four-bar is massively over-constrained!



A brief digression…

- It turns out that there are only three four revolute linkages that exist and can move with one degree of freedom. These are
  - The planar four-revolute mechanism (commonly known as “the four-bar”)
  - The spherical four-revolute mechanism
  - The Bennett mechanism

A brief digression…

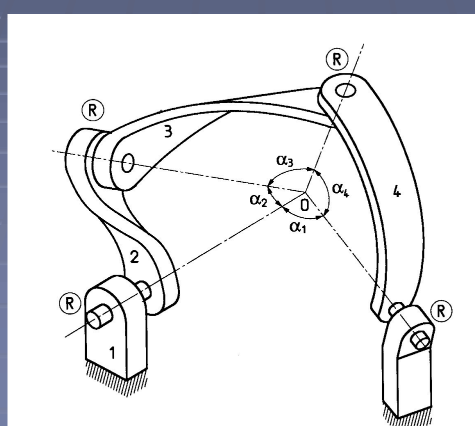
- Planar four-bars are unique in that all four revolute axes are parallel to one another and perpendicular to the plane of motion.
- The axes all intersect at infinity.
- That is why the mechanisms work even though the formula shows them having minus two degrees of freedom!

A brief digression…

- Spherical four-bars are also uniquely proportioned.
- All four revolute axes intersect at a common point.
- They have a lot in common with their planar cousins.
- They are just mapped onto a sphere
- That is why they also work even though the formula shows them having minus two degrees of freedom!

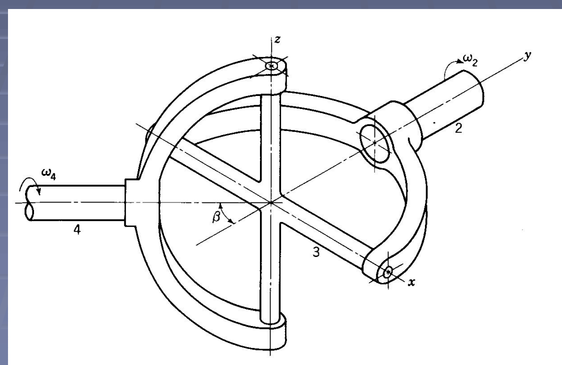
A brief digression…

- Here's what a fairly general spherical four-bar looks like:



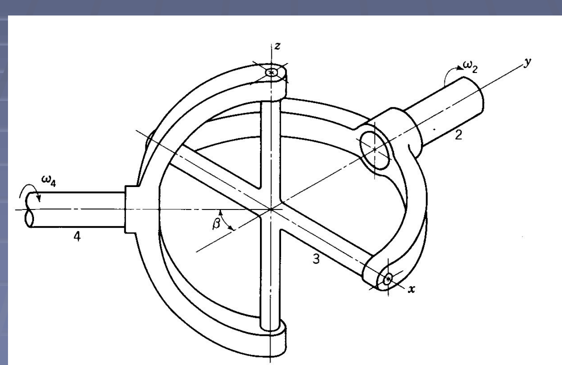
A brief digression…

- Here's what the most common special-case spherical four-bar looks like:



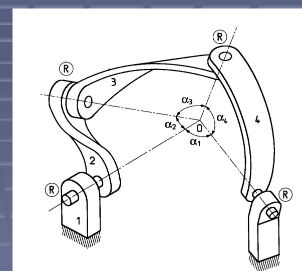
A brief digression…

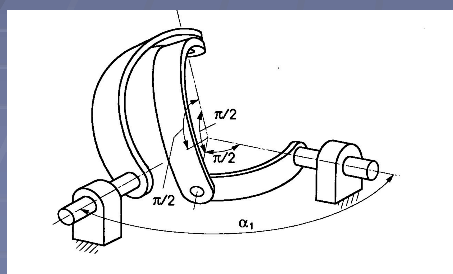
- This is the "Hooke" or "Cardan" universal joint

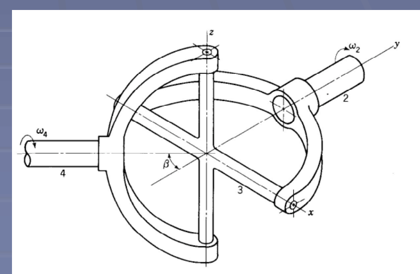


A brief digression…

- A Hooke joint is a special case four-revolute spherical linkage with all four of it's spherical angles equal to □/2

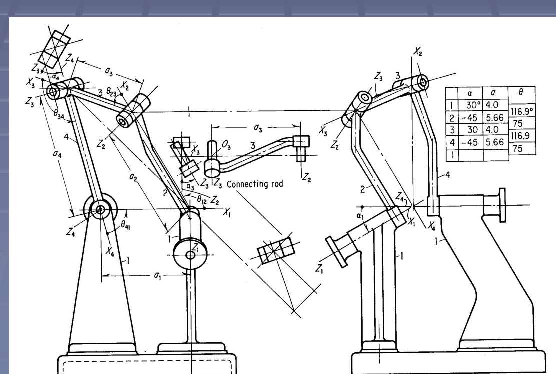






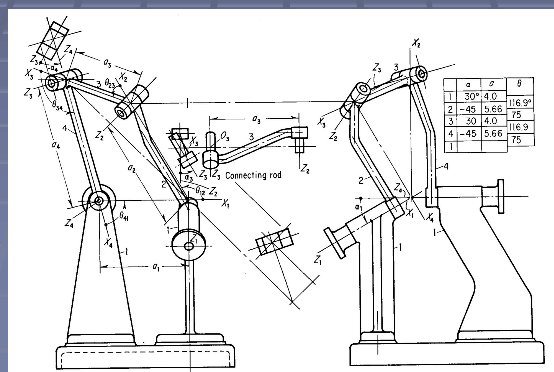
A brief digression…

- The Bennett mechanism is a singularly useless special case spherical mechanism.



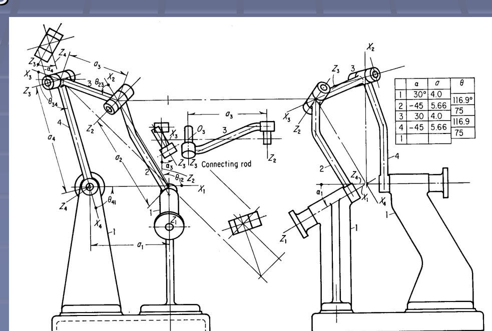
A brief digression…

- It's opposite sides are equal in length and the twists of opposite links are the same.
- Thus, it has a lot in common with a planar parallelogram linkage.



A brief digression…

- The Bennett mechanism has one interesting virtue- namely it has no dead center positions.
- When all the x axes are collinear, the output torque is produced by bending and torsion stresses in the connecting rod and frame.

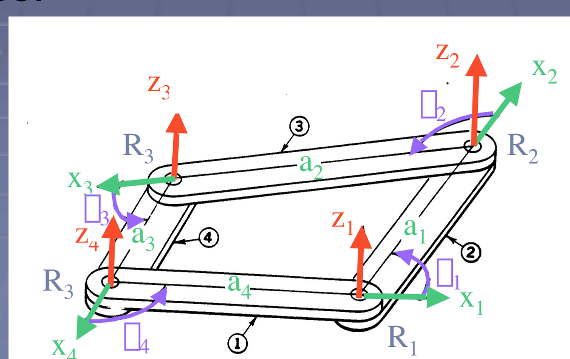


Symbolic Representation Example

- Now that you know a little bit about spatial four-revolute linkages, let's see how we can analyze them using the Hartenberg-Denavit method.

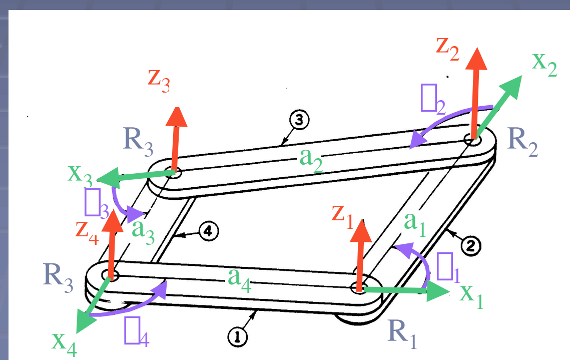
Symbolic Representation Example

- Here's a planar four-bar for example.
- The z axes are all oriented with the same sense.



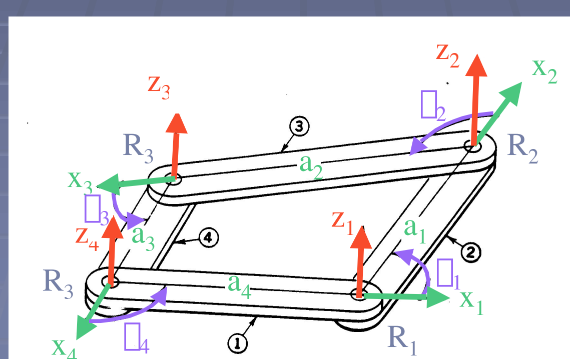
Symbolic Representation Example

- Successive common perpendiculars form the four x axes.



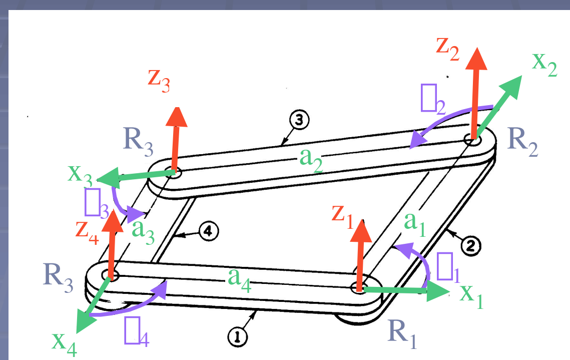
Symbolic Representation Example

- They axes aren't shown but would complete the four right-handed coordinate systems.



Symbolic Representation Example

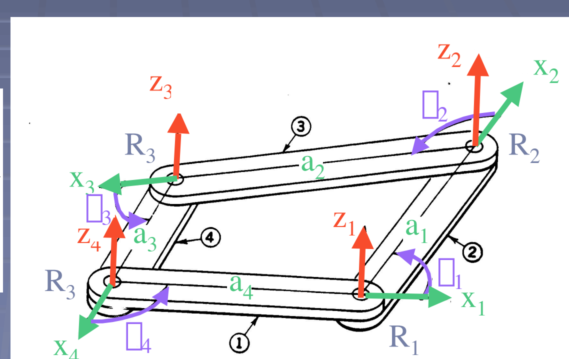
- Note that the $x_1y_1z_1$ system is fixed in link 1, the $x_2y_2z_2$ system is fixed in link 2, and so on.



Symbolic Representation Example

- Here's the symbolic equation for this planar four-bar:

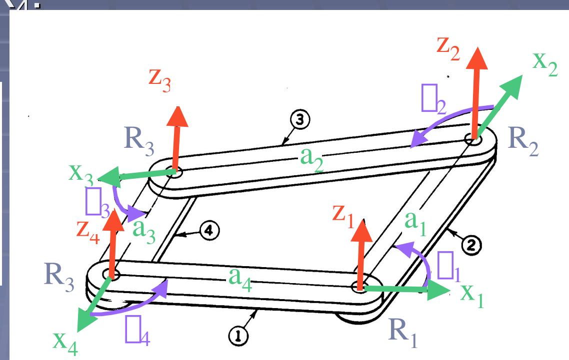
$$R_1\begin{vmatrix} a_1 \\ 0 \\ \theta_1 \\ 0 \end{vmatrix} R_2\begin{vmatrix} a_2 \\ 0 \\ \theta_2 \\ 0 \end{vmatrix} R_3\begin{vmatrix} a_3 \\ 0 \\ \theta_3 \\ 0 \end{vmatrix} R_4\begin{vmatrix} a_4 \\ 0 \\ \theta_4 \\ 0 \end{vmatrix} = I$$



Symbolic Representation Example

- Comparing it with the general form we see that the four pairs are the four revolutes R_1 R_2 R_3 and R_4.

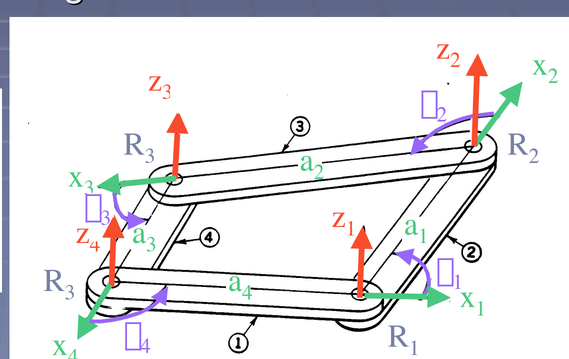
$$R_1 \begin{vmatrix} a_1 \\ 0 \\ \square_1 \\ 0 \end{vmatrix} R_2 \begin{vmatrix} a_2 \\ 0 \\ \square_2 \\ 0 \end{vmatrix} R_3 \begin{vmatrix} a_3 \\ 0 \\ \square_3 \\ 0 \end{vmatrix} R_4 \begin{vmatrix} a_4 \\ 0 \\ \square_4 \\ 0 \end{vmatrix} = I$$



Symbolic Representation Example

- The link lengths are the parameters a_1, a_2, a_3, and a_4 and are the distances between the z axes measured along the common perpendiculars.

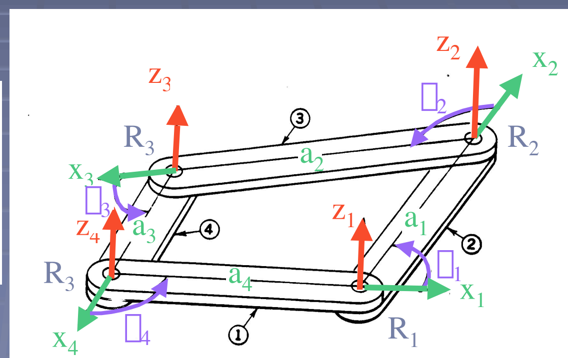
$$R_1 \begin{vmatrix} a_1 \\ 0 \\ \square_1 \\ 0 \end{vmatrix} R_2 \begin{vmatrix} a_2 \\ 0 \\ \square_2 \\ 0 \end{vmatrix} R_3 \begin{vmatrix} a_3 \\ 0 \\ \square_3 \\ 0 \end{vmatrix} R_4 \begin{vmatrix} a_4 \\ 0 \\ \square_4 \\ 0 \end{vmatrix} = I$$



Symbolic Representation Example

- The angles □ are all zero since the axes are all parallel.

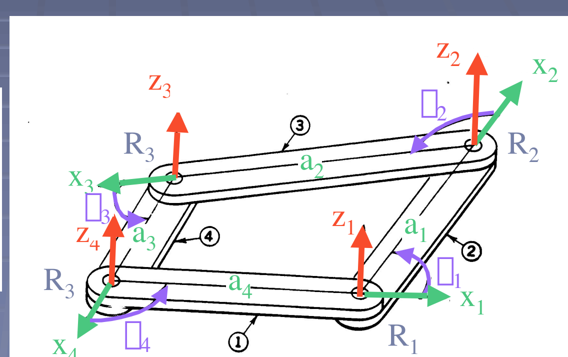
$$R_1 \begin{bmatrix} a_1 \\ 0 \\ \square_1 \\ 0 \end{bmatrix} R_2 \begin{bmatrix} a_2 \\ 0 \\ \square_2 \\ 0 \end{bmatrix} R_3 \begin{bmatrix} a_3 \\ 0 \\ \square_3 \\ 0 \end{bmatrix} R_4 \begin{bmatrix} a_4 \\ 0 \\ \square_4 \\ 0 \end{bmatrix} = I$$



Symbolic Representation Example

- The angles □1, □2, □3, and □4 are the pair variables of the revolute joints.

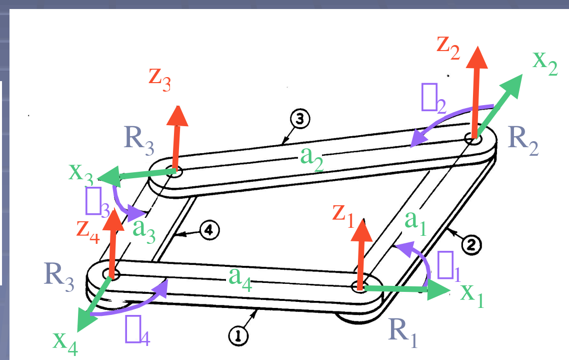
$$R_1 \begin{bmatrix} a_1 \\ 0 \\ \square_1 \\ 0 \end{bmatrix} R_2 \begin{bmatrix} a_2 \\ 0 \\ \square_2 \\ 0 \end{bmatrix} R_3 \begin{bmatrix} a_3 \\ 0 \\ \square_3 \\ 0 \end{bmatrix} R_4 \begin{bmatrix} a_4 \\ 0 \\ \square_4 \\ 0 \end{bmatrix} = I$$



Symbolic Representation Example

- The s distances are also all zero, since the successive x axes were chosen so as to intersect.

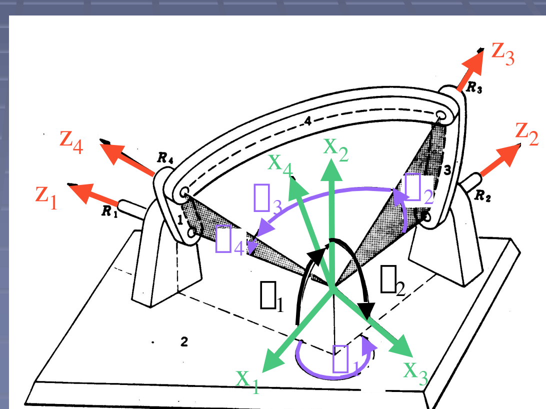
$$R_1 \begin{vmatrix} a_1 \\ 0 \\ \square_1 \\ 0 \end{vmatrix} R_2 \begin{vmatrix} a_2 \\ 0 \\ \square_2 \\ 0 \end{vmatrix} R_3 \begin{vmatrix} a_3 \\ 0 \\ \square_3 \\ 0 \end{vmatrix} R_4 \begin{vmatrix} a_4 \\ 0 \\ \square_4 \\ 0 \end{vmatrix} = I$$



Another Symbolic Representation Example

- Let's now look at a spherical four-revolute mechanism:

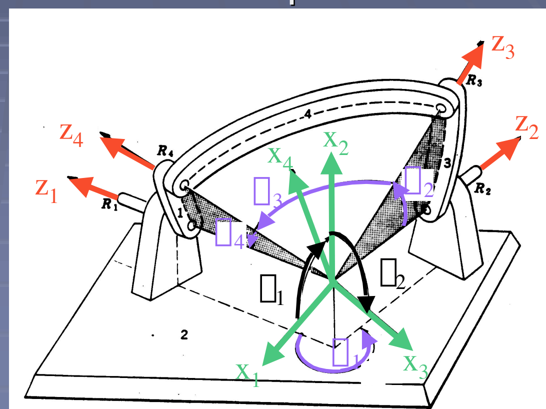
$$R_1 \begin{vmatrix} 0 \\ \square_1 \\ \square_1 \\ 0 \end{vmatrix} R_2 \begin{vmatrix} 0 \\ \square_2 \\ \square_2 \\ 0 \end{vmatrix} R_3 \begin{vmatrix} 0 \\ \square_3 \\ \square_3 \\ 0 \end{vmatrix} R_4 \begin{vmatrix} 0 \\ \square_4 \\ \square_4 \\ 0 \end{vmatrix} = I$$



Another Symbolic Representation Example

- In this case, all the z axes intersect.
- For this reason, all the a and s parameters are zero.

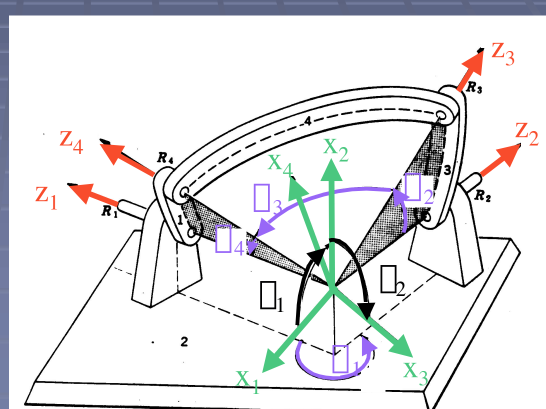
$$R_1 \begin{vmatrix} 0 \\ \square_1 \\ \square_1 \\ 0 \end{vmatrix} R_2 \begin{vmatrix} 0 \\ \square_2 \\ \square_2 \\ 0 \end{vmatrix} R_3 \begin{vmatrix} 0 \\ \square_3 \\ \square_3 \\ 0 \end{vmatrix} R_4 \begin{vmatrix} 0 \\ \square_4 \\ \square_4 \\ 0 \end{vmatrix} = I$$



Another Symbolic Representation Example

- The angles □ define the link dimensions.

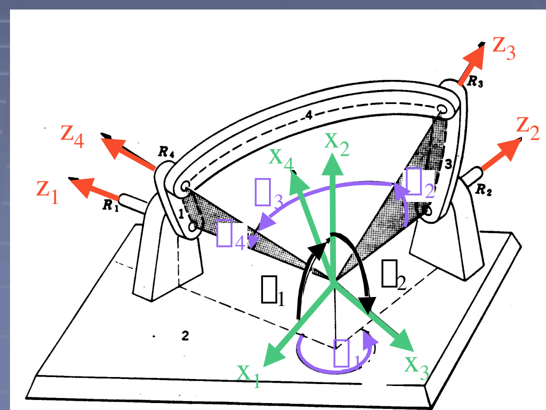
$$R_1 \begin{vmatrix} 0 \\ \square_1 \\ \square_1 \\ 0 \end{vmatrix} R_2 \begin{vmatrix} 0 \\ \square_2 \\ \square_2 \\ 0 \end{vmatrix} R_3 \begin{vmatrix} 0 \\ \square_3 \\ \square_3 \\ 0 \end{vmatrix} R_4 \begin{vmatrix} 0 \\ \square_4 \\ \square_4 \\ 0 \end{vmatrix} = I$$



Another Symbolic Representation Example

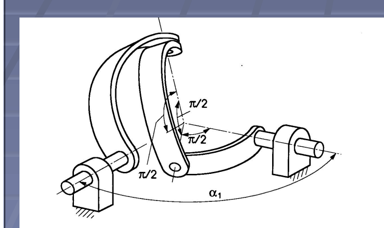
- The angles □ are the pair variables of the revolutes.

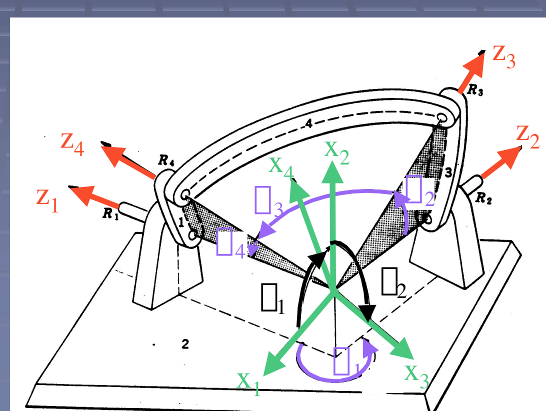
$$R_1\begin{vmatrix}0\\ \square_1\\ \square_1\\ 0\end{vmatrix} R_2\begin{vmatrix}0\\ \square_2\\ \square_2\\ 0\end{vmatrix} R_3\begin{vmatrix}0\\ \square_3\\ \square_3\\ 0\end{vmatrix} R_4\begin{vmatrix}0\\ \square_4\\ \square_4\\ 0\end{vmatrix} = I$$



Another Symbolic Representation Example

- The Hooke joint is a special case of this with $\square_2 = \square_3 = \square_4 = 90^o$

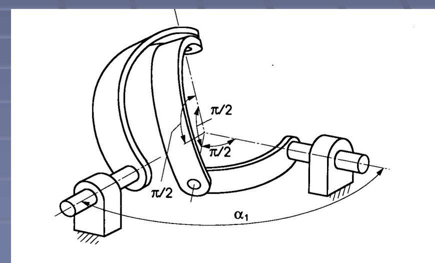




Another Symbolic Representation Example

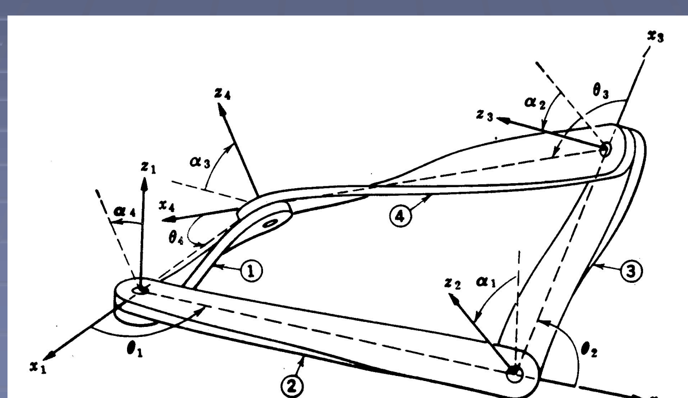
- The symbolic equation for the Hooke universal joint is:

$$R_1\begin{vmatrix}0\\ \square_1\\ \square_1\\ 0\end{vmatrix} R_2\begin{vmatrix}0\\ 90^o\\ \square_2\\ 0\end{vmatrix} R_3\begin{vmatrix}0\\ 90^o\\ \square_3\\ 0\end{vmatrix} R_4\begin{vmatrix}0\\ 90^o\\ \square_4\\ 0\end{vmatrix} = I$$



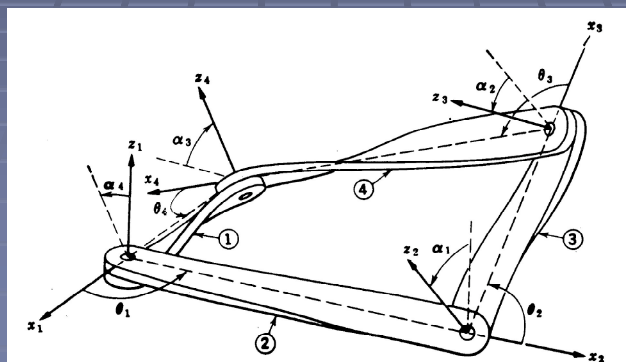
Another Symbolic Representation Example

- The Bennett mechanism has opposite links with equal twists (□ and □) and equal lengths (a and b).



Another Symbolic Representation Example

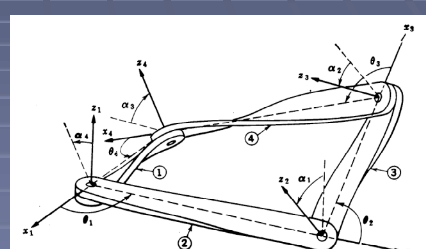
- The Bennett mechanism's x axes all intersect, so the s parameters are all zero.
- Again, the pair variables are the θ's.



Another Symbolic Representation Example

- The symbolic equation for the Bennett mechanism is:

$$R_1\begin{vmatrix} a \\ \alpha \\ \theta_1 \\ 0 \end{vmatrix} R_2\begin{vmatrix} b \\ \beta \\ \theta_2 \\ 0 \end{vmatrix} R_3\begin{vmatrix} a \\ \alpha \\ \theta_3 \\ 0 \end{vmatrix} R_4\begin{vmatrix} b \\ \beta \\ \theta_4 \\ 0 \end{vmatrix} = I$$



- An additional condition is that

$$\frac{a}{\sin\alpha} = \pm\frac{b}{\sin\beta}$$

Carrying out the Matrix Method of Analysis

- Once a linkage has been described by a symbolic equation, the coordinate transformation from one link's coordinate system to the next may be represented by a 4x4 matrix involving the four parameters a, α, θ, and s.

Carrying out the Matrix Method of Analysis

- This coordinate transformation from system k+1 to system k can be shown to be in the form:

$$A_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_k \cos\theta_k & \cos\theta_k & -\cos\alpha_k \sin\theta_k & \sin\alpha_k \sin\theta_k \\ a_k \sin\theta_k & \sin\theta_k & \cos\alpha_k \cos\theta_k & -\sin\alpha_k \cos\theta_k \\ s_k & 0 & \sin\alpha_k & \cos\alpha_k \end{bmatrix}$$

Carrying out the Matrix Method of Analysis

- Multiplying together matrices of this form in the right order can take you from one coordinate system to the next as you go around the loops of a closed-loop kinematic chain.

$$A_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_k \cos\theta_k & \cos\theta_k & -\cos\alpha_k \sin\theta_k & \sin\alpha_k \sin\theta_k \\ a_k \sin\theta_k & \sin\theta_k & \cos\alpha_k \cos\theta_k & -\sin\alpha_k \cos\theta_k \\ s_k & 0 & \sin\alpha_k & \cos\alpha_k \end{bmatrix}$$

Carrying out the Matrix Method of Analysis

- For instance, to go from the coordinate system on link 3 to the coordinate system on link 1 you would perform the matrix multiplication A_1A_2

Carrying out the Matrix Method of Analysis

- For the four-link examples given earlier (planar and spherical four revolutes or the Bennett mechanism), $A_1A_2A_3A_4$ would take you around the closed loop of the mechanism and back to the starting number one coordinate system.

Carrying out the Matrix Method of Analysis

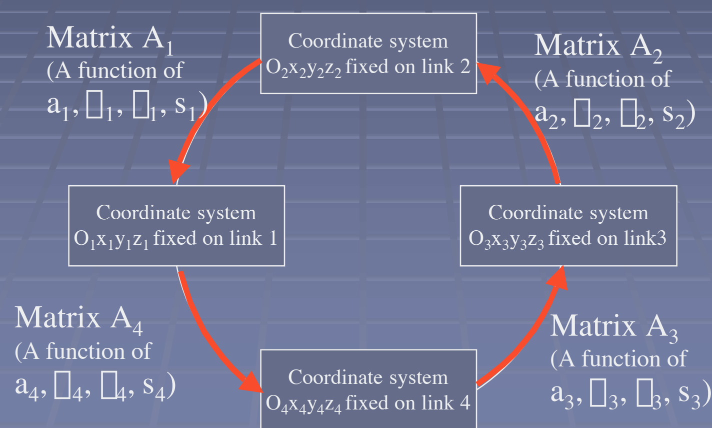
- Since you are back to the original #1 coordinate system, the product of these transformation matrices must be the identity matrix.

$$A_1 A_2 A_3 A_4 = I$$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Carrying out the Matrix Method of Analysis

$A_1 A_2 A_3 A_4 = I$



Carrying out the Matrix Method of Analysis

- All the remaining displacement relations relating the pair variables can then be extracted from this matrix equation:

$A_1 A_2 A_3 A_4 = I$

Example: Analysis of the Hooke Joint

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_1 & -\cos\alpha_1 \sin\theta_1 & \sin\alpha_1 \sin\theta_1 \\ 0 & \sin\theta_1 & \cos\alpha_1 \cos\theta_1 & -\sin\alpha_1 \cos\theta_1 \\ 0 & 0 & \sin\alpha_1 & \cos\alpha_1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & \sin\theta_2 & 0 & -\cos\theta_2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_3 & 0 & \sin\theta_3 \\ 0 & \sin\theta_3 & 0 & -\cos\theta_3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_4 & 0 & \sin\theta_4 \\ 0 & \sin\theta_4 & 0 & -\cos\theta_4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Carrying out the Matrix Method of Analysis

- To reduce the number of matrix products involved, both sides of this equation can be multiplied by the inverse matrix A_1^{-1}.

$$A_1^{-1} A_1 A_2 A_3 A_4 = A_1^{-1} I = A_1^{-1}$$

$$A_2 A_3 A_4 = A_1^{-1}$$

Carrying out the Matrix Method of Analysis

- The inverse matrix A_1^{-1} in this case can be obtained by simply interchanging rows and columns in A_1 and is simply:

$$A_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 & 0 \\ 0 & -\cos\theta_1 \sin\theta_1 & \cos\theta_1 \cos\theta_1 & \sin\theta_1 \\ 0 & \sin\theta_1 \sin\theta_1 & -\sin\theta_1 \cos\theta_1 & \cos\theta_1 \end{bmatrix}$$

Carrying out the Matrix Method of Analysis

- After carrying out the matrix products we get:

$$A_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 & 0 \\ 0 & -\cos\theta_1 \sin\theta_1 & \cos\theta_1 \cos\theta_1 & \sin\theta_1 \\ 0 & \sin\theta_1 \sin\theta_1 & -\sin\theta_1 \cos\theta_1 & \cos\theta_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2 \cos\theta_3 \cos\theta_4 + \sin\theta_2 \sin\theta_4 & \cos\theta_2 \sin\theta_3 & -\cos\theta_2 \cos\theta_3 \sin\theta_4 - \sin\theta_2 \cos\theta_4 \\ 0 & \sin\theta_2 \cos\theta_3 \cos\theta_4 - \cos\theta_2 \sin\theta_4 & \sin\theta_2 \sin\theta_3 & -\sin\theta_2 \cos\theta_3 \sin\theta_4 + \cos\theta_2 \cos\theta_4 \\ 0 & \sin\theta_3 \cos\theta_4 & -\cos\theta_3 & \sin\theta_3 \sin\theta_4 \end{bmatrix}$$

Carrying out the Matrix Method of Analysis

- Corresponding elements in both matrices must be equal.

$$A_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 & 0 \\ 0 & -\cos\alpha_1 \sin\theta_1 & \cos\alpha_1 \cos\theta_1 & \sin\alpha_1 \\ 0 & \sin\alpha_1 \sin\theta_1 & -\sin\alpha_1 \cos\theta_1 & \cos\alpha_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2\cos\theta_3\cos\theta_4 + \sin\theta_2\sin\theta_4 & \cos\theta_2\sin\theta_3 & -\cos\theta_2\cos\theta_3\sin\theta_4 - \sin\theta_2\cos\theta_4 \\ 0 & \sin\theta_2\cos\theta_3\cos\theta_4 - \cos\theta_2\sin\theta_4 & \sin\theta_2\sin\theta_3 & -\sin\theta_2\cos\theta_3\sin\theta_4 + \cos\theta_2\cos\theta_4 \\ 0 & \sin\theta_3\cos\theta_4 & -\cos\theta_3 & \sin\theta_3\sin\theta_4 \end{bmatrix}$$

Carrying out the Matrix Method of Analysis

- Suppose that θ_1 is the input variable and is known.
- We seek relations giving θ_2, θ_3, and θ_4 in terms of θ_1.

$$A_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 & 0 \\ 0 & -\cos\alpha_1 \sin\theta_1 & \cos\alpha_1 \cos\theta_1 & \sin\alpha_1 \\ 0 & \sin\alpha_1 \sin\theta_1 & -\sin\alpha_1 \cos\theta_1 & \cos\alpha_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2\cos\theta_3\cos\theta_4 + \sin\theta_2\sin\theta_4 & \cos\theta_2\sin\theta_3 & -\cos\theta_2\cos\theta_3\sin\theta_4 - \sin\theta_2\cos\theta_4 \\ 0 & \sin\theta_2\cos\theta_3\cos\theta_4 - \cos\theta_2\sin\theta_4 & \sin\theta_2\sin\theta_3 & -\sin\theta_2\cos\theta_3\sin\theta_4 + \cos\theta_2\cos\theta_4 \\ 0 & \sin\theta_3\cos\theta_4 & -\cos\theta_3 & \sin\theta_3\sin\theta_4 \end{bmatrix}$$

Carrying out the Matrix Method of Analysis

- Equating the ratios of these sets of elements gives θ_2 in terms of θ_1:

$$A_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 & 0 \\ 0 & -\cos\theta_1 \sin\theta_1 & \cos\theta_1 \cos\theta_1 & \sin\theta_1 \\ 0 & \sin\theta_1 \sin\theta_1 & -\sin\theta_1 \cos\theta_1 & \cos\theta_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2 \cos\theta_3 \cos\theta_4 + \sin\theta_2 \sin\theta_4 & \cos\theta_2 \sin\theta_3 & -\cos\theta_2 \cos\theta_3 \sin\theta_4 - \sin\theta_2 \cos\theta_4 \\ 0 & \sin\theta_2 \cos\theta_3 \cos\theta_4 - \cos\theta_2 \sin\theta_4 & \sin\theta_2 \sin\theta_3 & -\sin\theta_2 \cos\theta_3 \sin\theta_4 + \cos\theta_2 \cos\theta_4 \\ 0 & \sin\theta_3 \cos\theta_4 & -\cos\theta_3 & \sin\theta_3 \sin\theta_4 \end{bmatrix}$$

$$\tan\theta_2 = \cos\theta_1 \cot\theta_1$$

Carrying out the Matrix Method of Analysis

- Equating this pair of elements gives θ_3 in terms of θ_1:

$$A_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 & 0 \\ 0 & -\cos\theta_1 \sin\theta_1 & \cos\theta_1 \cos\theta_1 & \sin\theta_1 \\ 0 & \sin\theta_1 \sin\theta_1 & -\sin\theta_1 \cos\theta_1 & \cos\theta_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2 \cos\theta_3 \cos\theta_4 + \sin\theta_2 \sin\theta_4 & \cos\theta_2 \sin\theta_3 & -\cos\theta_2 \cos\theta_3 \sin\theta_4 - \sin\theta_2 \cos\theta_4 \\ 0 & \sin\theta_2 \cos\theta_3 \cos\theta_4 - \cos\theta_2 \sin\theta_4 & \sin\theta_2 \sin\theta_3 & -\sin\theta_2 \cos\theta_3 \sin\theta_4 + \cos\theta_2 \cos\theta_4 \\ 0 & \sin\theta_3 \cos\theta_4 & -\cos\theta_3 & \sin\theta_3 \sin\theta_4 \end{bmatrix}$$

$$\cos\theta_3 = \sin\theta_1 \cos\theta_1$$

Carrying out the Matrix Method of Analysis

- And equating the ratios of this pair of elements gives θ_4 in terms of θ_1:

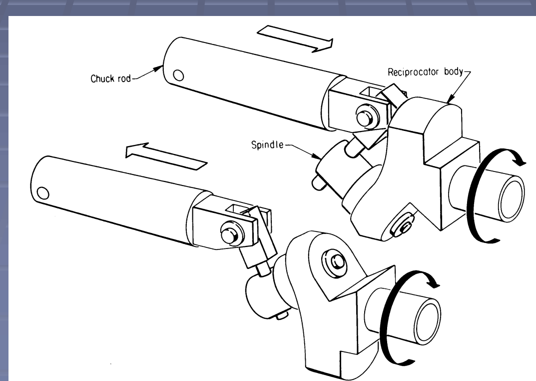
$$A_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 & 0 \\ 0 & -\cos\alpha_1 \sin\theta_1 & \cos\alpha_1 \cos\theta_1 & \sin\alpha_1 \\ 0 & \sin\alpha_1 \sin\theta_1 & -\sin\alpha_1 \cos\theta_1 & \cos\alpha_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2 \cos\theta_3 \cos\theta_4 + \sin\theta_2 \sin\theta_4 & \cos\theta_2 \sin\theta_3 & -\cos\theta_2 \cos\theta_3 \sin\theta_4 - \sin\theta_2 \cos\theta_4 \\ 0 & \sin\theta_2 \cos\theta_3 \cos\theta_4 - \cos\theta_2 \sin\theta_4 & \sin\theta_2 \sin\theta_3 & -\sin\theta_2 \cos\theta_3 \sin\theta_4 + \cos\theta_2 \cos\theta_4 \\ 0 & \sin\theta_3 \cos\theta_4 & -\cos\theta_3 & \sin\theta_3 \sin\theta_4 \end{bmatrix}$$

$$\tan\theta_4 = \frac{1}{\tan\theta_1 \sin\alpha_1}$$

Hartenberg-Denavit Homework:

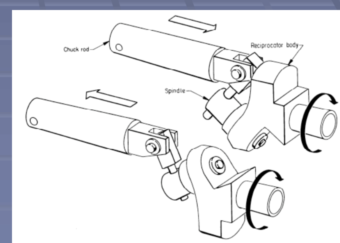
- The illustration shows an RRCRC saw drive mechanism reproduced from an article in Machine Design Magazine for Sept 24, 1964.



Hartenberg-Denavit Homework:

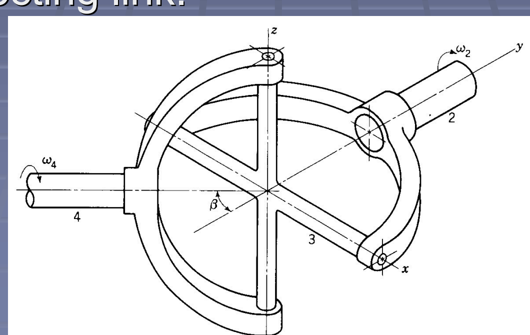
- Choose an appropriate coordinate system, take note of special proportions (such as 90º angles, zero lengths, etc. and derive the output versus input relation from the Hartenberg-Denavit matrix equation

$$[A_5]\ [A_4]\ [A_3]\ [A_2]\ [A_1] = [I]$$



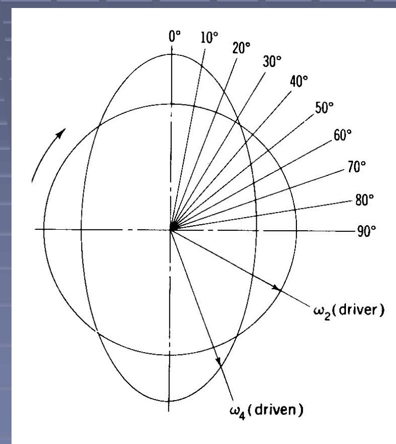
More Hartenberg-Denavit Homework: (based on Shigley & Uicker)

- The Hooke joint consists of two yokes (which are the driving and driven members) and a cross which is the connecting link.



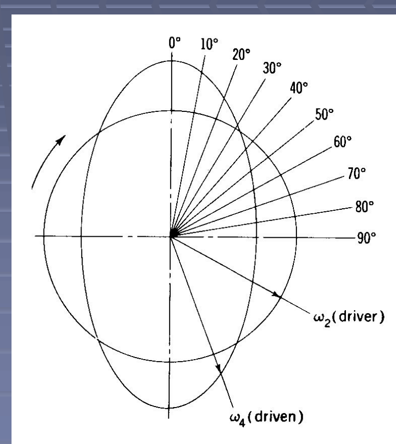
More Hartenberg-Denavit Homework: (based on Shigley & Uicker)

- One disadvantage of this joint is that the velocity ratio fluctuates during rotation.
- This is a polar angular velocity diagram for one complete rotation of the driver and driven links of the joint.



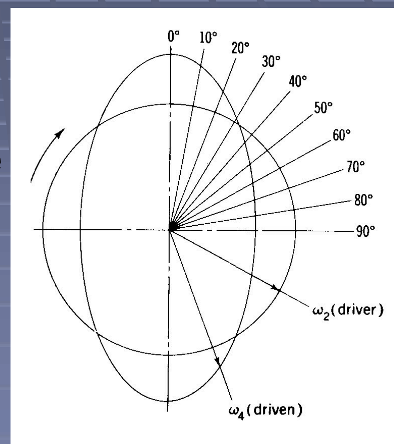
More Hartenberg-Denavit Homework: (based on Shigley & Uicker)

- Since the driver is assumed to have a constant angular velocity, its polar diagram is a circle.
- The diagram for the output is an ellipse which crosses the driver circle at four places.



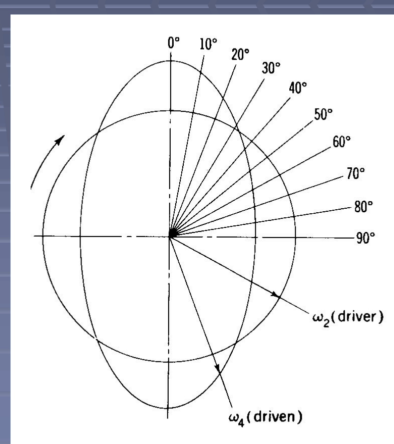
More Hartenberg-Denavit Homework: (based on Shigley & Uicker)

- This means there are four instants during each rotation when the angular velocities of the two shafts are equal.
- The rest of the time, the output rotates faster or slower.



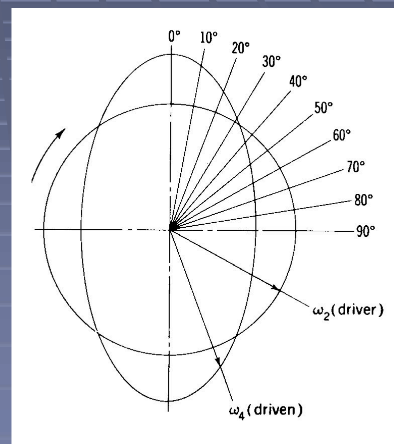
More Hartenberg-Denavit Homework: (based on Shigley & Uicker)

- Think of the drive shaft as having an inertia load at each end– the flywheel and engine spinning at constant speed at one end and the weight of the car running at high speed at the other end.



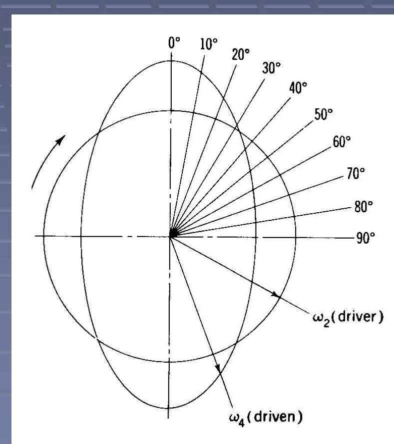
More Hartenberg-Denavit Homework: (based on Shigley & Uicker)

- If a single universal joint were used in a car either the speed of the engine or the speed of the car would need to vary during each rotation of the drive shaft.



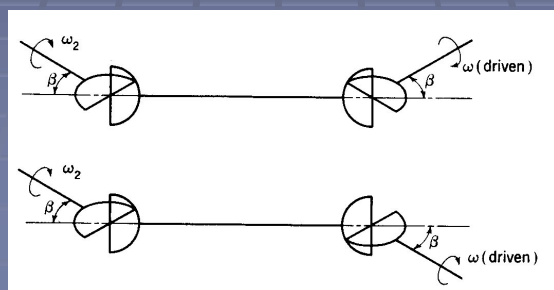
More Hartenberg-Denavit Homework: (based on Shigley & Uicker)

- Both inertias resist this so the tires would need to slip and the parts of the power transmission would be highly stressed.



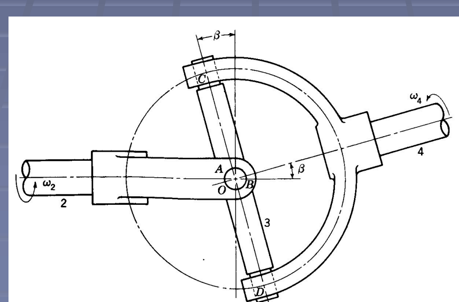
More Hartenberg-Denavit Homework: (based on Shigley & Uicker)

- To attain a uniform angular velocity ratio, actual drive shafts use a pair of universal joints arranged in one of these two configurations.
- This causes the speed fluctuations to cancel and a uniform velocity ratio from input to output.



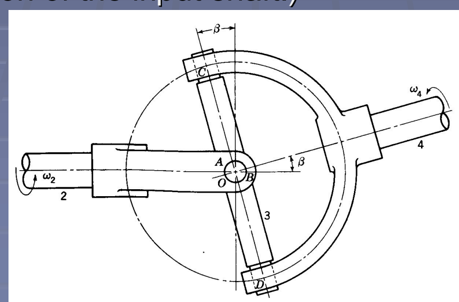
More Hartenberg-Denavit Homework: (based on Shigley & Uicker)

- Using the Hartenberg-Denavit method, develop an expression for the ratio of ω_{2}/ω_{4} in terms of the angle of shaft misalignment.



More Hartenberg-Denavit Homework: (based on Shigley & Uicker)

- Then use that expression to develop a table showing the ratio of the output angular velocity to the input angular velocity for a single universal joint at running at shaft misalignments of 0º, 5º, 10º, 15º, 30º, and 45º.
- (Data can be plotted at 15º increments if you like over just 90º rotation of the input shaft.)



More Hartenberg-Denavit Homework: (based on Shigley & Uicker)

- If the differences between the maximum and minimum ratios is expressed as a percent and plotted against the shaft angle a curve such as this one will result: