{ csci 3|6907 | Lecture 7 }

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Announcements

- HOMEWORK: HW4 due after spring break
- PRESENTATIONS: Apr 3 (?), 10, 17, 24.
- TODAY: probabilistic method & derandomization

PART I | derandomization

Derandomization

- NAMELY, remove the randomness from randomized algorithms.
  - Eliminate errors, from *w.h.p.* to *always*
  - Theoretical interest: does randomness *really* help in the design of algorithms?
- TWO TECHNIQUES. method of conditional expectations & pairwise independence.

Finding a Large Cut

- THEOREM: Any graph G with m edges has a cut of size $\geq \frac{m}{2}$.

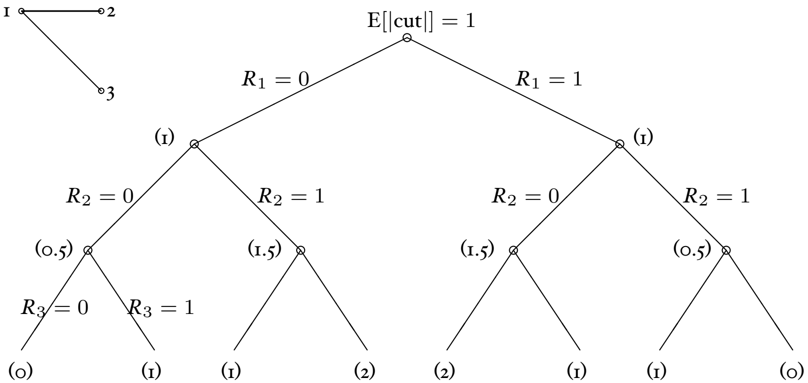
LargeCut Algorithm

Input: a graph $G = ([n], E)$

1. Flip n coins $r_1, r_2, \ldots, r_n$, put vertex i in S if $r_i = 0$ and in T otherwise.
2. Output (S, T).

- by averaging argument, there exists a choice of $r_1, r_2, \ldots, r_n$ that leads to a cut of size at least $\frac{m}{2}$.
- find a good sequence of coin tosses “bit by bit”.

LargeCut tree



QUESTION. How to compute labels of the “internal” nodes?

Deterministic LargeCut

- DEFINITION. Define the conditional expectation

$$e(r_1, r_2, \ldots, r_i) = \mathrm{E}_{R_1,\ldots,R_n}\Big[|\mathrm{cut}(S,T)| \mid R_1 = r_1, R_2 = r_2, \ldots, R_i = r_i\Big]$$

- BASE CASE. $e(\lambda) = |E|/2$.
- INDUCTIVE CASE. $e(\lambda) = \mathrm{E}_{R_1}[e(R_1)]$.
- more generally, $e(r_1, \ldots, r_i) = \mathrm{E}_{R_{i+1}}[e(r_1, \ldots, r_i, R_{i+1})]$
  i.e. $e(r_1, \ldots, r_i) = \frac{1}{2}(e(r_1, \ldots, r_i, 0) + e(r_1, \ldots, r_i, 1))$
- CLAIM. There exists $r_1, r_2, \ldots, r_n$ such that
  $e(\lambda) \leq e(r_1) \leq e(r_1, r_2) \leq \cdots \leq e(r_1, r_2, \ldots, r_n)$
- Q. which is bigger? $e(r_1, \ldots, r_i, 0)$ vs $e(r_1, \ldots, r_i, 1)$

Deterministic LargeCut

Deterministic LargeCut Algorithm I

1. Set $S = \emptyset, T = \emptyset$
2. For $i = 0, \ldots, n-1$:
   - 2.1 If $|\text{cut}(\{i+1\}, S)| > |\text{cut}(\{i+1\}, \bar{S})|$, set $T \leftarrow T \cup \{i+1\}$,
   - 2.2 else set $S \leftarrow S \cup \{i+1\}$.

- REMARK. This is the "natural" greedy algorithm. Method of conditional expectations tells us which objective function to optimize locally.

Derandomization via pairwise independence

- ANALYSIS, REVISITED.

$$\mathrm{E}[|\mathrm{cut}(S,T)|] = \sum_{(i,j)\in E} \Pr[R_i \neq R_j] = |E|/2$$

- OBSERVATION. suffices that $\Pr[R_i \neq R_j] = 1/2$ for each $i \neq j$; that is, *pairwise independent*
  - e.g., $N = 3$ vertices, R_1, R_2 independent, $R_3 = R_1 \oplus R_2$
  - each R_i is an unbiased random bit;
  - for each $i \neq j$, R_i is independent from R_j

- QUESTION. Can we generate N pairwise independent bits using less than N truly random bits?

Pairwise independent bits

- CONSTRUCTION. Let $B_1, \ldots, B_k$ be k independent unbiased random bits. For each nonempty set $A \subseteq [k]$, let R_A be the r.v. $\oplus_{i \in A} B_i$.
- CLAIM. The $2^k - 1$ random variables R_A are pairwise independent unbiased random bits.
  - Clear that each R_A is unbiased.
  - For pairwise independence, consider any $A \neq A' \subseteq [k]$. A’hen,

$$R_A = R_{A \cap A'} \oplus R_{A \setminus A'}; \quad R_{A'} = R_{A \cap A'} \oplus R_{A' \setminus A}$$

  - $R_{A \cap A'}, R_{A \setminus A'}, R_{A' \setminus A}$ are independent and at least two are non-empty.
  - Hence, $R_A, R_{A'}$ takes each value in $\{0, 1\}^2$ with prob. $1/4$.
- Can generate N pairwise independent bits from $\lceil \log(N + 1) \rceil$ independent random bits.

Deterministic LargeCut, II

Deterministic LargeCut Algorithm II

1. For all sequences of bits $b_1, b_2, \ldots, b_{\lceil \log(n+1) \rceil}$, run the randomized LargeCut algorithm using coin tosses $(r_A = \oplus_{i \in A} b_i)_{A \neq \emptyset}$ and choose the largest cut thus obtained.

PART 2 | sample and modify

Sample and Modify

- STAGE ONE. Construct a random structure that does not have the required properties.
- STAGE TWO. Modify the structure to have the required property.
- APPLICATION. Obtain bounds on the size of the largest independent set in a graph (set of vertices with no edges between them).

Existence of large independent sets

- THEOREM: Any graph G with n vertices and m edges has an independent set with at least $n^2/4m$ vertices (provided $m \geq n/2$).
- CONSTRUCTION.
  1. Delete each vertex of G (and its incident edges) with prob $1-p$.
  2. Remove all remaining edges along with one of its adjacent vertices.
- CLAIM. Let X and Y be resp. the # of vertices and edges that survive step 1. Then, construction outputs an I.S. of size at least $X - Y$.
- ANALYSIS. $\mathrm{E}[X] = np$ and $\mathrm{E}[Y] = mp^2$, so $\mathrm{E}[X-Y] = np - mp^2$. maximized at $n - 2mp = 0$, i.e. $p = n/2m$ and $\mathrm{E}[X-Y] = n^2/m$.

THE END | next, random walks