$\left\{ \text{Csc 80030} \;\middle|\; \text{Lecture 5} \right\}$

Probabilistic Analysis &
Randomized Algorithms

Hoeteck Wee · hoeteck@cs.qc.edu

PART 0

- HOMEWORK: HW2 due; HW3 is out
- TODAY: random graphs & probabilistic method

PART 1 | random graphs

Random Graphs

- Random graph model $\mathcal{G}_{n,p}$
  - Distribution over undirected graphs on n vertices
  - Every edge occurs with probability p
  - Graph with given set of m edges has probability

$$p^m(1-p)^{\binom{n}{2}-m}$$

- Basic properties
  - Expected number of edges is $p\binom{n}{2}$
  - Each vertex has expected degree $p(n-1)$

Threshold behavior for triangles

- TODAY: show that for random graph model $\mathcal{G}_{n,p}$:

$$\Pr[G \text{ contains a triangle}] \xrightarrow{n\to\infty} \begin{cases} 1 & \text{if } p = \omega(\frac{1}{n}) \\ 0 & \text{if } p = o(\frac{1}{n}) \end{cases}$$

  - If p grows faster than $\frac{1}{n}$, almost *every* graph contains a triangle
  - If p grows slower than $\frac{1}{n}$, almost *no* graph contains a triangle
- THRESHOLD BEHAVIOR: holds for many properties, e.g. "is connected", "contains a clique of size 4", with difference choices of "$\frac{1}{n}$"
- "STEP" ZERO: Let X be r.v. for # of triangles in a random graph in $\mathcal{G}_{n,p}$.

Threshold behavior for triangles

- GOAL: show that for random graph model $\mathcal{G}_{n,p}$:

$$\Pr[X \geq 1] \xrightarrow{n \to \infty} \begin{cases} 1 & \text{if } p = \omega(\frac{1}{n}) \\ 0 & \text{if } p = o(\frac{1}{n}) \end{cases}$$

- CLAIM 1: $\Pr[X \geq 1] \leq o(1) \quad \text{if } p = o(\frac{1}{n})$

  IDEA: by Markov's, $\Pr[X \geq 1] \leq \mathrm{E}[X]$.

- CLAIM 2: $\Pr[X \leq 0] \leq o(1) \quad \text{if } p = \omega(\frac{1}{n})$

  IDEA: use Chebyshev's to argue that $\Pr[|X - \mathrm{E}[X]| \geq \mathrm{E}[X]] = o(1)$

Triangles in expectation

- COMPUTING $\mathrm{E}[X]$: $X = \sum_S X_S$, S ranges over subsets of 3 vertices

$$X_S = \begin{cases} 1 & \text{if } S \text{ corresponds to a triangle in } G \\ 0 & \text{otherwise} \end{cases}$$

- $\mathrm{E}[X_S] = p^3$ and $\mathrm{E}[X] = \binom{n}{3} p^3$

$$\mathrm{E}[X] = \begin{cases} \omega(1) & \text{if } p = \omega(\frac{1}{n}) \\ \Theta(1) & \text{if } p = \Theta(\frac{1}{n}) \\ o(1) & \text{if } p = o(\frac{1}{n}) \end{cases}$$

- THUS: $\Pr[X \geq 1] \leq \mathrm{E}[X] = o(1) \quad \text{if } p = o(\frac{1}{n})$

Computing variance

- FACT: $\mathrm{Var}\left[\sum_S X_S\right] = \sum_S \mathrm{Var}[X_S] + \sum_{S\neq T} \mathrm{Cov}[X_S, X_T]$
- $\mathrm{Var}[X_S] = p^3(1-p^3)$
- $\mathrm{Cov}[X_S, X_T] = \mathrm{E}[X_S X_T] - \mathrm{E}[X_S]\,\mathrm{E}[X_T] = \Pr[X_S X_T = 1] - (p^3)^2$.
  - Case 1: $|S \cap T| \leq 1$:
    $$\Pr[X_S X_T = 1] = p^6 \quad \Rightarrow\ \mathrm{Cov}[X_S, X_T] = 0$$
  - Case 2: $|S \cap T| = 2$:
    $$\Pr[X_S X_T = 1] = p^5 \quad \Rightarrow\ \mathrm{Cov}[X_S, X_T] = p^5 - p^6$$
  - # pairs (S, T) fall into Case 2? $\binom{n}{2}(n-2)(n-3)$
- $\mathrm{Var}[X] = \binom{n}{3}p^3(1-p^3) + \binom{n}{2}(n-2)(n-3)(p^5-p^6) \leq \Theta(n^3p^3 + n^4p^5)$

Completing the analysis

- CLAIM 2: $\Pr[X \leq 0] \leq o(1) \quad \text{if } p = \omega(\frac{1}{n})$
- By Chebyshev's, $\Pr[X \leq 0] \leq \Pr[|X - \mathrm{E}[X]| \geq \mathrm{E}[X]] \leq \frac{\mathrm{Var}[X]}{(\mathrm{E}[X])^2}$.

  $\frac{\mathrm{Var}[X]}{(\mathrm{E}[X])^2} \leq \Theta\left(\frac{n^3p^3 + n^4p^5}{(n^3p^3)^2}\right) = \Theta\left(\frac{1}{n^3p^3} + \frac{p}{n^2p^2}\right) = o(1) \text{ if } p = \omega(\frac{1}{n})$

PART 2 | probabilistic method

Basic counting argument

- FACT: If $\Pr_{x \in \mathcal{U}}[x \text{ has property } P] > 0$, then $\exists\, x \in \mathcal{U}$ with property P.
- THEOREM: If $\binom{n}{k} 2^{-\binom{k}{2}+1} < 1$, then it is possible to color the edges of K_n with two colors so that it has no monochromatic K_k subgraph.
  - can set $k \approx 2 \log n$, e.g. exists a 2-coloring of the edges of K_{1000} with no monochromatic K_{20}.
  - here, $\mathcal{U} =$ all 2-colorings of the edges of K_n and
    $P =$ contains no monochromatic K_k subgraph
  - will show $\Pr_{x \in \mathcal{U}}[x \text{ contains a monochromatic } K_k \text{ subgraph}] < 1$

Avoiding monochromatic subgraphs

- CLAIM: If $\binom{n}{k}2^{-\binom{k}{2}+1} < 1$ and $\mathcal{U} =$ all 2-colorings of the edges of K_n, then $\Pr_{x \in \mathcal{U}}[x \text{ contains a monochromatic } K_k \text{ subgraph}] < 1$.
- STEP ONE: fix a K_k subgraph corresponding to a subset S of k vertices.
  - Picking random $x \in \mathcal{U} \equiv$ coloring each edge independently at random.
  - $\Pr_{x \in \mathcal{U}}[\text{edges of } S \text{ form a monochromatic } K_k \text{ subgraph in } x] = 2^{1-\binom{k}{2}}$
- STEP TWO: take a union bound.
  - $\Pr_{x \in \mathcal{U}}[x \text{ contains a monochromatic } K_k \text{ subgraph}] \leq \binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$

Averaging argument

- INFORMALLY: "not everyone is better than average"
- EXAMPLE: can show

  $E_{x \in \mathcal{U}}[\#$ of monochromatic K_k subgraphs in $x] < 1$.
- THEOREM: Any graph G with m edges has a cut of size $\geq \frac{m}{2}$.
  - $\mathcal{U} =$ all 2^n possible (vertex) partitions/cuts
  - Claim: $E_{x \in \mathcal{U}}[\text{size of the cut } x \text{ in } G] = \frac{m}{2}$.
  - Picking random $x \in \mathcal{U} \equiv$ picking each vertex on a random side of the cut

THE END | next, power of two choices