$\left\{ \text{Csc 80030} \mid \text{Lecture 10} \right\}$

Probabilistic Analysis & Randomized Algorithms

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PART 0

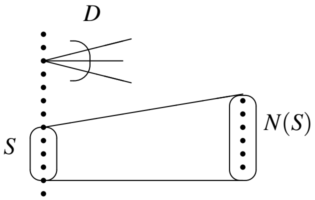
- HOMEWORK: HW5 due May 6
- PRESENTATIONS:
  - May 6 – Michael (balls and bins), Ben (sublinear-time algorithm), Peter (hiring problem)
  - May 13 – Valia (balls and bins), Justin
  - Homework problem requirement
- TODAY: expander graphs

PART 1 | expander graphs

Expander graphs

1. “sparse”
   - constant degree $\implies$ linear # of edges

2. “well-connected”
   - (K, A) vertex expansion: for all sets S of $\leq K$ vertices, the neighborhood $N(S)$ has size $\geq A \cdot |S|$.



   - would like $K = \Omega(N)$ and $A \sim D$.

Expander graphs

1. “sparse”
   - constant degree $\implies$ linear # of edges
2. “well-connected”
   - (K, A) vertex expansion: for all sets S of $\leq K$ vertices, the neighborhood $N(S)$ has size $\geq A \cdot |S|$.
   - edge expansion: # edges leaving S is large (i.e. no small cuts)
   - random walks: converge quickly to uniform, i.e., 2nd eigenvalue is small

QUESTION.

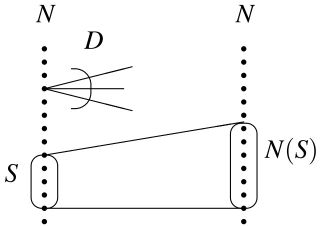
- Do good expanders exist?
- Specifically, (K, A) vertex expanders with $K = \Omega(N), A = 1 + \Omega(1)$?

Random graphs are good expanders

- THEOREM. For all $D \geq 3$, there is a constant $\alpha > 0$ s.t. a random D-regular graph on N vertices is an $(\alpha N, D - 1.01)$ vertex expander with prob. $\geq 1/2$.
  - $D \geq 3$ is necessary – every graph of degree 2 is a poor expander (being a union of cycles and chains).
  - will only prove a simpler result for bipartite expanders.

Random graphs are good expanders

- THEOREM. For all $D \geq 3$, there is a constant $\alpha > 0$ s.t. a random D-regular graph on N vertices is an $(\alpha N, D - 1.01)$ vertex expander with prob. $\geq 1/2$.
- DEFINITION. A bipartite graph is a (K, A) vertex expander if for all sets of $\leq K$ left vertices, the neighborhood $N(S)$ has size $\geq A \cdot |S|$.



- THEOREM. For all $D \geq 3$, there is a constant $\alpha > 0$ s.t. a random D-left-regular (N, N)-bipartite graph is an $(\alpha N, D - 2)$ vertex expander

Random bipartite expanders

- THEOREM. For all $D \geq 3$, there is a constant $\alpha > 0$ s.t. a random D-left-regular (N, N)-bipartite graph is an $(\alpha N, D-2)$ vertex expander with prob. $\geq 1/2$.

- STEP 1. Fix S of size $K \leq \alpha N$. Show that

$$\Pr\Big[|N(S)| \leq (D-2)\cdot K\Big] \leq \tbinom{KD}{2K}\left(\tfrac{KD}{N}\right)^{2K}$$

- STEP 2. Take a union bound over S and K.

$$\sum_{K=1}^{\alpha N}\binom{N}{K}\binom{KD}{2K}\left(\frac{KD}{N}\right)^{2K} \leq \sum_{K=1}^{\alpha N}\left(\frac{Ne}{K}\right)^{K}\left(\frac{KDe}{2K}\right)^{2K}\left(\frac{KD}{N}\right)^{2K} = \sum_{K=1}^{\alpha N}\left(\frac{e^3D^4K}{4N}\right)^{K}$$

$$\leq \sum_{K=1}^{\alpha N}\left(\frac{1}{4}\right)^{K} < \frac{1}{2} \quad \text{setting } \alpha = \tfrac{1}{e^3D^4}.$$

why

- Applications
  - construction of fault-tolerant networks
  - sorting in $O(\log n)$ time in parallel
  - derandomization (e.g. of UNDIRECTED S-T CONNECTIVITY)
  - error-correcting codes
  - distributed routing
  - data structures
  - negative results for integrality gaps for linear programming relaxations and metric embeddings
- Need *explicit* constructions (i.e. deterministic and efficient)

PART 2 | expansion and eigenvalues

Second eigenvalue

- DEFINITION. For an n-vertex regular graph G with random-walk matrix M, we define

$$\lambda(G) = \max_{\pi \in [0,1]^N} \frac{\|\pi M - u\|}{\|\pi - u\|}$$

- INTERPRETATION. If $\lambda(G)$ is small, then random walk on G converges quickly to uniform.
  - $\|\pi M - u\| \leq \lambda \|\pi - u\|$
  - $\|\pi M^k - u\| \leq \lambda^k \|\pi - u\|$

Second eigenvalue

- DEFINITION. For an n-vertex regular graph G with random-walk matrix M, we define
$$\lambda(G) = \max_{\pi \in [0,1]^N} \frac{\|\pi M - u\|}{\|\pi - u\|}$$

- CLAIM. Let $1 = \lambda_1, \lambda_2, \ldots, \lambda_N$ be the eigenvalues of M, with $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_N|$. Then, $\lambda(G) = |\lambda_2|$ ("2nd eigenvalue").
  - Write $\pi = u + c_2 v_2 + \cdots + c_N v_N$.
  - Hence, $\pi M = u + c_2 \lambda_2 v_2 + \cdots + c_N \lambda_N v_N$.

$$\begin{aligned} \text{Now,} \quad \|\pi M - u\|^2 &= c_2^2 \lambda_2^2 \|v_2\|^2 + \cdots + c_N^2 \lambda_N^2 \|v_N\|^2 \\ &\leq \lambda_2^2 (c_2^2 \|v_2\|^2 + \cdots + c_N^2 \|v_N\|^2) \\ &= \lambda_2^2 \|\pi - u\|^2 \end{aligned}$$

Expansion and eigenvalues

- THEOREM. If G is a regular graph with 2nd eigenvalue λ, then G is an $(N/2, 2-\lambda)$ vertex expander.
  - smaller λ, better expansion.
  - $2-\lambda$ is "best possible."
  - In fact, $\forall\ \alpha \in [0,1]$, G is an $(\alpha N, 1/((1-\alpha)\lambda^2+\alpha)$ vertex expander.

Expansion and eigenvalues

- THEOREM. If G is a regular graph with 2nd eigenvalue λ, then $\forall$ subsets S of size $\leq \alpha N$, $|N(S)| \geq |S|/(1-\alpha)\lambda^2 + \alpha)$.

- DEFINITIONS. For a prob. distribution π,
  - The *collision probability* of π is the prob. that two independent samples are equal, i.e., $\mathrm{CP}(\pi) = \sum_x \pi_x^2$.
  - The *support* of π is $\mathrm{Supp}(\pi) = \{x : \pi_x > 0\}$.

- CLAIMS. For any prob. distribution $\pi \in [0,1]^N$,
  - $\mathrm{CP}(\pi) = \|\pi\|^2 = \|\pi - u\|^2 + 1/N$.
  - $\mathrm{CP}(\pi) \geq 1/\,\mathrm{Supp}(\pi)$. (since $(\sum_x \pi_x^2) \cdot (\sum_x 1) \geq (\sum_x \pi_x)^2$)

- KEY IDEA. set π to be uniform over S.

Expansion and eigenvalues

- THEOREM. If G is a regular graph with 2nd eigenvalue λ, then $\forall$ subsets S of size $\leq \alpha N$, $|N(S)| \geq |S|/(1-\alpha)\lambda^2 + \alpha)$.
  - STEP 1. for all π, $\|\pi M - u\|^2 \leq \lambda^2 \|\pi - u\|^2$.
    1st claim $\implies \quad \mathrm{CP}(\pi M) - \frac{1}{N} \leq \lambda^2(\mathrm{CP}(\pi) - \frac{1}{N})$.
  - STEP 2. set π to be uniform over S. Then,
    2nd claim $\implies \quad \mathrm{CP}(\pi M) \geq \mathrm{Supp}(\pi M) = 1/|N(S)|$.
    Also, $\mathrm{CP}(\pi) = 1/|S|$.
  - COMBINING, $\frac{1}{|N(S)|} - \frac{1}{N} \leq \lambda^2(\frac{1}{|S|} - \frac{1}{N})$

An equivalence

- THEOREM. Let $\mathcal{G}$ be an infinite family of D-regular multigraphs for a constant D. Then, the following two conditions are equivalent:
  - "good" vertex expansion:
    $\exists$ constant $\delta_1 < 1$ s.t. every $G \in \mathcal{G}$ is an $(N/2, 2 - \delta_1)$ vertex expander.
  - "small" 2nd eigenvalue:
    $\exists$ constant $\delta_2 < 1$ s.t. every $G \in \mathcal{G}$ has 2nd eigenvalue at most δ_2.
- NOTE. $(2) \implies (1)$ as before; the other direction is "hard" (omitted).
- The term "expander" typically refers to a family of regular graphs of constant degree satisfying one of the two equivalent conditions.

PART 3 | towards explicit constructions

Iterative constructions of expanders

- APPROACH. iterative constructions

  1. Start with a "constant size" expander H.
  2. Repeatedly apply graph operations to get better expanders.

     increase # nodes, keep degree and 2nd eigenvalue bounded.

- DEFINITION. a (N, D, λ)-graph is a D-regular graph on N vertices with 2nd eigenvalue $\leq \lambda$.

Squaring of graphs

- GRAPH SQUARING. If $G = (V, E)$ is a D-regular graph, then $G^2 = (V, E')$ is a D^2-regular graph on the same vertex set, E' comprises pairs of vertices connected by a path of length 2 in G.

- LEMMA. If G is a (N, D, λ)-graph, then G^2 is a (N, D^2, λ^2)-graph.
  - If M is random-walk matrix for G, then M^2 is random-walk matrix for G^2.

THE END | next, ...