A Bayesian Paired Comparison Approach for Relative Accident Probability Assessment with Covariate Information.

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Discuss **Bayesian methodology** for assessing **relative accident probabilities** and their uncertainty using **paired comparisons** to elicit expert judgments. Approach is illustrated using expert judgment data elicited for **The Washington State Ferry Risk Assessment** in 1999

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1. INTRODUCTION

- An important class of elicitation techniques consists of the psychological scaling models that use the concept of paired comparisons. Origins can be traced back to **Thurstone's (1927) and Bradley (1953)).**

- Another popular paired comparison elicitation technique is called **the Analytical Hierarchy Process (AHP)** developed by **Saaty (1977, 1980).** The AHP Process is primarily used for the construction of value functions $V(\underline{X})$ involving multiple contributing factors $\underline{X} = (X_1, X_2, \ldots, X_p)$ (see, e.g. Foreman and Selly (2002)).

- The popularity of the paired comparison method can perhaps be contributed to the observation that experts are more comfortable making comparisons rather than directly assessing a quantity of interest.

- To the best of our knowledge, **Pulkkinen (1993, 1994)** was first to introduce a Bayesian paired comparison aggregation method for the elements of a multivariate random vector $\underline{\beta} = (\beta_1, \beta_2, \ldots, \beta_p)$ by multiple experts. Pulkkinen's (1993, 1994) exposition is mainly theoretical and limited to a discussion of mathematical properties.

- Similar to the AHP process, we are interested in the functional relationship between **contributing factors** $\underline{X} = (X_1, X_2, \ldots, X_p)$ and an accident probability $Pr(Accident| Incident, \underline{X})$ defined by

$$Pr(Accident|Incident, \underline{X}) = P_0 Exp(\underline{\beta}^T \underline{X}). \quad (1)$$

- $\underline{X} = (X_1, X_2, \ldots, X_p)$ describes **a system state** during which an incident (e.g. a mechanical failure) occurred.

- The accident probability model (1) resembles the well-known **proportional hazards model** originally proposed by **Cox (1972)** and builds on the assumption that accident risk behaves **exponentially** rather than linearly with changes in covariate values.

- Our goal is to establish the uncertainty distribution of the accident probability $Pr(Accident|Incident, \underline{X})$ in entirety rather than a point estimate.

"Since the truth is, we always have uncertainty, we say that speaking in probability curves is telling the truth".
(see, e.g., Kaplan, 1997, p. 412)

2. ACCIDENT PROBABILITY MODEL

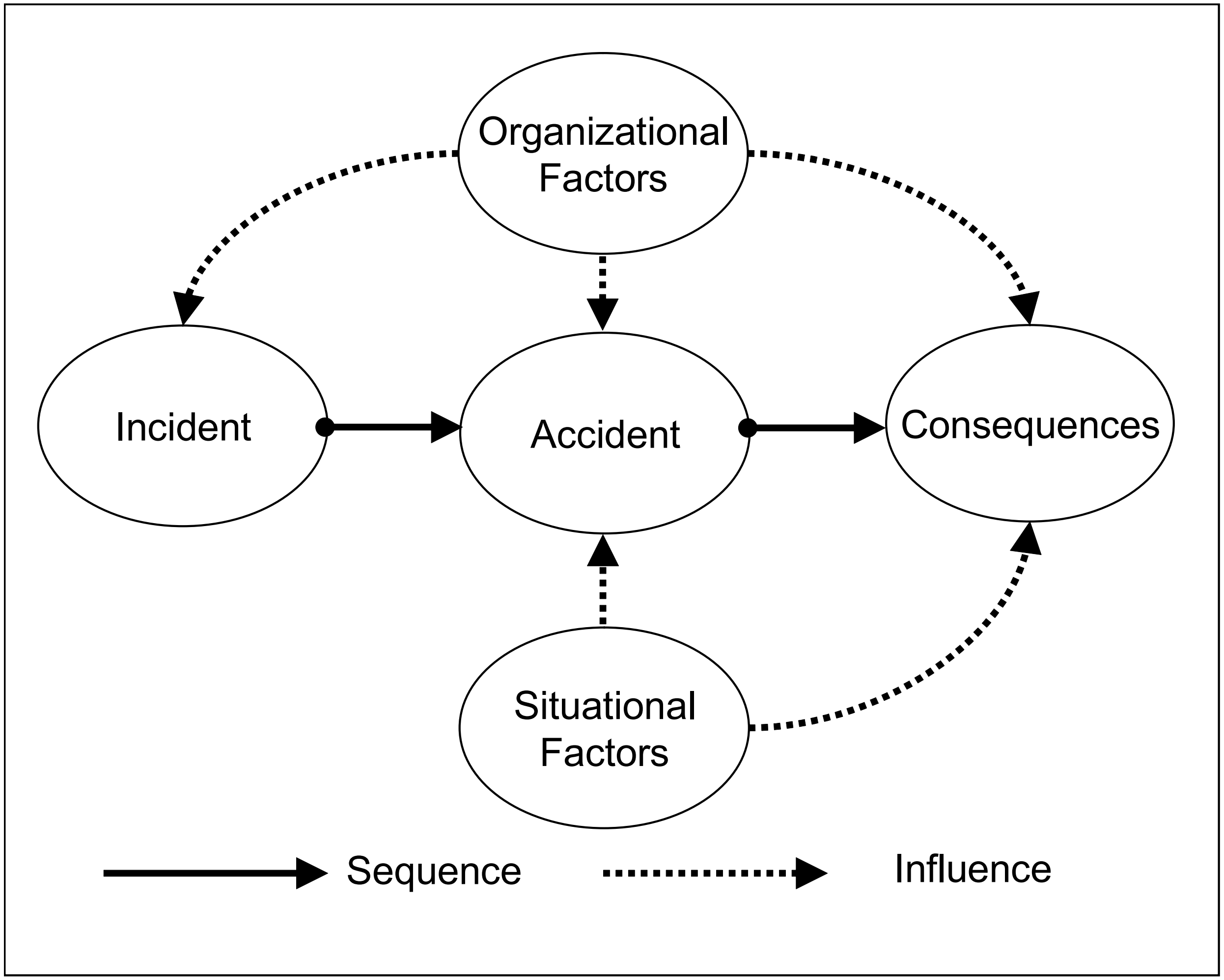

Figure 1. The accident probability model

Table 1. Description of 10 contributing factors to $Pr(Accident \mid Incident, \underline{X})$ in WSF Risk Assessment

	Designation	*Description*	*Discretization*
X_1	FR_FC	Ferry route-class combination	26
X_2	TT_1	1st interacting vessel type	13
X_3	TS_1	Scenario of 1st interaction	4
X_4	TP_1	Proximity of 1st interaction	*Binary*
X_5	TT_2	2nd interacting vessel type	5
X_6	TS_2	Scenario of 2nd interaction	4
X_7	TP_2	Proximity of 2nd interaction	*Binary*
X_8	VIS	Visibility	*Binary*
X_9	WD	Wind direction	*Binary*
X_{10}	WS	Wind speed	*Continuous*

- $\underline{X} \in [0,1]^p$, $\underline{\beta} \in \mathbb{R}^p$ and $P_0 \in (0,1)$. The covariate $X_i,\ i = 1, \ldots, p$ are **normalized** so that $X_i = 1$ describes the **"worst" case scenario** and $X_i = 0$ describes the **"best" case scenario.**

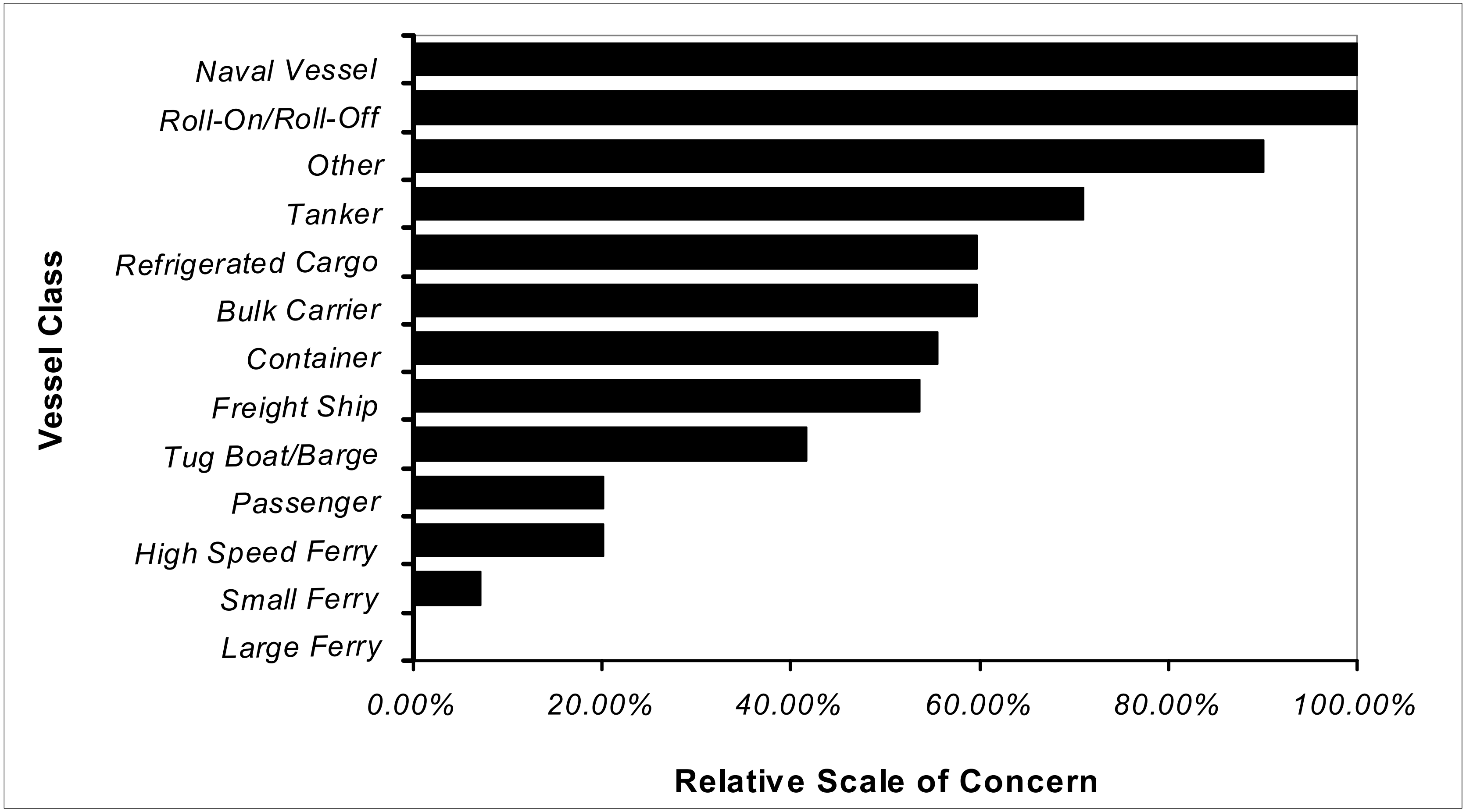

Figure 2. Constructed Covariate Scale for Interacting Vessels

Question: 32 48

Situation 1	Attribute	Situation 2
Super	**Ferry Class**	-
SEA-BAI	**Ferry Route**	-
Naval Vessel	**1st Interacting Vessel**	-
Crossing the bow	**Traffic Scenario 1st Vessel**	-
1 to 5 miles	**Traffic Proximity 1st Vessel**	-
Deep Draft	**2nd Interacting Vessel**	-
Crossing the bow	**Traffic Scenario 2nd Vessel**	-
1 to 5 miles	**Traffic Proximity 2nd Vessel**	-
more than 0.5 mile	**Visibility**	less than 0.5 mile
Along Ferry	**Wind Direction**	-
40 knots	**Wind Speed**	-
	9 8 7 6 5 4 3 2 1 2 3 4 5 6 7 8 9	
Situation 1 is worse	**<===================X===================>**	**Situation 2 is worse**

Figure 3. An example question appearing in one of the questionnaires used in the WSF risk assessment

$$P(\underline{X}^1, \underline{X}^2 \,|\, \underline{\beta}\,) = Exp\{\underline{\beta}^T(\underline{X}^1 - \underline{X}^2)\} \in [0, \infty]. \tag{2}$$

$$Log\{P(\underline{X}^1, \underline{X}^2 \,|\, \underline{\beta}\,)\} = \underline{\beta}^T(\underline{X}^1 - \underline{X}^2) \in (-\infty, \infty) \tag{3}$$

3. THE LIKELIHOOD OF A SINGLE EXPERT'S RESPONSE

$$Y_j = \text{Experts response to ratio } \frac{Pr(Accident|Incident,\ \underline{X}_j^1)}{Pr(Accident|Incident,\ \underline{X}_j^2)},$$

$$Z_j = Log\, Y_j, j = 1, \ldots, n.$$

The response of the expert to such a question is uncertain and will assumed to be **normal distributed** such that

$$(Z_j \mid \mu_j, r) \sim N(\mu_j, r),\ r = 1/\sigma^2 \quad (4)$$

$$\mu_j = q_j^T \underline{\beta}, \underline{q}_j = (\underline{X}_j^1 - \underline{X}_j^2) \quad (5)$$

$$f_{Z_j}(z_j) \propto \sqrt{r}\ exp\left\{ -\frac{r}{2}(z_j - \mu_j)^2 \right\}. \quad (6)$$

• Expert answers ***n* paired comparison questions** defined by $\underline{q}_j = (\underline{X}_j^1 - \underline{X}_j^2)$, $j = 1, \ldots, n$, Define Q to be the $p \times n$ matrix and $\mathcal{Z}$ to be the vector with log responses of expert

$$Q = [\underline{q}_1, \ldots, \underline{q}_n],\ \mathcal{Z} = (z_1, \ldots, z_n). \qquad (7)$$

• **Likelihood of an expert responding $\mathcal{Z}$ to questionnaire Q**, may be derived from (6) as being proportional to

$$\mathcal{L}(\mathcal{Z} \mid \underline{\beta}, r, Q) \propto r^{\frac{n}{2}} exp\left\{ -\frac{r}{2}(c - 2\,\underline{b}^T \underline{\beta} + \underline{\beta}^T A \underline{\beta}) \right\}. \qquad (9)$$

where

$$A = \sum_{j=1}^{n} \underline{q}_j \underline{q}_j^T ;\ \underline{b} = \sum_{j=1}^{n} \underline{q}_j z_j ;\ c = \sum_{j=1}^{n} z_j^2 \qquad (10)$$

If columns of Q span $\mathbb{R}^p$ the matrix A can be shown to be symmetric, positive definite and henceforth invertible.

4. PRIOR DISTRIBUTION

• To allow for a conjugate Bayesian analysis **a multivariate normal/gamma prior** is proposed for the joint distribution of $(\underline{\beta}\,, r)$ similar to the one described in **West and Harrison (1989).**

$$\prod(r\,|\alpha,\nu) = \frac{\frac{\nu}{2}^{\frac{\alpha}{2}}}{\Gamma(\frac{\alpha}{2})} r^{\frac{\alpha}{2}-1} exp(-\frac{r}{2}\nu), \text{ i.e. } Gamma(\frac{\alpha}{2},\frac{\nu}{2}). \qquad (11)$$

$$\prod(\underline{\beta}\,|\,r) \propto r^{\frac{p}{2}} exp\left\{-\frac{r}{2}(\underline{\beta}-\underline{m})^T\Delta(\underline{\beta}-\underline{m})\right\}, \text{ i.e. } MVN(\underline{m}, r\Delta). \quad (12)$$

Hence, **the joint prior distribution** on $(\underline{\beta}\,, r)$ follows from (11) and (12) to be

$$\prod(\underline{\beta}\,, r) \propto r^{\frac{\alpha}{2}-1} exp(-\frac{r}{2}\nu) \times r^{\frac{p}{2}} exp\left\{-\frac{r}{2}(\underline{\beta}-\underline{m})^T\Delta(\underline{\beta}-\underline{m})\right\}. \quad (13)$$

- The marginal distribution of $\underline{\beta}$ may be derived from (14), yielding

$$\prod(\underline{\beta}) \propto \left[1+\frac{1}{\nu}(\underline{\beta}-\underline{m})^T\Delta(\underline{\beta}-\underline{m})\right]^{-\frac{\alpha+p}{2}} \qquad (14)$$

and is recognized as a ***p*-dimensional multivariate *t*-distribution** with α degrees of freedom, location vector $\underline{m}$ and precision matrix $\frac{\alpha}{\nu}\Delta$.

- From (14) and (3) follows that the **log-relative probability** $Log\{P(\underline{X}^1,\underline{X}^2\,|\,\underline{\beta})\}$ has a **prior *t*-distribution** with mean and precision

$$\underline{m}^T(\underline{X}^1-\underline{X}^2),\ \frac{\alpha}{\nu}(\underline{X}^1-\underline{X}^2)^T\Delta(\underline{X}^1-\underline{X}^2) \qquad (15)$$

4.1. Prior Parameter Specification

- A **prior chi-squared distribution** with α degrees of freedom (equivalent to a gamma distribution $Gamma(\frac{\alpha}{2},\frac{\nu}{2})$ with $\nu=1$) and $E[r|\alpha,\nu{=}1]=\alpha$.

- The prior parameter α will be set equal to **the reciprocal of the variance** of **an expert responding at random** and depends on the scale that is used in the paired comparison questions to collect the expert responses.

$$\alpha = E[r|\alpha,\nu{=}1] = \frac{1}{\frac{2}{17}\sum_{k=2}^{9}\{Log(k)\}^2} \approx 0.380341. \qquad (16)$$

- For distribution of $(\underline{\beta}|r)$ we may select **a location vector** and the **unit precision matrix**

$$\underline{m} = (0,\ldots,0)^T, \Delta = \begin{pmatrix} 1 & & \emptyset \\ & \ddots & \\ \emptyset & & 1 \end{pmatrix}, \qquad (17)$$

as long as the prior distribution on the relative accident probabilities (2) are flat.

- The **pdf of the relative accident probability** in Figure 4C is one of a **log-*t* distribution** (see, e.g., McDonald and Butler (1987)) with prior parameters (cf. (19) and (20))

$$\underline{m}^T(\underline{X}^1 - \underline{X}^2) = 0, \alpha = 0.380341,\ \nu = 1\,,\ \delta_{ii} = (\underline{X}^1 - \underline{X}^2)^T \Delta(\underline{X}^1 - \underline{X}^2) = 4.$$

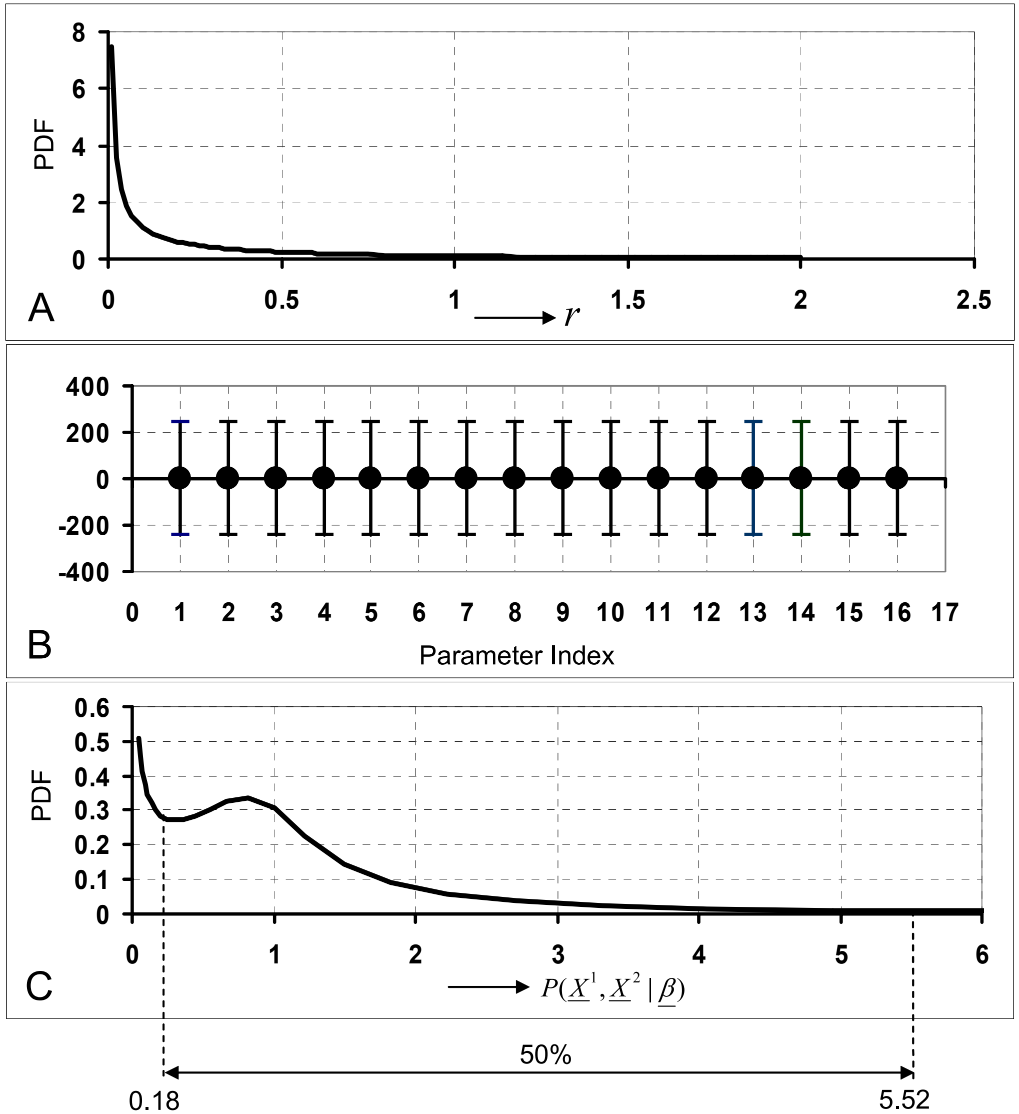

Figure 4. Prior on $(\underline{\beta}, r)$ and $P(\underline{X}^1, \underline{X}^2 \mid \underline{\beta})$ of question in Figure 3 (cf. (2))

- **The prior median of $P(\underline{X}^1, \underline{X}^2 \mid \underline{\beta})$ equals 1** (indicating indifference in collision likelihood between system states $\underline{X}^1$ and $\underline{X}^2$).

- **A 50% credibility interval** of $P(\underline{X}^1, \underline{X}^2 \mid \underline{\beta})$ in Figure 4A equals $[0.181, 5.515]$. **A 75% credibility interval** of $P(\underline{X}^1, \underline{X}^2 \mid \underline{\beta})$ equals $[2.012 \cdot 10^{-5},\ 4.971 \cdot 10^{4}]$ (which is quite wide).

Table 2. Interaction Variables associated with the contributing factors in Table 1.

	Name	*Description*	*Discretization*
X_{11}	FR_FC · TT_1	Interaction	13
X_{12}	FR_FC · TS_1	Interaction	13
X_{13}	FR_FC · VIS	Interaction	4
X_{14}	TT_1 · TS_1	Interaction	*Binary*
X_{15}	TT_1 · VIS	Interaction	13
X_{16}	TS_1 · VIS	Interaction	4

5. POSTERIOR ANALYSIS

Applying Bayes theorem utilizing the likelihood (9) , the prior distribution (13) and it follows that the posterior distribution $\prod(\underline{\beta}, r \mid \mathcal{Z}, Q)$ is proportional to

$$\prod(\underline{\beta}, r \mid \mathcal{Z}, Q) \propto r^{\frac{\alpha+n}{2}-1} exp\left\{-\frac{r}{2}\left(1+c+\underline{m}^T \Delta \underline{m}\right)\right\} \times \qquad (18)$$

$$r^{\frac{p}{2}} exp\left\{-\frac{r}{2}\left(-2\left[\underline{b}+\Delta \underline{m}\right]^T \underline{\beta} + \underline{\beta}^T \left[A+\Delta\right]\underline{\beta}\right)\right\}.$$

Defining Δ^u to be $\Delta^u = A + \Delta$ and implicitly defining $\underline{m}^u$ satisfying

$$\left[\underline{b}+\Delta\,\underline{m}\right]^T \underline{\beta} = \left[\Delta^u \underline{m}^u\right]^T \underline{\beta} \qquad (19)$$

for all $\underline{\beta}$, it follows that

$$\underline{b}+\sum \underline{m} = \Delta^u \underline{m}^u \Leftrightarrow \underline{m}^u = \left(\Delta^u\right)^{-1}\left(\underline{b}+\Delta\,\underline{m}\right). \qquad (20)$$

Utilizing (20) and $\Delta^u = A + \Delta$ we derive from (18) that

$$\prod(\underline{\beta}, r \mid \mathcal{Z}, Q) \propto r^{\frac{\alpha+n}{2}-1} exp\Big\{-\frac{r}{2}\Big(1 + c + \underline{m}^T \Delta \underline{m} - \big[\underline{m}^u\big]^T \Delta^u \underline{m}^u\Big)\Big\} \times \quad (21)$$

$$r^{\frac{p}{2}} exp\Big\{-\frac{r}{2}\big[\underline{\beta} - \underline{m}^u\big]^T \Delta^u \big[\underline{\beta} - \underline{m}^u\big]\Big\}.$$

From (21) it follows that $(\underline{\boldsymbol{\beta}} \mid \boldsymbol{\mathcal{Z}}, \boldsymbol{Q}) \sim \boldsymbol{MVN}(\underline{\boldsymbol{m}}^{\boldsymbol{u}}, \boldsymbol{r}\boldsymbol{\Delta}^{\boldsymbol{u}})$ where

$$\begin{cases} \Delta^u = \sum_{j=1}^{n} \underline{q}_j \underline{q}_j^T + \Delta \\ \underline{m}^u = \Big(\Delta^u\Big)^{-1} \Big(\sum_{j=1}^{n} \underline{q}_j z_j + \Delta\, \underline{m}\Big) \end{cases} \quad (30)$$

and $(\boldsymbol{r} \mid \boldsymbol{\mathcal{Z}}, \boldsymbol{Q}) \sim \boldsymbol{Gamma}(\frac{\boldsymbol{\alpha^u}}{\boldsymbol{2}}, \frac{\boldsymbol{\nu^u}}{\boldsymbol{2}})$ with

$$\begin{cases} \alpha^u = \alpha + n \\ \nu^u = \nu + \sum_{j=1}^{n} z_j^2 + \underline{m}^T \Delta \underline{m} - \big[\underline{m}^u\big]^T \Delta^u \underline{m}^u \end{cases} \quad (31)$$

6. EXAMPLE FROM WSF RISK ASSESSMENT

- **8 Experts** were selected amongst WSF captains and WSF first mates who had extensive experience with all 13 different ferry routes over an extended period of time (more than 5 years). **Combination** of the responses of these 8 experts follows naturally by **exploiting the conjugacy of the analysis** in Section 3, 4 and 5 through **sequential updating.**

Table 3. Expert Response to the Paired Comparison in Figure 3

Expert Index	1	2	3	4	5	6	7	8
Response	5	5	3	9	7	9	3	0.5

- **During the WSF risk assessment in 1998** expert responses were aggregated by taking **geometric means of their responses** and using them in a **classical log linear regression analysis** approach to assess relative accident probabilities given by (2). **Classical point estimates** for the parameters $\beta_j, j = 1, \ldots, 16$ associated with the contribution factors (the so-called main effects) in Table 1 and interaction effects in Table 2 **will be compared** to their **Bayesian counterparts** following our Bayesian aggregation method.

- Expert were instructed to assume that **a navigation equipment failure** had occurred on the **Washington State Ferry** and were next asked to assess **how much more likely a collision is to occur** in Situation 1 (good visibility in Figure 3) as compared to Situation 2 (bad visibility in Figure 3) taking into account the value of all the contributing factors. **Total of 60 Questions.** The questions were **randomized** in order and were **distributed evenly over the 10 contributing factors** in Table 1 (i.e. 6 questions per changing contributing factor).

6.1. The elements A, $\underline{b}$ and c of the likelihood given by (10)

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{32}$$

where **A_{11} is a 10×10 diagonal matrix** with diagonal elements

$$(4.56, 4.33, 2.89, 6, 1.5, 2.44, 6, 6, 6, 0.375) \tag{33}$$

and associated with **the contributing factors $X_1, \ldots, X_{10}$.** (The matrix A_{11} in (32) is a diagonal matrix since the paired comparison scenarios $\underline{X}^1$ and $\underline{X}^2$ **only differed in one covariate** (see Figure 3)).

The matrix $\boldsymbol{A_{22}}$ in (32) is a symmetric 6×6 matrix with elements

$$\begin{bmatrix} 3.45 & 0.33 & 0 & 1.44 & 0.76 & 0 \\ 0.33 & 3.45 & 0.44 & 0.33 & 0 & 1 \\ 0 & 0.44 & 4.11 & 0 & 1 & 2.39 \\ 1.44 & 0.33 & 0 & 1.89 & 0.36 & 0.08 \\ 0.76 & 0 & 1 & 0.36 & 3.02 & 2 \\ 0 & 1 & 2.39 & 0.08 & 2 & 6.67 \end{bmatrix} \quad (34)$$

and associated with the **interaction effects $\boldsymbol{X_{11}, \ldots, X_{16}}$.** Finally, **the matrix $\boldsymbol{A_{21} = A_{12}^T}$** is a sparse 10×6 matrix

$$\begin{bmatrix} 1 & 2.82 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2.26 & 0 & 2.12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.13 & 0 & 0 & 0 & 0 & 0 & 0 & 3.06 & 0 & 0 \\ 0 & 2.13 & 0.52 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.02 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1.56 & 0 & 0 & 0 & 0 & 5.33 & 0 & 0 \end{bmatrix} \quad (35)$$

with **only positive elements** associated with the contributing factors X_1, X_2, X_3 and X_8 that are **included in the interaction effects** $X_{11}, \ldots, X_{16}$.

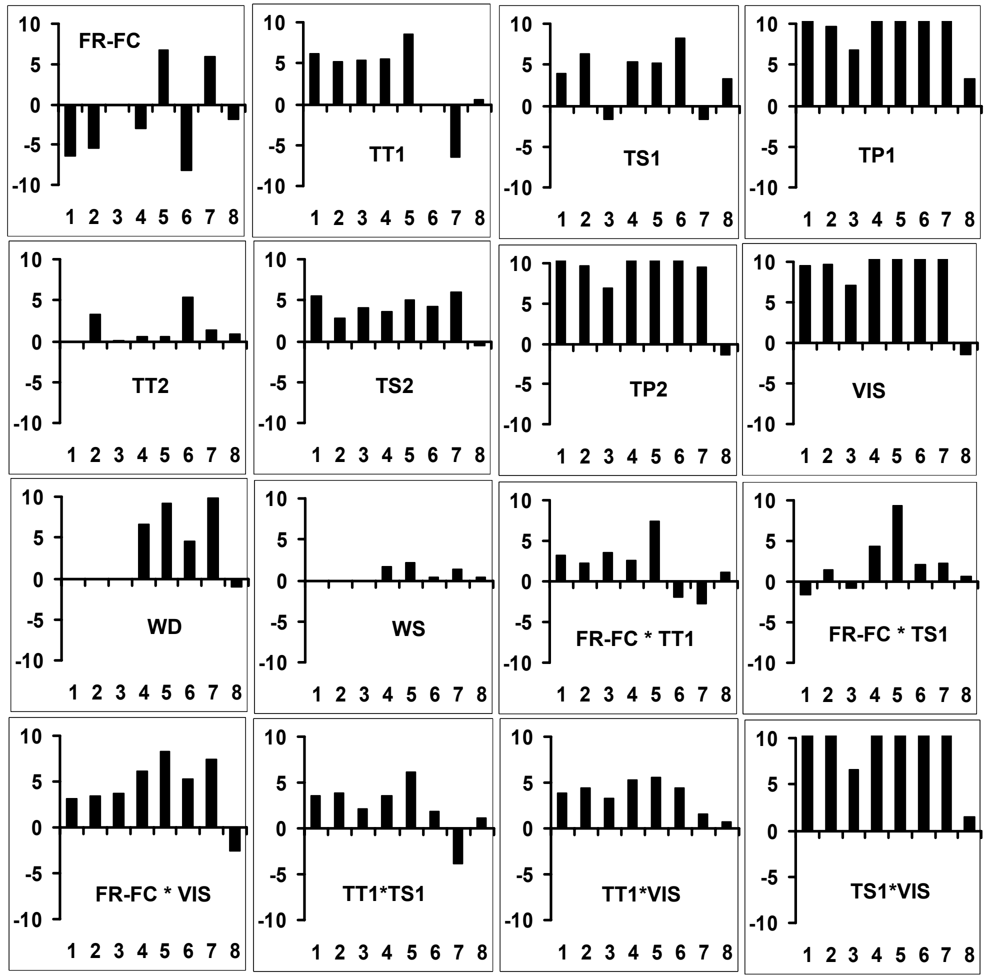

Figure 5. Summary of Individual Expert Response for 8 WSF experts in terms of i-th element of the vector $\underline{b}$ (cf. (11) for each of the contributing factors $X_i, i = 1, \ldots, 10$ in Table 1 and interaction effects X_i, $i = 11, \ldots, 16$ in Table 2.

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Table 4. Values for c (cf. (11)) for the 8 individual experts.

Expert Index	1	2	3	4	5	6	7	8
c	149.07	95.28	55.74	147.93	185.71	177.30	147.12	44.94

6.2. Posterior Analysis

The resulting posterior parameters for the precision $r \sim \; Gamma(\frac{\alpha^u}{2}, \frac{\nu^u}{2})$ are

$$\alpha^u = 480.38,\ \nu^u = 530.95 \qquad (36)$$

The posterior distribution of the parameter vector $\underline{\beta}$ is a multivariate t distribution with location vector $\underline{m}^u$ and precision matrix $\frac{\alpha^u}{\nu^u}\Delta^u$, where α^u, ν^u are given by (36),

$$\Delta^u = \Delta + 8A$$

and location vector $\underline{m}^u$ is depicted in the following figure.

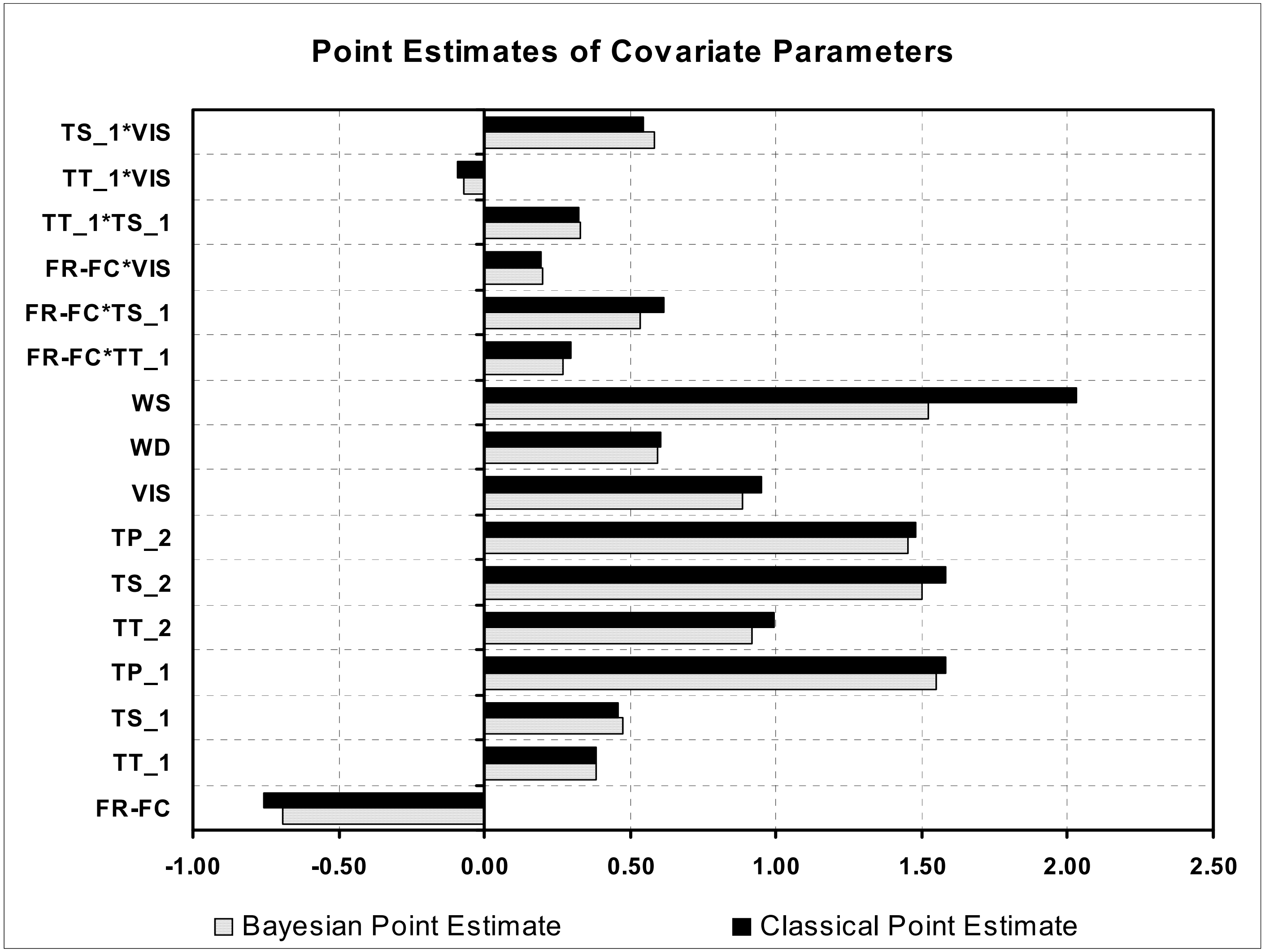

Figure 6. Comparison of Bayesian and Classical Point Estimates of the parameters β_i, $i = 1, \ldots, 16$.

- It can be concluded from Figure 6 that **traffic proximity of the first and second interacting vessel** (X_4 and X_7, respectively), **traffic scenario of the second interacting vessel** X_7 and wind speed X_{10} are the largest contributing factors to accident risk. In addition, **the manner in which the first interacting vessel approaches the ferry route - ferry class combination** (X_{12}), i.e. crossing, passing or overtaking, and in what visibility conditions (X_{16}) are the largest interacting factors.

- The posterior location vector $\underline{m}^u$ is displayed in Figure 7 together with their classical counterpart estimated via a log-linear regression method utilizing the geometric means of the expert responses. **A remarkable agreement** should be noted between the **Bayesian** and **classical point estimates** provided in Figure 6, except for a discrepancy associated with the contributing factor WS (Wind Speed). From Figure 7, however, it follows that the classical point estimate associated with WS in Figure 6 is well within the 90% credibility bounds of β_{10} depicted in Figure 7.

- Figure 6C displays the posterior distribution of the relative probability $P(\underline{X}^1, \underline{X}^2 \mid \underline{\beta})$ associated with Figure 3.

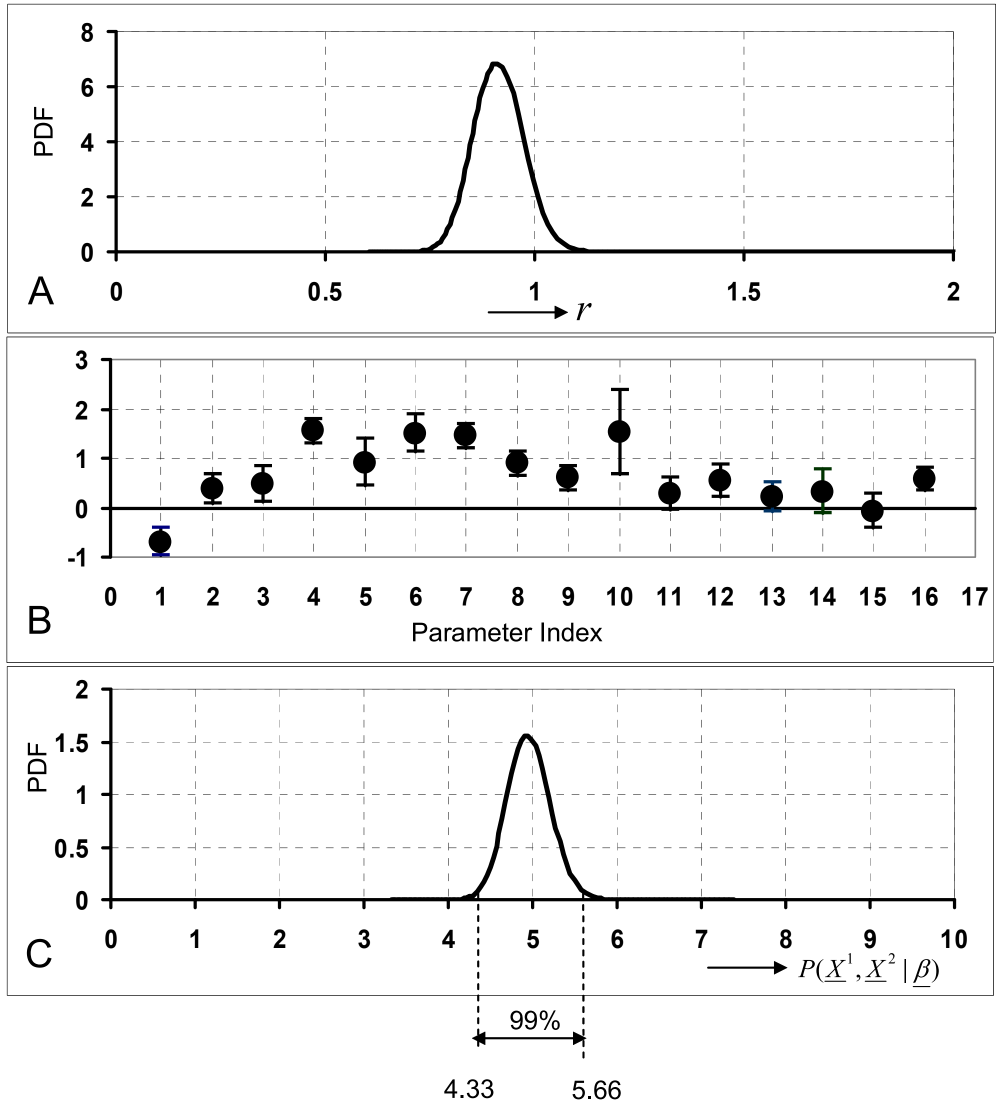

Figure 7. Posterior on $(\underline{\beta}, r)$ and $P(\underline{X}^1, \underline{X}^2 \mid \underline{\beta})$ of question in Figure 3 (cf. (2)).

- Compare the **50% posterior credibility interval of $P(\underline{X}^1, \underline{X}^2 | \underline{\beta})$ of $[4.78, 5.13]$** to the **50% prior one of $[0.18, 5.52]$** in Figure 4C. In addition, the **99% posterior credibility interval of $[4.33, 5.66]$** is indicated in Figure 6C, which is remarkably narrow compared to the prior **75% credibility interval of $[2.012 \cdot 10^{-5},\ 4.971 \cdot 10^{4}]$**

- **The median point estimate of $P(\underline{X}^1, \underline{X}^2 | \underline{\beta})$** equals **4.94.** Hence, Situation 2 in Figure 3 is approximately 5 times more likely to result in a collision than Situation 1 given that a navigation equipment failure occurred on the ferry.

- Utilizing **posterior distributional results for the parameter vector $\underline{\beta}$** credibility statements can be made for any arbitrary paired comparison. For example, setting **Situation 1** in (2) to the **best possible scenario ($\underline{X}^1 = \underline{0}$)** and **Situation 2** to the **worst possible scenario ($\underline{X}^2 = \underline{1}$)** a **99% credibility interval of $P(\underline{X}^1, \underline{X}^2 | \underline{\beta})$** equals **$[31142, 36749]$.** Therefore, collision risk in the worst possible scenario differs at least by **4 orders of magnitude** to that of the best possible scenario **while taking uncertainty** of the expert judgments **into account.**

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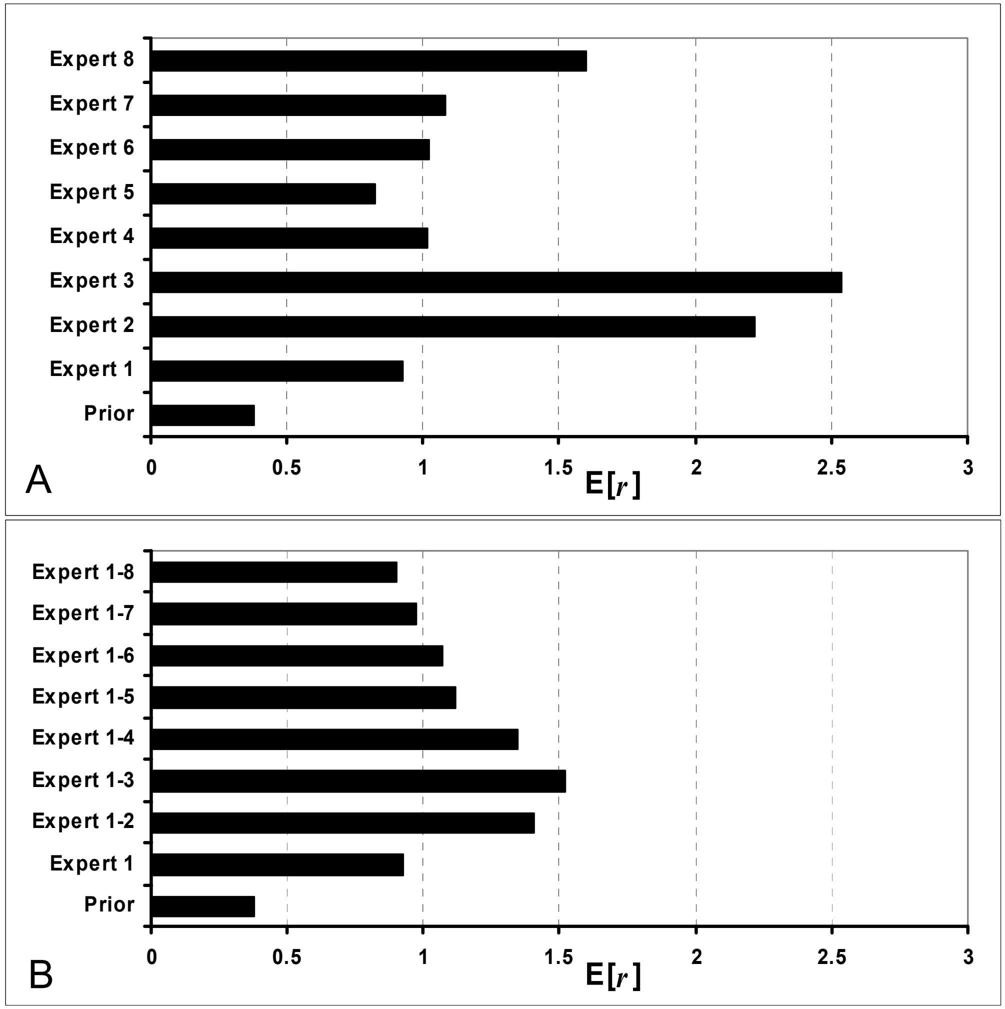

Figure 8. Prior and Posterior points estimates of the precision r (cf. (4) and (13))
A: Individual posterior estimates for Experts i, $i = 1, \ldots, 8$;
B: Sequential Posterior estimates involving Experts 1 through i, $i = 1, \ldots, 8$.

7. CONCLUDING REMARKS

- **Bayesian aggregation method** has been developed using responses from multiple experts to a **paired comparison questionnaire** to assess the distribution of **relative accident probabilities.** **The classical analysis** conducted during the WSF risk assessment **only resulted in point estimates** of relative accident probabilities.

- **Worst case scenario's** however may have a very **low incidence of occurrence**, which is why **all conditional probabilities** in Figure 1 and their uncertainties **need to be estimated** to assess the distribution of collision risk on for example **a per year basis.** This paper **only** provided distributional results for the relative probability $P(\underline{X}^1, \underline{X}^2 | \underline{\beta})$. Merrick et al (2003) assesses **the distribution of $Pr(OF, SF)$** using **Bayesian Simulation techniques.** A subsequent paper will integrate the approach herein with that of Merrick et al (2003) to assess collision risk and its uncertainty in a Bayesian manner.

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