JSPI 2241 pp: 1-20 (col.fig.: Nil) PROD. TYPE: TYP ED: HR PAGN: Indu – SCAN: Ravi

ARTICLE IN PRESS

ELSEVIER

Journal of Statistical Planning and Inference ▮▮▮ (▮▮▮▮) ▮▮▮–▮▮▮

journal of statistical planning and inference

www.elsevier.com/locate/jspi

A general Bayes exponential inference model for accelerated life testing

J. René Van Dorp*, Thomas A. Mazzuchi

Department of Engineering Management and Systems Engineering, School of Engineering and Applied Science, The George Washington University, 1776 G Street, N.W., Suite 110, Washington, DC 20052, USA

Received 6 January 2002; received in revised form 12 August 2002

Abstract

This article develops a general Bayes inference model for accelerated life testing assuming failure times at each stress level are exponentially distributed. Using the approach, Bayes point estimates as well as probability statements for use-stress life parameters may be inferred from the following testing scenarios: regular life testing, fixed-stress testing, step-stress testing, profile-stress testing, and also mixtures thereof. The inference procedure uses the well known Markov chain Monte Carlo (MCMC) methods to derive posterior quantities and accommodates both the interval data sampling strategy and type I censored sampling strategy for the collection of ALT test data. The approach is illustrated with an example.

Keywords: Ordered Dirichlet; Markov chain Monte Carlo

1. Introduction

In the case of highly reliable items, mean times to failure (MTTF) exceeding a year is not uncommon, see e.g. Fornell (1991). The use of these items, however, may still require reliability demonstration or verification testing, especially when used for military or high-risk public applications. With such MTTFs, it is often both time consuming and costly to practically test these items in their use (or nominal) environment due to the length of time required to generate a meaningful number of failures for analysis. It has therefore become a standard procedure in MIL-STD-781C (1977) to test these items under more severe environments than experienced in actual use. Such tests, referred to

* Corresponding author. Tel.: +1-202-9946638; fax: +1-202-9940245.
E-mail address: dorpjr@seas.gwu.edu (J. René Van Dorp).

0378-3758/02/$ - see front matter
PII: S0378-3758(02)00411-1

as accelerated life tests (ALTs), are becoming more frequent than ordinary life tests and thus the design and analysis of such tests are important problems.

The design of an ALT deals with issues of increasing the rate of failure of high reliability items in an optimal and meaningful manner. As such, ALT design involves the selection of independent variables (called "stress variable") such as temperature, vibration, humidity, voltage, etc., which define the operating environment, and the determination of optimal testing levels for these variables to produce an environment which accelerates failure in a meaningful way (see, e.g. Chaloner and Larntz, 1992; Khamis and Higgins, 1996; Khamis, 1997; Erkanli and Soyer, 2000). There are several common test scenarios considered in the design of life-tests, including testing in a constant environment (e.g. regular lifetime testing or fixed-stress ALT), a continuously changing environment (continuous-stress ALT), or an environment that changes in a step-like pattern (step-stress ALT or profile ALT). The focus of this paper, however, is on the statistical inference problem, i.e. on how to make inference about the reliability in the use environment from failure data obtained from a prescribed ALT.

There is a host of literature on the subject of ALT inference. Most of the ALT inference methods to date are based on the use of maximum likelihood estimation which may require large sample sizes for meaningful statistical ALT inference, see e.g. Nelson (1980), Lin and Fei (1991), Tyoskin and Krivolapov (1996), Bai et al. (1997), Khamis and Higgins (1996), Khamis (1997), Meeker and Esocbar (1998) and Gouno (2001). The ALT inference problem, however, typically deals with smaller sample sizes (see e.g. MClinn, 1998) which is suitable for a Bayes approach, see e.g. DeGroot and Goel (1988), Mazzuchi and Singpurwalla (1988), Mazzuchi and Soyer (1992), Van Dorp et al. (1996) and Mazzuchi et al. (1997). Typically, inference for ALT methods have been developed assuming that: (i) only a single test scenario is considered for all test items, (ii) the scale parameter of the life distribution is related to the stress environment via a pre-specified parametric function known as a time transformation function, (iii) the lifetime distribution in a constant stress environment belongs to a common parametric family of distributions.

The ALT inference procedure to be developed in this paper is a comprehensive procedure allowing variation of ALT and/or regular life testing scenarios between test items, a common practice amongst reliability engineers (see e.g., Meeker and Hahn, 1978; Nelson, 1980; Luvalle and Hines, 1992; Thomas and Gaines, 1978). In addition, the inference procedure allows for the combination of regular life testing data as well as most of the common testing scenarios used in reliability engineering. The development of such a flexible inference procedure is new and provides greater flexibility in both design and analysis of ALTs. Also, the inference procedure allows for comparative analysis from one ALT testing scenario to another from an inference point of view (e.g. fixed-stress testing versus step-stress testing) within a common modeling framework. To date the authors are not aware of such a common modeling framework.

The ALT inference procedure presented here is Bayesian in nature, allowing small sample sizes and relying on the use of engineering judgment to specify prior distributions used in the inference. With regards to the third assumption, inference procedures will be developed using the exponential failure time model (see, e.g. Cohen et al., 1999). The exponential failure time model can be found amongst several application in an ALT setting but particularly for electronic components (see e.g. Denson, 1995; Gouno, 2001). With regard to the second assumption, the ALT inference procedure is free from the restriction of the use of a parametric time transformation functions. Instead, it is assumed that the testing environments can be rank ordered with respect to severity (a less restrictive assumption) thus implying an ordering of the failure rates in the testing environments. Preserving the ordering in the analysis by defining a multivariate prior distribution for the failure rates over an *ordered region*, may be loosely interpreted as a non-parametric time transformation function. This rank ordering induces positive dependence between the failure rates within each testing environment and thus determines how test data obtained in an accelerated environment affects the failure rate in the use stress environment.

Closed form expressions for the multivariate posterior distribution obtained within the Bayesian paradigm by updating the prior with ALT data cannot be obtained. Until recently the lack of closed form expressions for posterior distributions severely restricted the use of the Bayesian paradigm in complex problem settings. However, the advancement of the well known Markov chain Monte Carlo (MCMC) approach by Casella and George (1992) (which does not require closed form expressions for the posterior distribution) has resulted in the widespread use of the Bayesian paradigm (see e.g., Gilks et al., 1996). The resulting approximation of the multivariate posterior distribution of the ordered failure rates can be used for inference with respect test environments.

The likelihood of ALT data allowing variation of ALT scenario's amongst test items is formulated in Section 2. The prior distribution preserving rank ordering of the failure rates for the exponential failure time model is motivated and developed in Section 3. Posterior analysis using the MCMC methods is briefly discussed in Section 4. Section 5 presents a comprehensive example to illustrate the approach. Finally, Section 6 provides some concluding remarks.

2. A general likelihood function for accelerated life tests

Any statistical inference procedure involves developing a likelihood. The flexibility of the likelihood formulation drives the flexibility of the statistical inference procedure in terms of its applicability to different testing scenarios. The likelihood model developed in this section allows for a comprehensive representation and combination of regular fixed-stress, progressive step-stress, regressive step-stress and profile step-stress life testing (see for example Fig. 1). In addition, the likelihood model accounts for the possibility of gradual environment changes (ramping) between steps (see, e.g., Bai et al., 1997). Thus, the likelihood model accommodates continuously stress ALT as well.

Typically in ALT, an environment can be described by a collection of stress variables and their levels. In this model, it is assumed that a total of K environments $E_1, \ldots, E_K$ are preselected as candidate test environments within *minimum* and *maximum* design ranges of stresses to ensure that the predominant failure modes at use environments and accelerated environments are the same. It will be assumed that these candidate test

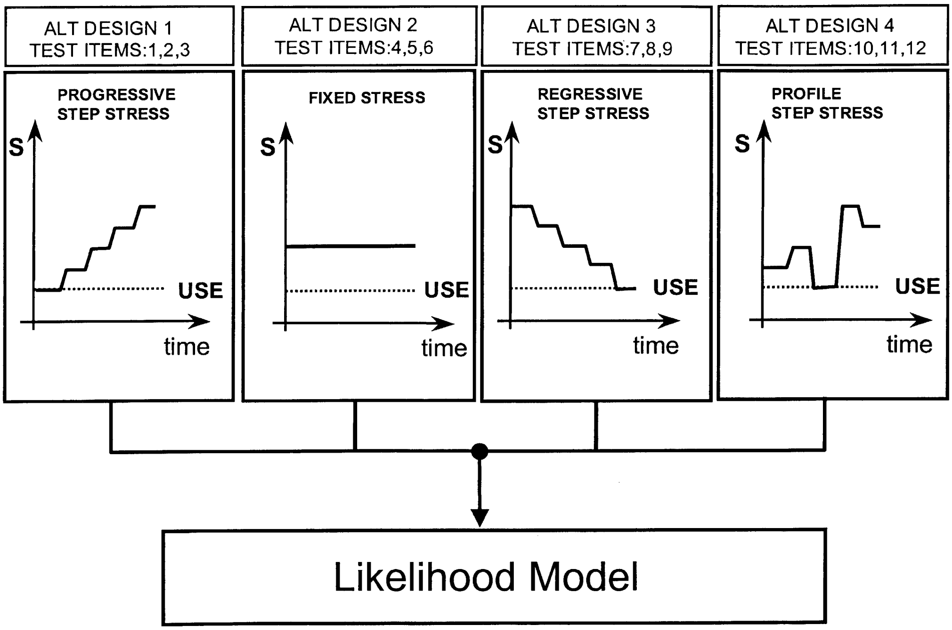

Fig. 1. Allowing each test items to follow a separate ALT scenario.

environments may be rank ordered with respect to severity using engineering knowledge, i.e. $E_1(E_K)$ coincides with the least (most) severe environment.

The lifetime distribution in a constant stress environment E_e will be modeled as an exponential life time distribution with failure rate λ_e, $e = 1, \ldots, K$. The index of the use environment or nominal environment will be denoted by ε and, typically, $\varepsilon = 1$. However, $\varepsilon > 1$ will be allowed as well, in case lower stress environments than use stress are used in the ALT. The ordering of the test environments in terms of severity induces the same ordering in the associated failure rates, i.e.

$$0 \equiv \lambda_0 < \lambda_1 < \cdots < \lambda_K < \lambda_{K+1} \equiv \infty. \tag{1}$$

Let a total of N test items be available for testing and let each test item j be subjected to an ALT with m_j test intervals $[t_{i-1,j}, t_{i,j})$, $i = 1, \ldots, m_j$, with $t_{0,j} \equiv 0$, $t_{m_j+1,j} \equiv \infty$ for all j. If item j has not failed by time $t_{m_j,j}$, it is removed (censored) from testing. As testing may proceed in a variety of step patterns, the actual test environment in $[t_{i-1,j}, t_{i,j})$ will be denoted by $E_{a_{i,j}}$, $a_{i,j} \in \{1, \ldots, K\}$. The change from one environment to the next may not be instantaneous but gradual (as is the case with temperature). The amount of time to change gradually from one environment to the next in $[t_{i-1,j}, t_{i,j})$, referred to as ramp-time, will be denoted by $\rho_{i,j}$.

With the above setup, and the assumption that the failure rate over the ramp period may be approximated by a linear function, the failure rate of test item j, $j = 1, \ldots, N$, over the course of its ALT follows as:

$$h_j(t\,|\,\underline{\lambda}) = \begin{cases} \lambda_{a_{i,j}}, & 0 \leqslant t < t_{1,j}, \\ \frac{\lambda_{a_{i,j}} - \lambda_{a_{i-1,j}}}{\rho_{i,j}}(t - t_{i-1,j}) & \\ \quad + \lambda_{a_{i-1,j}}, & t_{i-1,j} \leqslant t < t_{i-1,j} + \rho_{i,j},\ \ i = 2, \ldots, m_j, \\ \lambda_{a_{i,j}}, & t_{i-1,j} + \rho_{i,j} \leqslant t < t_{i,j},\ \ i = 2, \ldots, m_j, \end{cases} \tag{2}$$

where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_K)$. Example profiles of the failure function given (2) are provided in Fig. 1. The reliability and failure time distribution of test item j over the course of its ALT can be derived from the failure rate given by (2) using well known reliability expressions. Specifically, letting the life length of test item j over the course of its ALT be denoted by T_j, it follows that:

$$\phi(i, j, t\,|\,\underline{\lambda}) = \Pr(T_j \geqslant t\,|\,T_j \geqslant t_{i-1,j}\}$$
$$= \begin{cases} \exp\{-\lambda_{a_{i-1,j}}(t - t_{i-1,j}) & \\ \quad -\frac{(t - t_{i-1,j})^2}{2\rho_{i,j}}(\lambda_{a_{i,j}} - \lambda_{a_{i-1,j}})\}, & t_{i-1,j} < t \leqslant t_{i-1,j} + \rho_{i,j}, \\ \exp\{-\frac{1}{2}\lambda_{a_{i-1,j}}\rho_{i,j} - \lambda_{a_{i,j}} & \\ \quad (t - t_{i-1,j} - \frac{1}{2}\rho_{i,j})\}, & t_{i-1,j} + \rho_{i,j} < t \leqslant t_{i,j}, \end{cases} \tag{3}$$

where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_K)$ and $i = 1, \ldots, m_j$.

Introducing the transformation

$$\lambda_e = -\frac{\mathrm{Ln}(u_e)}{c}, \tag{4}$$

where $c > 0$ is a preset transformation constant, (2) and (3) may be expressed in terms of u_e for mathematical convenience (see Section 3). Substituting (4) into (2) and (3), yields

$$h_j(t\,|\,\underline{u}) = \begin{cases} -\frac{\mathrm{Ln}(u_{a_{i,j}})}{c}, & t_{i-1,j} \leqslant t < t_{i,j},\ \ i = 1, \\ \frac{\mathrm{Ln}(u_{a_{i-1,j}}) - \mathrm{Ln}(u_{a_{i,j}})}{c\rho_{i,j}} & \\ \quad \times (t - t_{i-1,j}) - \frac{\mathrm{Ln}(u_{a_{i-1,j}})}{c}, & t_{i-1,j} \leqslant t < t_{i-1,j} + \rho_{i,j},\ \ i = 2, \ldots, m_j, \\ -\frac{\mathrm{Ln}(u_{a_{i,j}})}{c}, & t_{i-1,j} + \rho_{i,j} \leqslant t < t_{i,j},\ \ i = 2, \ldots, m_j, \end{cases} \tag{5}$$

and

$$\phi(i,j,t\,|\,\underline{u}) = \begin{cases} (u_{a_{i-1,j}})^{-(t-t_{i-1,j})^2+2\rho_{i,j}(t-t_{i-1,j})/2c\rho_{i,j}} \\ \quad \times (u_{a_{i,j}})^{(t-t_{i-1,j})^2/2c\rho_{i,j}}, & t_{i-1,j} < t \leqslant t_{i-1,j}+\rho_{i,j}, \\ (u_{a_{i-1,j}})^{\frac{1}{2}\rho_{i,j}/c}(u_{a_{i,j}})^{(t-t_{i-1,j}-\frac{1}{2}\rho_{i,j})/c}, & t_{i-1,j}+\rho_{i,j} < t \leqslant t_{i,j}, \end{cases} \tag{6}$$

where $\underline{u}=(u_1,\ldots,u_K)$ and u_e is given by (4). Expressions (5) and (6) will be used in the derivation of the likelihood for interval ALT data and type I censored ALT data.

For flexibility of ALT likelihood formulation, the likelihood will be formulated as a product over the environment index e rather than a product over the interval index i. To accomplish such a formulation, let

$$n_{e,j} = \text{the number of times that test item } j \text{ visits environment } E_e, \tag{7}$$

and

$$v_{e,j}^k = \text{interval index for which item } j \text{ visits } E_e \text{ for the } k\text{th time}. \tag{8}$$

2.1. Interval data

Assuming that the failure of test items can only be monitored at the end of a step-interval, the probability of test item j failing in ALT step-interval q_j equals

$$\prod_{e=1}^{K}\prod_{k=1}^{n_{e,j}} f(e,j,k\,|\,\underline{u},\underline{q}), \tag{9}$$

where

$$f(e,j,k\,|\,\underline{u},\underline{q}) = \begin{cases} \phi(v_{e,j}^k, j, t_{v_{e,j}^k,j}\,|\,\underline{u}), & v_{e,j}^k < q_j, \\ 1-\phi(v_{e,j}^k, j, t_{v_{e,j}^k,j}\,|\,\underline{u}), & v_{e,j}^k = q_j, \end{cases} \tag{10}$$

where $\underline{q}=(q_1,\ldots,q_N)$, $\phi(\cdot\,|\,\underline{u})$ is given by (6) and the following conventions; $\prod_{k=1}^{0}\{\cdot\}\equiv 1$; and $q_j=m_j+1$, if the test item is censored at t_{m_j}. With (6), (9), (10) and assuming conditional independence between the failure times of the test items conditioned on knowing $\underline{u}$, it follows that the likelihood given interval data $(N,\underline{q})$ equals

$$\mathscr{L}\{\underline{u};(N,\underline{q})\} = \prod_{j=1}^{N}\prod_{e=1}^{K}\prod_{k=1}^{n_{e,j}} f(e,j,k\,|\,\underline{u},\underline{q}), \tag{11}$$

where $n_{e,j}$ is defined by (7).

2.2. Type I censored data

The interval data sampling strategy has the disadvantage that failure time information is obscured by only using the interval number of the failure rather than the exact failure time. In the type I censored sampling strategy, test items are continuously monitored over the course of the ALT and thus this strategy does not suffer from that disadvantage. In the case of type I censored data, the failure time r_j of test item j is known exactly if the test item fails in $[0, t_{m_j})$. Knowing the failure times r_j, the step intervals q_j in which the items failed may be determined. Using a similar approach as in (11), the likelihood given the data $(N, \underline{r}, \underline{q})$, where $\underline{r} = (r_1, \ldots, r_N)$, $\underline{q} = (q_1, \ldots, q_N)$ follows as:

$$\mathscr{L}\{\underline{u}; (N, \underline{r}, \underline{q})\} = \prod_{e=1}^{K} \prod_{j=1}^{N} \prod_{k=1}^{n_{e,j}} g(e, j, k \mid \underline{u}, \underline{r}, \underline{q}), \tag{12}$$

where

$$g(e, j, k \mid \underline{u}, \underline{r}, \underline{q}) = \begin{cases} \phi(v_{e,j}^k, j, t_{v_{e,j}^k, j} \mid \underline{u}), & v_{e,j}^k < q_j, \\ h_j(r_j \mid \underline{u}) \phi(v_{e,j}^k, j, r_j \mid \underline{u}), & v_{e,j}^k = q_j, \end{cases} \tag{13}$$

$h_j(\cdot \mid \underline{u}), \phi(\cdot \mid \underline{u})$ are given by (5) and (6), respectively, and $n_{e,j}$, $v_{e,j}^k$ are defined by (7) and (8), respectively.

3. Prior distribution

Using the inverse transformation of (4), i.e.

$$u_e = \exp(-c\lambda_e), \tag{14}$$

where $c > 0$ is a present transformation constant and (1) it follows that:

$$0 \equiv u_{K+1} < u_K < \cdots < u_1 < u_0 \equiv 1. \tag{15}$$

The motivation and a method for selecting the preset transformation constant c is discussed in Section 3.2. Rather than defining a prior distribution for $\underline{\lambda}$ exhibiting property (1), one may equivalently define a prior for $\underline{u} = (u_1, \ldots, u_K)$ exhibiting property (15). Concentrating on $\underline{u} = (u_1, \ldots, u_K)$, a prior distribution which is: (i) mathematically tractable, (ii) is defined over the region specified in (15), and (iii) imposes *no other restrictions* on $\underline{u}$, is the multivariate ordered Dirichlet distribution

$$\Pi\{\underline{u} \mid \eta, \underline{v}\} = \frac{\prod_{e=1}^{K+1} (u_{e-1} - u_e)^{\eta v_e - 1}}{\mathbb{D}(\eta, \underline{v})}, \tag{16}$$

where, $\eta > 0$, $v_e > 0$, $e = 1, \ldots, K+1$, and

$$\mathbb{D}(\eta, \underline{v}) = \frac{\prod_{e=1}^{K+1} \Gamma(\eta v_e)}{\Gamma(\eta)}, \tag{17}$$

where $\sum_{e=1}^{K+1} v_e = 1$. Using the transformation given by (4) allows one to take full advantage of the mathematical properties of the ordered Dirichlet distribution. For example, from (16) the correlation between u_e and u_f, $1 \leqslant e \leqslant f \leqslant K$ may be derived as

$$\varrho_{ef} = \mathrm{Cor}(u_e, u_f) = \sqrt{\frac{v_{e}\bullet(1 - v_{f}\bullet)}{(1 - v_{e}\bullet)v_{f}\bullet}}, \tag{18}$$

where $v_{e}\bullet = \sum_{j=1}^{e} v_j$. Expression (18) may be interpreted as a measure of positive dependence between the failure behavior over the various stress environments. In addition from (16), the prior marginal distribution for any u_e is obtained as a beta distribution given by

$$\Pi\{u_e\} = \frac{(u_e)^{\eta(1-v_{e}\bullet)-1}(1-u_e)^{\eta v_{e}\bullet - 1}}{\mathbb{B}(\eta(1-v_{e}\bullet), \eta v_{e}\bullet)}, \tag{19}$$

where $\mathbb{B}(\cdot,\cdot)$ is the well known beta constant. From (19) it may be derived that

$$E[u_e] = (1 - v_{e}\bullet) \tag{20}$$

and

$$\mathrm{Var}(u_e) = \frac{(1 - v_{e}\bullet)v_{e}\bullet}{\eta + 1}. \tag{21}$$

Hence, with (18), (20) and (21) it follows that the parameters v_e, $e = 1,\ldots,K+1$ determine location of u_e and the degree of positive dependence between the elements of $\underline{u}$ (and thus via (14) also between the failure rates in $\underline{\lambda}$), whereas the parameter η given the parameters v_e completely determines the variance in the marginal prior distributions given by (19). Both the prior variance specified in (21) and the prior correlations specified in (18) are indicative for the effect of data in accelerated environments on the failure behavior in the use stress environment. As such, the multivariate prior distribution given by (16) together with its prior parameters specifies an *implicit* time transformation function, rather than an explicit time transformation function common to most ALT inference procedures to date.

3.1. Expert judgment for prior parameter specification

Typically, to define the prior parameters, expert judgment concerning quantities of interest are elicited and equated to their theoretical expression for central tendency such as mean, median, or mode, see e.g. Cooke (1991). In addition, some quantification of the quality of the expert judgment is often given by specifying a variance or a probability interval for the prior quantity. Solving these equations generally leads to the desired parameter estimates.

It is desirable, for the design of a meaningful elicitation procedure to engineers, that elicited information can be easily related to observables, see e.g. Chaloner and Duncan (1983). Mission time reliabilities can be related to observables by asking about the number of test items that survive the mission out of a fixed sample. The prior for u_e given by (19) can be used to make prior probability statements concerning mission time reliabilities at the different stress levels due to the one-to-one relationships of these quantities to u_e via the transformation defined by (4). Specifically,

$$R(\tau \mid u_e) = \Pr\{X > \tau \mid u_e\} = (u_e)^{\tau/c}, \tag{22}$$

where X is the lifetime of a test item exposed to environment E_e and $R(\tau \mid u_e)$ denotes the respective reliability for a pre-specified mission time τ.

An advantage of eliciting the median of mission time reliabilities instead of the mean or the mode is that it allows for the use of betting strategies in an *indirect* elicitation procedure, see e.g. Cooke (1991). The same holds for eliciting a lower or upper quantile of a mission time reliability instead of the variance to obtain a measure of variability. For these reasons it is assumed that estimates on median mission time reliabilities at the different stress levels $R^{\bullet}_1, \ldots, R^{\bullet}_K$ and a lower quantile on mission time reliability at use stress R^q_ε can be elicited through engineering judgment procedures for a preset mission time τ. The engineering judgment in terms of mission time reliabilities needs to be elicited from design, reliability and testing engineers, and requires the development of a structured elicitation procedure. Such procedures (see, e.g., Cooke et al., 1991) involves the careful design of a questionnaire and feedback to reduce elicitation bias. The design of the elicitation procedure is not a topic in this paper.

With (22), the prior marginal distribution of u_e given by (19), setting elicited values $R^{\bullet}_1, \ldots, R^{\bullet}_K$ equal to the medians of the mission time reliabilities and setting the elicited value R^q_ε equal to the qth quantile (e.g. the 0.05th quantile) of the mission time reliability at use stress, it follows that problem $\mathscr{P}$ below needs to be solved to specify the prior parameters $\Theta = (\eta, \underline{v})$. The method to solve problem $\mathscr{P}$ will be described in Section 3.3.

Problem P. Solve $\Theta = (\eta, \underline{v})$ from

1. $\Pr\{R(\tau \mid u_\varepsilon) \leqslant R^{\bullet}_\varepsilon \mid \Theta\} = 0.50$;
2. $\Pr\{R(\tau \mid u_\varepsilon) \leqslant R^q_\varepsilon \mid \Theta\} = q$;
3. $\Pr\{R(\tau \mid u_e) \leqslant R^{\bullet}_e \mid \Theta\} = 0.50,\ e = 1, \ldots, K,\ e \neq \varepsilon,\ 0 < q < 0.50$.

3.2. Motivation and selection of the transformation factor

With a preset transformation factor c and the medians of the mission time reliabilities $R^{\bullet}_1, \ldots, R^{\bullet}_K$, one may solve for medians $u^{\bullet}_1, \ldots, u^{\bullet}_K$ of the associated marginal prior distributions given by (19) utilizing (22). The motivation for the transformation parameter c, discussed below, follows from: (i) the value of the predictive medians $u^{\bullet}_e$, (ii) the marginal distributions of u_e given by (19), and (iii) the fact that no closed form expression is available for the incomplete beta function $B : [0,1] \to [0,1]$ given by

$$B(u \mid a, b) = \frac{1}{\mathbb{B}(a,b)} \int_0^u x^{a-1}(1-x)^{b-1}\,\mathrm{d}x \tag{23}$$

to be used for evaluations of the cdf associated with (19). Numerical algorithms exist to approximate the incomplete beta function given by (22), see e.g., Press et al. (1989). These approximations are well behaved for parameter values $a \geqslant 1,\ b \geqslant 1$. However, in case $a < 1$ ($b < 1$), the beta density *explodes* at $x = 0$ ($x = 1$), resulting in numerical instability for the approximation of $B(u \mid a, b)$ for values close to $u = 0$ ($u = 1$). To achieve maximum numerical stability in the analysis it suggested to solve for the transformation factor c in (22) by setting

$$(R_K^{\bullet})^{c/\tau} = 1 - (R_1^{\bullet})^{c/\tau}, \tag{24}$$

where τ is the mission time under consideration. The suggestion in (24) is to select c such that $u_K^{\bullet} = 1 - u_1^{\bullet}$ and that the $B(u\,|\,a,b)$ (see (23)) associated with $u_K^{\bullet}(u_1^{\bullet})$ is well behaved at $u = 0$ ($u = 1$). General root finding algorithms need be used to solve for c in (24). The analysis present herein is robust with respect to perturbations from the solution c given by (24), provided numerically stable analysis results. For example, deviations in the order of 10% from the solution c given by (24) did not affect the analysis results.

3.3. Solving for prior parameters

Using the preset mission time τ and having solved for the transformation constant c utilizing (24), the procedure to solve for prior parameters $\eta > 0$, $v_e > 0$, $e=1,\ldots,K+1$, using problem definition $\mathscr{P}$, can be organized in the following steps:

Step 1: transform medians $R_e^{\bullet}$ into medians $u_e^{\bullet} = (R_e^{\bullet})^{c/\tau}$.

Step 2: transform lower quantile R_ε^q at use stress into lower quantile $u_\varepsilon^q = (R_\varepsilon^q)^{c/\tau}$.

Step 3: solve for the prior parameters η and $v_{\varepsilon^{\bullet}}$ of (19), where $v_{\varepsilon^{\bullet}} = \sum_{j=1}^{\varepsilon} v_j$.

Step 4: solve for the prior parameters $v_{e^{\bullet}}$, $e = 1,\ldots,K$, $e \neq \varepsilon$, where $v_{e^{\bullet}} = \sum_{j=1}^{e} v_j$.

Step 5: solve for v_e utilizing $\sum_{j=1}^{K+1} v_j = 1$, yielding

$$\begin{cases} v_e = v_{e^{\bullet}}, & e = 1, \\ v_e = v_{e^{\bullet}} - v_{(e-1)^{\bullet}}, & e = 2,\ldots,K, \\ v_e = 1 - v_{e^{\bullet}}, & e = K+1. \end{cases} \tag{25}$$

Step 4 and Step 5 required solving for the parameters of the beta distribution (19) under two quantile constraints. Van Dorp and Mazzuchi (2000) showed that parameters of a beta distribution can be solved for under *any* lower and upper quantile constraint and developed a numerical procedure to solve for these parameters. The method in Van Dorp and Mazzuchi (2000) solves for a unique set of parameters of a beta distribution that satisfies the two-quantile constraints while maximizing uncertainty in the underlying beta distribution. Given the uniqueness of the latter solution and a preset transformation factor c, a unique solution to problem $\mathscr{P}$ exists.

4. Posterior approximation

The posterior distribution of $\underline{u}$ follows for interval data (type I censored data) by applying Bayes theorem to the multivariate prior (16) and the likelihood expressions (11) and (12). The posterior distribution of u_ε may be obtained by integrating the multivariate posterior density function over the variables u_e, $e \neq \varepsilon$. Closed form analysis of the posterior distribution of u_ε is intractable but for the case of identical ALT testing scenario's for all test items and interval data (see Van Dorp et al., 1996). Hence, the need for the use of the well known MGMC approach for posterior analysis when ALT scenarios may vary per test item and when Type I censored data is available.

The MCMC approach samples from the multivariate posterior distribution of $\underline{u}$ by sampling successively from the marginal posterior full conditionals $\Pi\{u_e \,|\, \underline{u}_{-e}, \mathscr{D}, \eta, \underline{v}\}$, where $\mathscr{D}$ represents data and

$$\underline{u}_{-e} = (u_1, \ldots, u_{e-1}, u_{e+1}, \ldots, u_K). \tag{26}$$

From the sample, approximations of the posterior distribution of $\underline{u}$, its marginal posterior distribution u_e (and thus $R(\tau \,|\, u_e)$) and relevant moments may be obtained. For an extensive discussion of the MCMC approach we refer to Casella and George (1992).

In case the marginal posterior full conditional $\Pi\{u_e \,|\, \underline{u}_{-e}, \mathscr{D}, \eta, \underline{v}\}$ is known in closed form, standard sampling methods may be used. When $\Pi\{u_e \,|\, \underline{u}_{-e}, \mathscr{D}, \eta, \underline{v}\}$ is only known up to a proportionality constant, the well known rejection sampling technique developed by Smith and Gelfand (1992) or the Metropolis Hastings Algorithm (see e.g. Robert and Casella, 1999) may be used to sample from the marginal posterior full conditional $\Pi\{u_e \,|\, \underline{u}_{-e}, \mathscr{D}, \eta, \underline{v}\}$. Although the Metropolis Hastings algorithm has shown to be more computationally efficient in general than the rejection sampling technique, the rejection sampling technique is simple and straightforward and through numerical analysis computational efficiency has not been shown to be an issue in this problem setting. The application of the rejection sampling technique requires sampling from the marginal prior full conditional $\Pi\{u_e \,|\, \underline{u}_{-e}, \eta, \underline{v}\}$ which can be obtained from the multivariate prior distribution specified by (16) as

$$\Pi\{u_e \,|\, \underline{u}_{-e}, \eta, \underline{v}\} = \frac{(u_e - u_{e+1})^{\eta v_{e+1}-1}(u_{e-1} - u_e)^{\eta v_e - 1}}{\mathbb{B}(\eta v_{e+1}, \eta v_e)(u_{e-1} - u_{e+1})^{\eta v_{e+1} + \eta v_e - 1}}. \tag{27}$$

Expression (27) may be recognized as a transformed beta distribution on the interval (u_{e+1}, u_{e-1}). The following section presents a comprehensive example of the ALT inference procedure. Additional details regarding of the MCMC method as applied to the ALT inference procedure in this paper are provided in Van Dorp (1997).

The posterior inference for use stress analysis is a function of the uncertainty characteristics of the use stress prior information, the prior accelerated environment information and the resulting ALT data. An issue of concern to some is the sensitivity of the posterior distribution of use stress parameters to the prior distribution. Sensitivity of posterior distribution to uncertainty characteristics of the prior distribution have been studied extensively in a general settings (see, e.g. DeGroot, 1989) and specifically for the ALT scenario in Van Dorp et al. (1996) by Dietrich et al. (1997). Similar sensitivity to the prior distribution may be observed in this model. If there is little data, reliance on the prior distribution is the only means of performing the analysis, which emphases the need for prior elicitation with strict adherence to a scientific method (see, e.g. Cooke, 1991). On the other hand, if there is an abundance of data, the posterior distribution is dominated by the observed data.

5. Example

The test system in this fictitious example is a new design of an electronic system of a radar manufacturer. The environments in the 5-step ALT are combinations of voltage and temperature stress and are specified in Table 1. The mission time is preset

Table 1
Environments, prior failure rate and mission time reliability estimates

E_e	Temp (°F)	Volt (VDC)	Prior $R_e^{\bullet}$	$E[R_e]$	Mode (R_e)
1	100	10.0	0.95	0.831	1.000
2	125	13.0	0.90	0.777	1.000
3	160	15.0	0.56	0.545	1.000
4	200	17.0	0.24	0.315	0.999
5	250	19.0	0.02	0.123	0.982
$R_\varepsilon^{5\%}$ at use stress	0.26				

to 1000 h. Mission time reliability estimates and a 5% quantile on the mission time reliability at the use stress level are available for specification of the prior parameters and are also given in Table 1.

The transformation factor c can be solved utilizing (24) and the mission time $\tau = 1000$ and follows as $c = 841.6124$. The prior parameters $(\eta, \underline{v})$ may be solved from the data provided in Table 1 using the method described in Section 3.3. The resulting parameter values are: $\eta = 1.6656$; $v_1 = 0.1522$; $v_2 = 0.0481$; $v_3 = 0.2198$; $v_4 = 0.2168$; $v_5 = 0.2109$; $v_6 = 0.1522$. Measures of positive dependence between the failure behavior in the use stress environment $(e = 1)$ and the higher stress environments $(e > 1)$ may be calculated utilizing (18) and follow with (29) as $\varrho_{12} = 0.8466$; $\varrho_{13} = 0.4978$; $\varrho_{14} = 0.3199$; $\varrho_{15} = 0.1795$. From these correlations it may be concluded that level of positive dependence decreases when the difference in stress between two environments increases. Hence, although ALT data allows to infer the failure behavior in use stress, there is a point where ALT data from a higher stress level has little to no information with respect to the failure behavior at the use stress level (as the correlation decreases). This behavior is consistent with the initial assumption that the predominant failure modes need to remain relatively the same when stress level is increased. From a reliability engineering perspective this is unlikely when stress is increased beyond e.g. the *design envelope*.

To show the flexibility of the ALT inference procedure developed in this paper, consider the ALT scenarios for twelve test items in Fig. 1. The 12 test items are grouped in 4 testing groups of three test items per group. The different testing groups are exposed to, (i) a progressive step-stress ALT, (ii) a regular life test at a higher stress level than use stress, (iii) a regressive step-stress ALT and (iv) a profile step-stress ALT, respectively. The test plan for test item $j = 1, \ldots, 12$ is summarized by the environment levels $\underline{a}_j = (a_{1j}, \ldots, a_{5j})$, step-interval endpoints $\underline{t}_j = (t_{1j}, \ldots, t_{5j})$ and step interval ramp-times $\underline{\rho}_j = (p_{1j}, \ldots, \rho_{5j})$, where

$$\underline{a}_j = \begin{cases} (1 \quad 2 \quad 3 \quad 4 \quad 5) & j = 1, 2, 3, \\ (3 \quad 3 \quad 3 \quad 3 \quad 3) & j = 4, 5, 6, \\ (5 \quad 4 \quad 3 \quad 2 \quad 1) & j = 7, 8, 9, \\ (2 \quad 3 \quad 1 \quad 5 \quad 4) & j = 10, 11, 12, \end{cases}$$

Table 2
ALT test data in terms of $\underline{r}$, $\underline{q}$

Test item	1	2	3	4	5	6	7	8	9	10	11	12
r_j	3598	2153	716	2879	3600	3600	3596	3600	3592	2157	3591	1438
q_j	5	3	1	4	6	6	5	6	5	3	5	2

$$\underline{t}_j = (720 \quad 1440 \quad 2160 \quad 2880 \quad 3600), \quad j = 1, \ldots, 12,$$

$$\underline{\rho}_j = (0 \quad 6 \quad 6 \quad 6 \quad 6), \quad j = 1, \ldots, 12. \tag{28}$$

The entries in $\underline{t}_j$ and $\underline{\rho}_j$ are specified in hours. The test data obtained from the ALT design in Fig. 3 is summarized in Table 2.

Note that, $q_j = 6$ indicates that the test item survived its ALT. Hence, 3 out of 12 of the test items in Table 2 survived their ALT.

Using the MCMC method described herein, a Gibbs sequence of length 100,000 was generated for the case of interval data in Table 2. Let $\underline{R}_i^{\mathrm{Seq}} = \{R_i^1, \ldots, R_i^{100,000}\}$ be such a sequence for candidate test environment i. An approximate i.i.d. posterior sample for candidate test environment i may be created by selecting a Gibbs burnin period M and a Gibbs lag L and thinning the Gibbs sequence $\underline{R}_i^{\mathrm{Seq}}$ for given M and L such that

$$\underline{R}_i^{\mathrm{Sample}} = \{R_i^j \in \underline{R}_{\mathrm{Seq}} \mid j = M + iL, i = 1, 2, 3, \ldots\}.$$

Cowles and Carlin (1996) give an overview of various diagnostic methods to detect convergence and burnin of the Gibbs sequence. We propose to select M and L based on a cusum path plot method introduced by Brooks (1998). By plotting a convergence statistic (referred to by Brooks (1998) as the *hairiness* statistic of the Gibbs sequence) against an upper control limit and a lower control limit in the usual cusum fashion (see e.g. Hawkins and David, 1998), the parameters M and L may be selected.

We first select the burnin period M by generating a cusum path plot of the convergence statistic for the raw Gibbs sequence $\underline{R}_1^{\mathrm{Seq}}$. A necessary condition for convergence of $\underline{R}_1^{\mathrm{Seq}}$ is that the convergence statistic settles down around a common value. Hence, with Fig. 2A the burnin period M for $\underline{R}_1^{\mathrm{Seq}}$ may be set to 500 iterations. Fig. 4B displays a cusum path plot of the convergence statistic for $\underline{R}_1^{\mathrm{Sample}}$ by setting $M = 500$ and $R = 1$. For an i.i.d. posterior sample the cusum path plot would have to fall within 95% normal confidence bounds. From Fig. 2B, it can be concluded that this is clearly not the case for a Gibbs lag $L = 1$, supporting the notion that $\underline{R}_1^{\mathrm{Sample}}$ is a dependent sample from the posterior distribution. We next successively thin $\underline{R}_1^{\mathrm{Seq}}$ for given $M = 500$ by increasing the value of L until we achieve a cusum path plot of the convergence statistic contained within 95% normal confidence bounds. Such a cusum path plot is achieved for $L = 50$ as displayed in Fig. 2C.

The resulting posterior sample size equals 1990 and the associated posterior sample $\underline{R}_1^{\mathrm{Sample}}$ of sample generated from the Gibbs sequence $\underline{R}_1^{\mathrm{Seq}}$ using a Gibbs burnin $M = 500$ and Gibbs lag $L = 50$ may be used for inference on the posterior mission time reliability at uses stress. Fig. 3A contains prior distributional results for the mission time

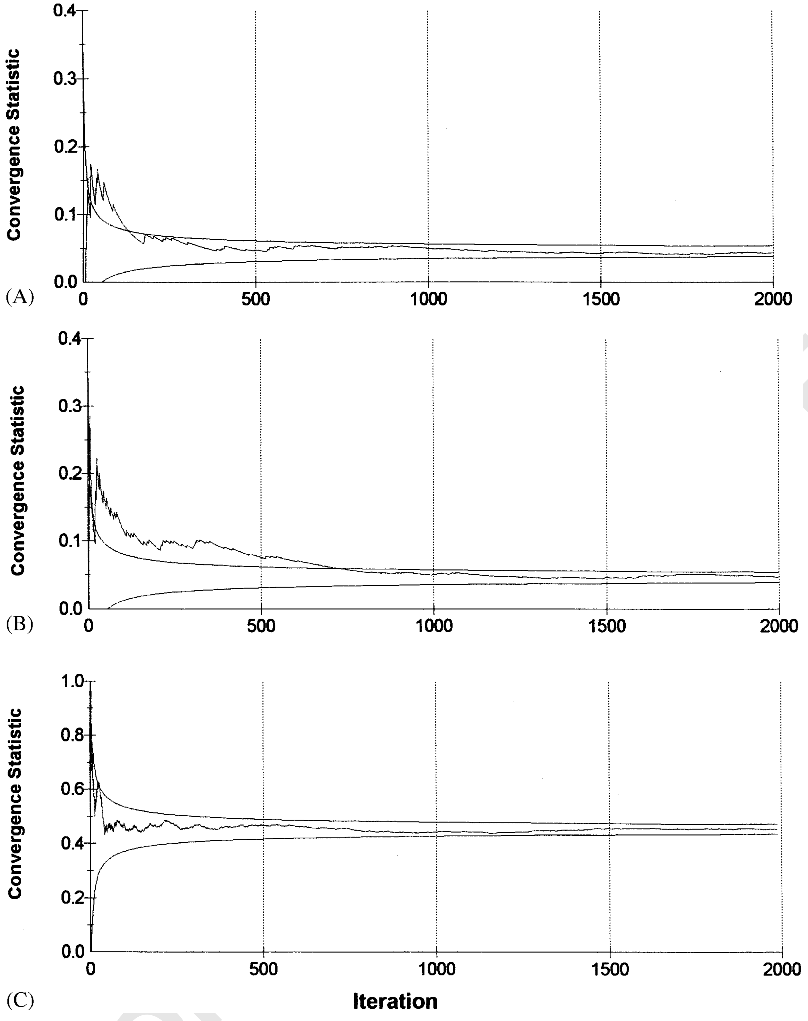

Fig. 2. (A) Cusum path plot of a convergence statistic for the raw Gibbs sequence $\underline{R}_1^{\text{Seq}}$; (B) Cusum path plot of a convergence statistic for $\underline{R}_i^{\text{Sample}}$: $M=500,\ L=1$; (C) Cusum path plot of a convergence statistic for $\underline{R}_i^{\text{Sample}}$: $M=500,\ L=50$.

reliability at use stress, Fig. 3B contains posterior distributional results for the mission time reliability at use stress using interval data and Fig. 3C contains posterior distributional results for the mission time reliability at use stress using type I censored data. Fig. 4 compares the cumulative distributions of mission time reliability at use stress using interval data and type I censored data.

From the interval data and type I censored data in Table 2 one observes that all failures have failed close to the end of a step interval. Comparing the posterior results

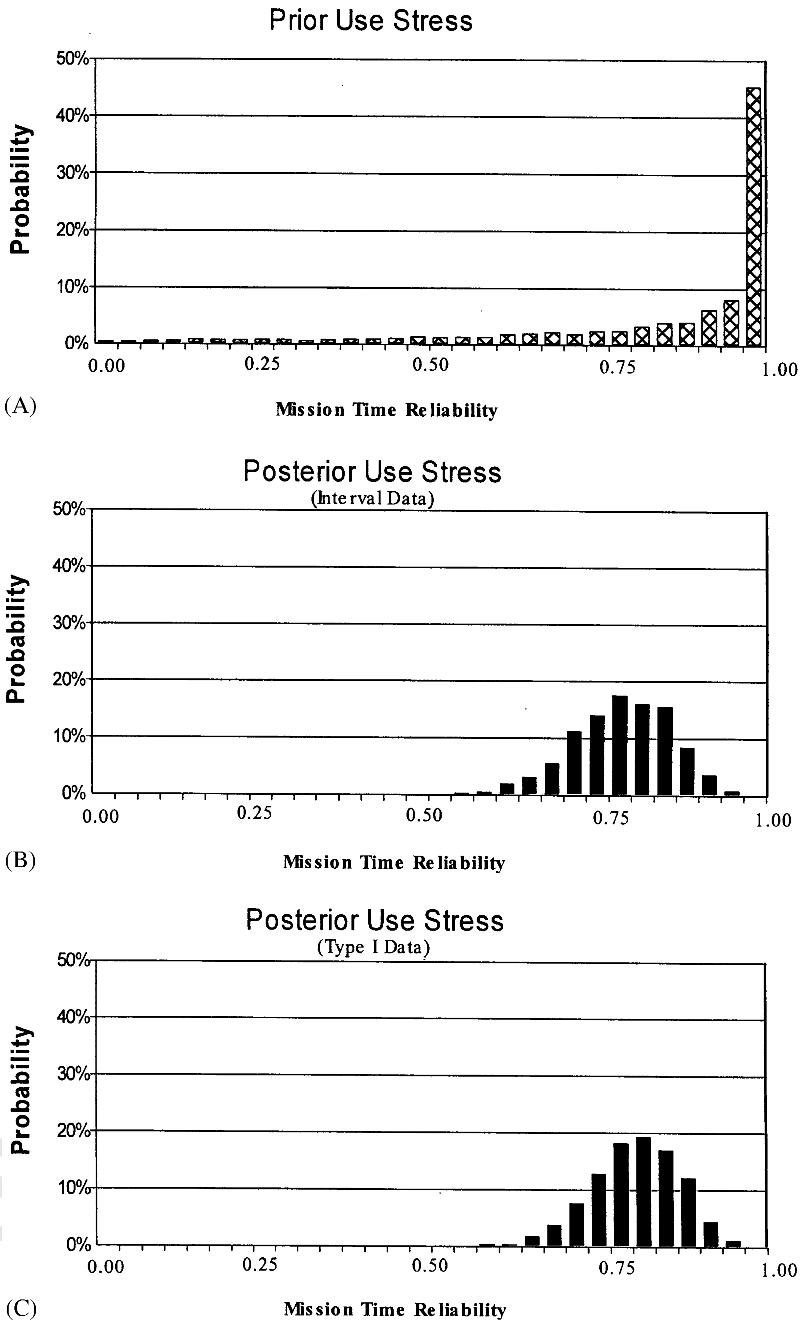

Fig. 3. Prior and posterior distribution for mission time reliability at use stress using interval data and type I censored data.

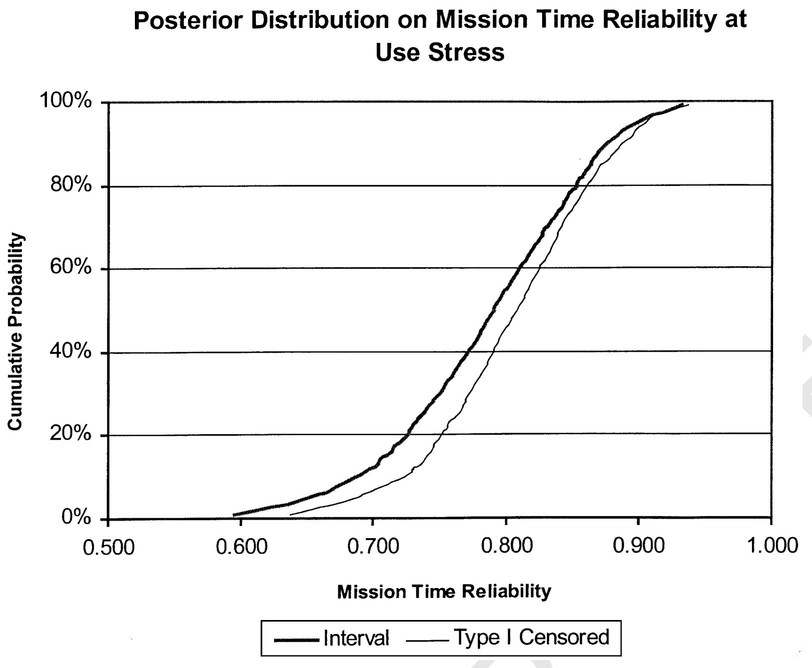

Fig. 4. Comparison of posterior cumulative distribution for mission time reliability at use stress using interval data or type I censored data.

Table 3
Comparison of posterior analysis using interval data and type I censored data

E_e	Gibbs Int. data, $R_e^{\bullet}$	Gibbs Type I data, $R_e^{\bullet}$	Gibbs Int. data, $E[R_e]$	Gibbs Type I data, $E[R_e]$	Gibbs Int. data, Mode (R_e)	Gibbs Type I data, Mode (R_e)
1	0.790	0.809	0.786	0.804	0.959	0.969
2	0.786	0.803	0.782	0.799	0.958	0.969
3	0.751	0.773	0.748	0.767	0.935	0.955
4	0.691	0.715	0.677	0.699	0.919	0.919
5	0.603	0.621	0.580	0.598	0.869	0.898

in Fig. 4 for interval data and type I censored data, it follows that the results are consistent with the above observation.

Indeed, the posterior mission time reliability is consistently lower with interval data compared to the results derived with type I censored data. The result in Fig. 4 indicates a more efficient use of available failure information in the case of type I censored data compared to interval data. The same conclusion is supported by Table 3 which contains: (1) posterior mission time reliability estimates (median, mean and mode)

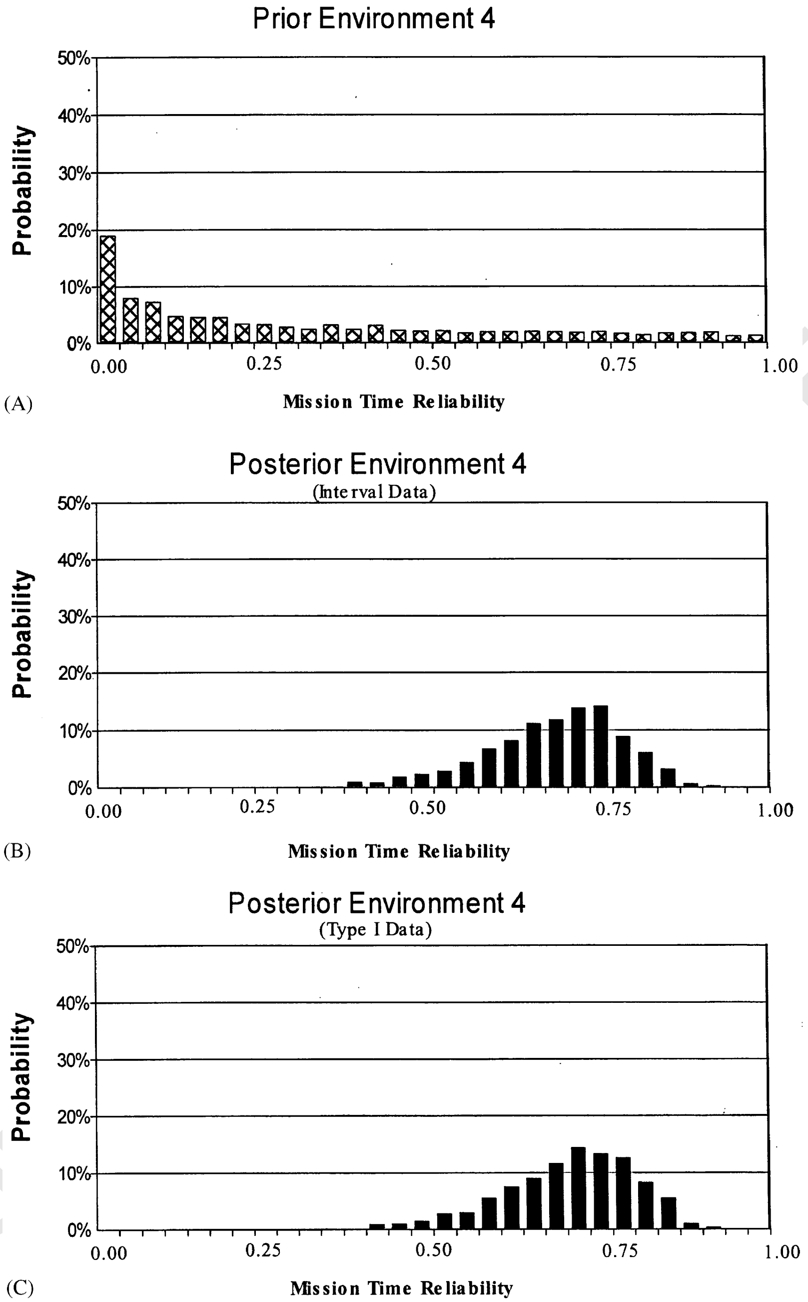

Fig. 5. Prior and posterior distribution for mission time reliability at environment 4 using interval data and type I censored data.

using interval data and (2) posterior mission time reliability estimates (median, mean and mode) using type I censored data for the five different stress environments.

Comparing the posterior results at use stress in Table 3 to the prior information in Table 1, it follows that the results indicate an over estimation of prior median mission reliability at use stress and candidate test environment 2. However, the results indicate an under estimation of prior median mission time reliability in candidate test environments 3–5. The same conclusion relative to use stress and candidate test environment 4 may be drawn from Fig. 3 and Figs. 5A–C which contain prior and posterior distributional results for the mission time reliability at candidate test environment 4 using interval data and type I censored data, respectively.

6. Conclusions

In this paper we developed a flexible Bayesian inference procedure for the analysis of accelerated life testing using the exponential failure model. The inference procedure covers a host of testing scenarios which are motivated by actual testing demands and constraints. More specifically, the technique is general to allow for different step patterns for different test items and mixtures of ALT scenarios between test items. Methods were derived to solve for the prior parameters using engineering judgment in terms of predictive mission time reliabilities. Providing these methods allows for a straightforward application of the ALT inference procedure in a practical and meaningful manner.

Acknowledgements

We are pleased to thank IIT Research institute for providing funding to perform the study in this paper. The research in this paper have been partially prepared under IITRI Purchase Order Number 1231964. The roots of the research in this paper were founded in previous research performed under USAF Contracts #F196092M5356 and #F1965092P3709. We are also indebted to the Executive Editor of JSPI and the referee for their valuable comments and suggestions which improved the presentation and content of the first version.

References

Bai, D.S., Chun, Y.R., Cha, M.S., 1997. Time-censored ramp tests with stress bound for Weibull life distribution. IEEE Trans. Reliability 46 (1), 99–107.

Brooks, S.P., 1998. Quantitative convergence assessment for Markov chain Monte Carlo via cusums. Statist. Comput. 8, 226–227.

Casella, G., George, E.I., 1992. Explaining the Gibbs sampler. Amer. Statist. 46 (3), 167–174.

Chaloner, K.M., Duncan, G.T., 1983. Assessment of a beta prior distribution: PM elicitation. Statistician 174–181.

Chaloner, K., Larntz, K., 1992. Bayesian design for accelerated life testing. J. Statist. Plann. Inference 33, 245–259.

Cohen, M.L., Steffey, D.L., Rolph, J.E. (Eds.), 1999. Statistics, Testing, and Defense Acquisition, Panel on Statistical Methods for Testing and Evaluating Defense Systems. National Research Council, National Academy Press, Washington DC.
Cooke, R.M., 1991. Experts in Uncertainty. Oxford University Press, New York.
Cooke, R.M., Dekker, R., van Dorp, J.R., van Noortwijk, J.M., Mazzuchi, T.A., 1991. The elicitation and use of expert judgment for maintenance optimization. The World Congress on Expert Systems Proceedings, pp. 888–896.
Cowles, M.K., Carlin, B.P., 1996. Markov chain Monte Carlo convergence diagnostics: a comparative review. J. Roy. Statist. Soc. Ser. B 57, 473–484.
DeGroot, M.H., 1989. Probability and Statistics, 2nd Edition. Addison-Wesley, Reading, MA.
DeGroot, M.H., Goel, P.K., 1988. Bayesian design and analysis of accelerated life testing with step-stress. Proceedings S.I.P. Course CII, Accelerated Life Testing and Experts' Opinion in Reliability, pp. 193–202.
Denson, W.K., 1995. The reliability of plastic encapsulated microcircuits. RAC J. 3, 2.
Department of Defense Mil-Std-781C, 1977. Reliability Design Qualification and Production Acceptance Tests: Exponential Case. United States Government Printing Office, Washington DC.
Dietrich, D., Somerville, I., Mazzuchi, T.A., 1997. Bayesian reliability analysis using the Dirichlet prior distribution with emphasis on accelerated life testing in random order. Nonlinear Anal. Theory Methods Appl. 30 (Part 7), 4415–4423.
Erkanli, A., Soyer, R., 2000. Simulation-based designs for accelerated life testing. J. Statist. Plann. Inference 90, 335–348.
Fornell, G.E., 1991. Life testing advances prove high reliability C^3I systems. Signal 62–69.
Gilks, W.R., Richardson, S., Spiegelhalter, D.J., 1996. Markov Chain Monte Carlo in Practice. Chapman & Hall, London.
Gouno, E., 2001. An inference method for temperature step-stress accelerated life testing. Quality Reliability Eng. Internat. 17, 11–18.
Hawkins, D.M., David, H.O., 1998. Cumulative Sum Charts and Charting for Quality Improvement. Springer, New York.
Khamis, I.H., 1997. Optimum M-step, step-stress test with k stress variables. Comm. Statist. Comput. Simulation 26, 1301–1313.
Khamis, I.H., Higgins, J.J., 1996. Optimum 3-step step-stress tests. IEEE Trans. Reliability 45 (2), 341–345.
Lin, Z., Fei, H., 1991. A non-parametric approach to progressive stress accelerated life testing. IEEE Trans. Reliability 40 (2), 173–176.
Luvalle, M.J., Hines, L.L., 1992. Using step stress to explore the kinetics of failure. Quality Reliability Eng. Internat. 8, 361–369.
Mazzuchi, T.A., Singpurwalla, N.D., 1988. Inference from accelerated life tests—some recent results. Proceedings S.I.P. Course CII, Accelerated Life Testing and Experts' Opinion in Reliability, pp. 181–192.
Mazzuchi, T.A., Soyer, R., 1992. A dynamic general linear model for inference from accelerated life tests. Naval Res. Logist. 39, 757–773.
Mazzuchi, T.A., Soyer, R., Vopatek, A., 1997. Linear Bayesian inference for accelerated Weibull model. Lifetime Data Anal. 3, 63–75.
MClinn, J.A., 1998. Ways to improve the analysis of multi-level accelerated life testing. Quality Reliability Eng. Internat. 14, 393–401.
Meeker, W.Q., Esocbar, L.A., 1998. Statistical Methods for Reliability Data. Wiley, New York.
Meeker, Hahn, 1978. A comparison of accelerated test plans to estimate the survival probability at a design stress. 20(3).
Nelson, W., 1980. Accelerated life testing—step-stress models and data analysis. IEEE Trans. Reliability R-29, 103–108.
Press, W.H., Flannery, B.P., Teukolsky, S.A., Vettering, W.T., 1989. Numerical Recipes in Pascal. Cambridge University Press, New York.
Robert, C.P., Casella, G., 1999. Monte Carlo Statistical Methods. Springer, New York.
Smith, A.F.M., Gelfand, A.E., 1992. Bayesian statistics without tears: a sampling–resampling perspective. Amer. Statist. 46 (2), 84–88.
Thomas, R.E., Gaines, G.B., 1978. Procedure for developing experimental designs for accelerated tests for service life prediction. Thirteenth IEEE Photovoltaic Specialists Conference, pp. 47–52.

Tyoskin, O.I., Krivolapov, S.Y., 1996. Non-parametric model for step-stress accelerated life testing. IEEE Trans. Reliability 45 (2), 346–350.

Van Dorp, J.R., 1997. A general Bayesian inference model for accelerated life testing. UMI Dissertation Services, Ann Arbor, MI.

Van Dorp, J.R., Mazzuchi, T.A., 2000. Solving for the parameters of Beta distribution under two quantile constraints. J. Statist. Comput. Simulation 67, 189–201.

Van Dorp, J.R., Mazzuchi, T.A., Fornell, G.E., Pollock, L.R., 1996. A Bayes approach to step-stress accelerated life testing. IEEE Trans. Reliability 45 (3), 491–498.