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Sequential inference and decision making for single mission systems development

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Abstract

There is a host of literature focusing on modeling product reliability during the development phase under the title *reliability growth modeling*. However, these models are seldom used to address the issue of when to terminate the development process. This is important to decision makers as it is directly related to budgeting for the development process, program monitoring, and predicting when the product will be ready for field use. Herein, we address these issues by developing a Bayesian decision theoretic framework for analyzing the problem of when to terminate testing for 'single shot' or 'single mission' systems and derive an optimal stopping rule. We illustrate the approach with an example. © 1997 Elsevier Science B.V.

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1. Introduction

During its development phase, a product will experience a series of Test, Analyze and Fix (TAAF) stages in which the product is tested and subsequently modified when unsatisfactory performance is observed. It is expected that, due to the modifications, the reliability of the product will increase (or at least not deteriorate) over these stages and thus this process is usually referred to as *reliability growth*. The reliability growth process has been studied by a host of authors considering the cases of *attribute* (pass or fail) and *variable* (failure time) test data. Herein, we focus on the attribute testing scenario as it applies to *single shot* or *single mission* systems. This has been considered

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in the literature from both the sampling theoretic (Lloyd and Lipow, 1962; Barlow and Scheuer, 1966; Finkelstein, 1983) and Bayesian (Smith, 1977; Pollock, 1968; Mazzuchi and Soyer, 1992; Mazzuchi and Soyer, 1993) perspectives.

To date, the literature on reliability growth, with the exception of Kenett and Pollak (1986), has focused on modeling the growth process for making inference on product reliability for each TAAF stage and has neglected an equally important issue of determining when to terminate the development process and release the system for field use. The latter issue is important to practitioners whose tasks include budgeting for the development process, program monitoring, and deciding when the product is ready for field use (see, for example, Meth, 1992). In addressing these issues, we formulate the problem of optimal stopping in single mission system development as a sequential decision problem, taking into account the costs and benefits associated with additional testing. Often, sequential decision problems of this type can become too difficult to analyze due to the reliance on preposterior analysis. Berger and Sung (1993), for example, have discussed this type of problem in the context of sequential reliability demonstration tests. Herein, using the reliability growth model of Mazzuchi and Soyer (1992) we are able to obtain tractable expressions for the random quantities of interest and determine a simple stopping rule under a reasonable class of loss functions.

In Section 2 we describe the reliability growth process and formulate the stopping problem in a sequential decision framework and develop an optimal stopping rule for a class of loss functions. In Section 3, we present an overview of the Mazzuchi–Soyer model and discuss sequential Bayes inference. In Section 4, we motivate specific forms for loss function relevant to the reliability growth process and demonstrate the applicability of the stopping rule. In Section 5, we illustrate our approach with an example.

2. An overview of the reliability growth process

Assume that the development phase consists of at most m TAAF stages in which identical copies of the product are tested one-at-a-time and the test may result in a failure or a successful completion. The testing in each TAAF stage continues until a failure occurs and, upon the discovery of a failure, the failure cause is analyzed and the product is subsequently modified to remedy the cause of failure.

Given the above testing scenario, the sampling model (or likelihood) for the number of items, N_i, tested during TAAF stage i follows a geometric distribution of the form

$$\Pr\{N_i = n_i \mid R_i\} = (1 - R_i)R_i^{n_i - 1}, \quad n_i = 1, 2, \ldots, \tag{2.1}$$

where R_i, $i = 1, \ldots, m + 1$, denotes the product reliability, after TAAF stage $i - 1$. Since it is expected that reliability does not deteriorate due to the modifications, it is reasonable to assume that

$$0 \leqslant R_1 \leqslant \cdots \leqslant R_m \leqslant R_{m+1} \leqslant 1. \tag{2.2}$$

At the end of each TAAF stage, a decision must be made whether to terminate the development program. Thus, after the completion of i TAAF stages, the decision of whether or not to stop testing will be based on $\mathscr{D}^{(i)} = \{\mathscr{D}^{(0)}, n_1, \ldots, n_i\}$, where $n_j, j = 1, \ldots, i$ represents the realization of the random quantity N_j and $\mathscr{D}^{(0)}$ represents available information prior to testing.

2.1. A decision theoretic setup for the optimal stopping problem

Any reasonable stopping rule must be based on maximization (minimization) of expected utility (loss). Intuitively, an appropriate loss function for the decision of when to terminate the development program and release the product must be a function of the number of items tested in each stage $\boldsymbol{N} = (N_1, \ldots, N_m)$ and the stage reliabilities $\boldsymbol{R} = (R_1, \ldots, R_{m+1})$. Such a function should reflect the tradeoff between the loss associated with extensive testing versus the loss associated with releasing a product with low reliability. In view of this tradeoff, let the loss associated with stopping and releasing the product after the ith TAAF stage is given by

$$\mathscr{L}_i(\boldsymbol{N}_i, R_{i+1}) = \sum_{j=1}^{i} \mathscr{L}_{\mathrm{T}}(N_j) + \mathscr{L}_S(R_{i+1}), \quad i = 0, \ldots, m, \tag{2.3}$$

where $\boldsymbol{N}_i = (N_1, \ldots, N_i)$, $\mathscr{L}_{\mathrm{T}}(\cdot)$ denotes the cost due to testing for one stage, and $\mathscr{L}_{\mathrm{S}}(\cdot)$ denotes the loss associated with stopping and releasing the product for field use. Note that with the convention that $\sum_{j=i+1}^{i}\{\cdot\} \equiv 0$, $\mathscr{L}_0$ (the loss associated releasing the product before any testing) is a function of R_1 alone.

Conceptually, the problem can be presented as a decision tree (as in Fig. 1) which represents the sequence of events during the development process. Using conventional representation (Lindley, 1985), random events in the process are denoted by circles (random nodes) and decisions by squares (decision nodes). The optimal decision path can be obtained using dynamic programming which entails taking expectation at random nodes and minimizing the expected loss at the decision nodes. At decision node i, the *additional expected loss* associated with the STOP and the TEST decisions are given by $E[\mathscr{L}_{\mathrm{S}}(R_{i+1}) \mid \mathscr{D}^{(i)}]$ and $E[\mathscr{L}_{\mathrm{T}}(N_{i+1}) \mid \mathscr{D}^{(i)}] + L_{i+1}^*$,

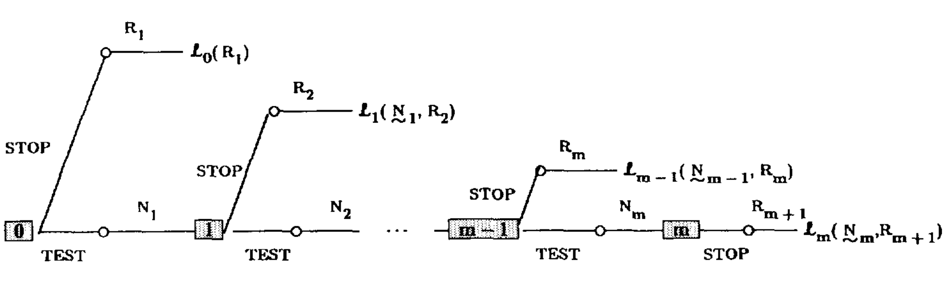

Fig. 1. Decision tree for reliability growth process.

respectively, where

$$L_i^* = \mathrm{MIN}\{E[\mathscr{L}_\mathrm{S}(R_{i+1})\,|\,\mathscr{D}^{(i)}], E[\mathscr{L}_\mathrm{T}(N_{i+1})\,|\,\mathscr{D}^{(i)}] + L_{i+1}^*\} \tag{2.4}$$

for $i = 0, \dots, m$ and $L_{m+1}^* = \infty$ at the terminal node. The optimal decision at decision node i then is the one associated with L_i^*. Eq. (2.4) is the well-known dynamic programming recursion relation.

Note that while (2.4) represents a finite set of relationships, the calculation of L_i^* is non-trivial as it involves a combination of expectations and minimizations. In determining an optimal stopping rule, it is convenient to rewrite (2.4) as

$$L_i^* = \underset{\delta = 0, \dots, m-i}{\mathrm{MIN}} \{L_i^{(\delta)}\}, \tag{2.5}$$

where

$$L_i^{(\delta)} = \sum_{j=1}^{\delta} E[\mathscr{L}_\mathrm{T}(N_{i+j})\,|\,\mathscr{D}^{(i)}] + E[\mathscr{L}_\mathrm{S}(R_{i+\delta+1})\,|\,\mathscr{D}^{(i)}] \tag{2.6}$$

is the additional expected loss associated with testing for δ more stages after TAAF stage i.

Given the above setup, the following optimal stopping rule can be developed.

Theorem 1. *Let $E[\mathscr{L}_\mathrm{T}(N_j)\,|\,\mathscr{D}^{(i)}]$ be increasing in j for $j = i+1, \dots, m$ and $E[\mathscr{L}_\mathrm{S}(R_j)\,|\,\mathscr{D}^{(i)}]$ be discrete convex in j for $j = i+1, \dots, m+1$; then after the completion of i TAAF stages, the following stopping rule is optimal with respect to $L_i^{(\delta)}$:*

$$\text{if} \begin{cases} L_i^{(1)} - L_i^{(0)} < 0 \Rightarrow \textit{continue testing}, \\ L_i^{(1)} - L_i^{(0)} \geqslant 0 \Rightarrow \textit{stop testing and release}. \end{cases} \tag{2.7}$$

Proof. If $L_i^{(1)} - L_i^{(0)} < 0$, then testing at least one more stage will yield improvement, so *continue testing* is the optimal decision.

If $L_i^{(1)} - L_i^{(0)} \geqslant 0$ then

$$E[\mathscr{L}_\mathrm{T}(N_{i+1})\,|\,\mathscr{D}^{(i)}] \geqslant E[\mathscr{L}_\mathrm{S}(R_{i+1})\,|\,\mathscr{D}^{(i)}] - E[\mathscr{L}_\mathrm{S}(R_{i+2})\,|\,\mathscr{D}^{(i)}]. \tag{2.8}$$

Under the assumption that $E[\mathscr{L}_\mathrm{T}(N_j)\,|\,\mathscr{D}^{(i)}]$ is increasing in j for $j = i+1, \dots, m$ and $E[\mathscr{L}_\mathrm{S}(R_j)\,|\,\mathscr{D}^{(i)}]$ is discrete convex in j, i.e. that the first-order differences $E[\mathscr{L}_\mathrm{S}(R_{j+1})\,|\,\mathscr{D}^{(i)}] - E[\mathscr{L}_\mathrm{S}(R_j)\,|\,\mathscr{D}^{(i)}]$ are increasing for $j = i+1, \dots, m$, (2.8) guarantees that

$$E[\mathscr{L}_\mathrm{T}(N_{i+\delta+1})\,|\,\mathscr{D}^{(i)}] \geqslant E[\mathscr{L}_\mathrm{S}(R_{i+\delta+1})\,|\,\mathscr{D}^{(i)}] - E[\mathscr{L}_\mathrm{S}(R_{i+\delta+2})\,|\,\mathscr{D}^{(i)}] \tag{2.9}$$

for all $\delta \geqslant 0$. Eq. (2.9) implies that $L_i^{(\delta+1)} - L_i^{(\delta)} \geqslant 0$, for $\delta \geqslant 0$ and thus from (2.5), $L_i^* = L_i^{(0)}$ and *stop testing and release* is the optimal decision. □

If after TAAF stage i, the optimal decision is to continue testing, then it may be important to estimate the expected time and cost necessary to complete the program. The additional expected loss associated with testing for δ more stages, is a discrete function over a finite number of integer values ($\delta = 1, \dots, m - i$) and thus a global minimum can be determined. Once the value δ^* which minimizes this expression is determined, the expected number of additional items tested is determined as

$$\sum_{k=i+1}^{i+\delta^*} E[N_k \mid \mathscr{D}^{(i)}]. \tag{2.10}$$

This can be used to estimate both the expected time and testing budget until completion since both are functions of testing effort. Note that when $i = 0$, (2.10) provides an estimate for the entire program length and budget. The term 'estimate' has been used because even though after TAAF stage i it appears that testing for δ^* more stages is optimal, this is only based on $\mathscr{D}^{(i)}$. Additional testing information at subsequent stage(s) will yield a revision of this estimate or may even result in a stopping decision before stage $i + \delta^*$.

Note that the above theorem implies that it is optimal to stop testing when the expected increase in loss due to testing an additional stage is greater than the expected decrease in loss due to the improvement in reliability resulting from testing an additional stage. In examining the assumptions of the theorem, it seems reasonable that the expected loss due to testing, $E[\mathscr{L}_{\mathrm{T}}(N_j) \mid \mathscr{D}^{(i)}]$, is increasing in j due to (2.2) and the sampling model (2.1); however, the reasonableness of the discrete convexity assumption for the loss due to release, $[\mathscr{L}_{\mathrm{S}}(R_j) \mid \mathscr{D}^{(i)}]$, is not so obvious. In the sequel, the reasonableness of this assumption will be conveyed via application of the above decision theoretic framework with the Bayesian attribute reliability growth model of Mazzuchi and Soyer (1992).

3. A Bayesian reliability growth model

3.1. The prior distribution

To model the reliability growth process, Mazzuchi and Soyer (1992) use the ordered Dirichlet distribution as the prior joint distribution for $\boldsymbol{R} = (R_1, \dots, R_{m+1})$. This distribution is defined over (2.2) and given by

$$\Pi(\boldsymbol{R} \mid \mathscr{D}^{(0)}) = \frac{\prod_{j=1}^{m+2} (R_j - R_{j-1})^{\beta\alpha_j - 1}}{\mathbb{D}(\beta, \boldsymbol{\alpha})}, \tag{3.1}$$

where $R_0 \equiv 0$, $R_{m+2} \equiv 1$, $\beta, \alpha_i > 0$, $\sum_{i=1}^{m+2} \alpha_i = 1$, and

$$\mathbb{D}(\beta, \boldsymbol{\alpha}) = \prod_{j=1}^{m+2} \Gamma(\beta\alpha_j)/\Gamma(\beta) \tag{3.2}$$

is the Dirichlet constant. The term $\mathscr{D}^{(0)}$ represents the prior information captured by the parameters β, α_i, $i = 1, \ldots, m+2$.

An advantage of using the ordered Dirichlet, is that, a priori, all relevant quantities of interest have closed-form distributions, for example,

$$[R_i \mid \mathscr{D}^{(0)}] \sim \text{Beta}(\beta\alpha_i^*, \beta(1-\alpha_i^*)), \quad i = 1, \ldots, m+1 \tag{3.3}$$

and

$$[R_j - R_i \mid \mathscr{D}^{(0)}] \sim \text{Beta}(\beta(\alpha_j^* - \alpha_i^*), \beta(1 - \alpha_j^* + \alpha_i^*)), \quad \text{for } i < j, \tag{3.4}$$

where $\alpha_i^* = \sum_{k=1}^{i} \alpha_k$. The above distributions provide meaningful interpretations for the parameters α_i and α_i^* in terms of the reliability growth scenario. Specifically, using the well-known expression for the mean of a beta random variable, the parameters α_i and α_i^* can be interpreted as the expected one-step improvement in reliability from stage $i-1$ to i, and expected reliability at stage i, respectively. This allows for different methods of elicitation and feedback for incorporating expert judgements into the prior (see, for example, Mazzuchi and Soyer, 1993).

3.2. Posterior inference and reliability growth monitoring

An additional feature of (3.1) is that all of the distributional forms are preserved as mixtures a posteriori. It can be shown that the posterior joint distribution of $\boldsymbol{R}_i = (R_i, \ldots, R_{m+1})$, is a mixture of ordered Dirichlet distributions as

$$\Pi(\boldsymbol{R}_i \mid \mathscr{D}^{(i)}) = \sum_{l_1=0}^{1} \cdots \sum_{l_i=0}^{1} \mathscr{W}(\boldsymbol{l}^i)\,\Pi(\boldsymbol{R}_i \mid \beta^u(\boldsymbol{l}^i), \boldsymbol{\alpha}^u(\boldsymbol{l}^i)), \tag{3.5}$$

where $\boldsymbol{l}^i = (l_1, \ldots, l_i)$,

$$\mathscr{W}(\boldsymbol{l}^i) = \frac{(-1)^{l_i^*}\left\{\prod_{j=1}^{i-1} \dfrac{\Gamma(S_j(\boldsymbol{l}^i) + \beta\alpha_j^*)}{\Gamma(S_j(\boldsymbol{l}^i) + \beta\alpha_{j+1}^*)}\right\}\left\{\dfrac{\Gamma(S_i(\boldsymbol{l}^i) + \beta\alpha_i^*)}{\Gamma(S_i(\boldsymbol{l}^i) + \beta)}\right\}}{\sum_{l_1=0}^{1} \cdots \sum_{l_i=0}^{1} (-1)^{l_i^*}\left\{\prod_{j=1}^{i-1} \dfrac{\Gamma(S_j(\boldsymbol{l}^i) + \beta\alpha_j^*)}{\Gamma(S_j(\boldsymbol{l}^i) + \beta\alpha_{j+1}^*)}\right\}\left\{\dfrac{\Gamma(S_i(\boldsymbol{l}^i) + \beta\alpha_i^*)}{\Gamma(S_i(\boldsymbol{l}^i) + \beta)}\right\}}, \tag{3.6}$$

$$\Pi(\boldsymbol{R}_i \mid \beta^u(\boldsymbol{l}^i), \boldsymbol{\alpha}^u(\boldsymbol{l}^i)) = \frac{R_i^{(\beta^u(\boldsymbol{l}^i)\alpha_i^u(\boldsymbol{l}^i)-1)} \prod_{j=i+1}^{m+2} (R_j - R_{j-1})^{\beta^u(\boldsymbol{l}^i)\alpha_j^u(\boldsymbol{l}^i)-1}}{\mathbb{D}(\beta^u(\boldsymbol{l}^i), \boldsymbol{\alpha}^u(\boldsymbol{l}^i))}, \tag{3.7}$$

and the updated parameters $\beta^u(\boldsymbol{l}^i)$ and $\boldsymbol{\alpha}^u(\boldsymbol{l}^i) = (\alpha_i^u(\boldsymbol{l}^i), \ldots, \alpha_{m+2}^u(\boldsymbol{l}^i))$ are given as

$$\beta^u(\boldsymbol{l}^i) = \beta + S_i(\boldsymbol{l}^i) \tag{3.8}$$

and

$$\alpha_j^u(\boldsymbol{l}^i) = \begin{cases} \dfrac{(S_i(\boldsymbol{l}^i) + \beta\alpha_j^*)}{\beta^u(\boldsymbol{l}^i)} & \text{for } j = i, \\ \dfrac{\beta\alpha_j}{\beta^u(\boldsymbol{l}^i)} & \text{for } j = i+1, \dots, m+2 \end{cases} \tag{3.9}$$

with

$$S_j(\boldsymbol{l}^i) = \sum_{k=1}^{j} (l_k + n_k - 1), \quad l_i^* = \sum_{k=1}^{i} l_k \text{ and } \prod_{j=1}^{0} \{\cdot\} \equiv 1.$$

The progress of reliability growth can be assessed by comparing prior and posterior estimates of the growth curve during the development program. The growth curve estimate after the ith TAAF stage is obtained as the plot of $E[R_k \mid \mathscr{D}^{(i)}]$ versus k for $k = 1, \dots, m+1$. Thus, after TAAF stage i, a comparison can be made of the most recent growth curve estimate, $E[R_k \mid \mathscr{D}^{(i)}]$, with that of some earlier stage $r < i$, $E[R_k \mid \mathscr{D}^{(r)}]$. Of particular interest are $r = 0$ (prior estimates) and $r = i - 1$ (estimates just prior to the ith testing stage).

Note that expressions $E[R_k \mid \mathscr{D}^{(i)}]$ where $k > i$ are forecasts for future stage reliabilities and can be obtained directly from (3.5), whereas for $k < i$ these are the smoothed estimates of past stage reliabilities and must be obtained from the full posterior of $\boldsymbol{R}_1$. These expressions can be obtained as

$$E[R_k \mid \mathscr{D}^{(i)}] = \begin{cases} \displaystyle\sum_{l_1=0}^{1} \cdots \sum_{l_i=0}^{1} \mathscr{W}(\boldsymbol{l}^i) \left\{ \prod_{j=k}^{i-1} \frac{S_j(\boldsymbol{l}^i) + \beta\alpha_j^*}{S_j(\boldsymbol{l}^i) + \beta\alpha_{j+1}^*} \right\} \frac{S_i(\boldsymbol{l}^i) + \beta\alpha_i^*}{S_i(\boldsymbol{l}^i) + \beta} & \text{for } k < i, \\ \displaystyle\sum_{l_1=0}^{1} \cdots \sum_{l_i=0}^{1} \mathscr{W}(\boldsymbol{l}^i) \left\{ \frac{S_i(\boldsymbol{l}^i) + \beta\alpha_k^*}{S_i(\boldsymbol{l}^i) + \beta} \right\} & \text{for } k \geqslant i. \end{cases} \tag{3.10}$$

4. The optimal stopping rule with specific loss functions

As it is reasonable to assume that $\mathscr{L}_{\mathrm{T}}$ is an increasing function of N_i, a candidate for this loss function is

$$\mathscr{L}_{\mathrm{T}}(N_i) = C_{\mathrm{T}} N_i, \tag{4.1}$$

where C_{T} is some positive constant which may be interpreted as the testing cost per item. As per the requirement of Theorem 1, $E[\mathscr{L}_{\mathrm{T}}(N_j) \mid \mathscr{D}^{(i)}]$ is increasing in $j = i+1, \dots, m$ when $\mathscr{L}_{\mathrm{T}}$ is defined by (4.1). This can be seen since, from (2.1),

$$E[\mathscr{L}_{\mathrm{T}}(N_j) \mid \mathscr{D}^{(i)}] = C_{\mathrm{T}} E\left[\frac{1}{1 - R_j} \mid \mathscr{D}^{(i)}\right] \tag{4.2}$$

and the fact that both prior and posterior are defined over (2.2).

In determining $\mathscr{L}_S$, it is expected that an unreliable product will yield a higher loss, and thus $\mathscr{L}_S$ should be a decreasing function of the reliability at the time of release. Two specific forms of $\mathscr{L}_S$ are worth noting,

$$\mathscr{L}_S(R_{i+1}) = K(1 - R_{i+1}) \tag{4.3}$$

and

$$\mathscr{L}_S(R_{i+1}) = K(1 - R_{i+1})^2, \tag{4.4}$$

where $K > 0$ is some positive constant. Note that in (4.3), $1 - R_{i+1}$ denotes the probability of field failure of the product if the product is released after i TAAF stages. For example, if the goal of the development program is to produce a batch of B products, then the number of field failures is a binomial random variable and (4.3) represents the expected cost due to field failures with $K = C_F B$ for C_F, the per unit failure cost. As for (4.4), consider a quadratic loss function for penalizing reliability values which deviate from the most desirable value of 1. The scaling constant K, then can be thought of a per unit loss for deviating reliabilities from 1.

It can be shown that both (4.3) and (4.4) satisfy the assumptions of Theorem 1 provided that $E[R_{j+1}|\mathscr{D}^{(0)}] - E[R_j|\mathscr{D}^{(0)}] = \alpha_{j+1}$ is a decreasing sequence in j for $j = 1, \ldots, m+2$. When using the Bayesian reliability growth model of Section 3, this is the same as requiring that $E[R_j|\mathscr{D}^{(0)}] = \alpha_j^*$ is discrete concave in j. Such an assumption implies that the reliability improvement diminishes over the development program. This is due, in part, to the fact that the more major product flaws will be discovered (and subsequently fixed via modification) during the earlier stages. Consider the following theorem.

Theorem 2. *When the prior distribution of* $\boldsymbol{R}$ *is given by* (3.1) *with* $E[R_j|\mathscr{D}^{(0)}] = \alpha_j^*$ *discrete concave in j, then the expected loss functions are discrete convex in j when loss is given by* (4.3) *or* (4.4)

Proof of (4.3). As in Theorem 1, for $j \geqslant i$, we may write the first difference of expected loss, $E[\mathscr{L}_S(R_j)|\mathscr{D}^{(i)}] - E[\mathscr{L}_S(R_{j+1})|\mathscr{D}^{(i)}]$ using (4.3) as

$$E[\mathscr{L}_S(R_j)|\mathscr{D}^{(i)}] - E[\mathscr{L}_S(R_{j+1})|\mathscr{D}^{(i)}] = K\{E[R_{j+1}|\mathscr{D}^{(i)}] - E[R_j|\mathscr{D}^{(i)}]\}.$$

Because $E[R_j|\mathscr{D}^{(i)}]$ is increasing in j (as both prior and posterior are defined over (2.2)), and K is a positive constant, the above difference is positive for all $j \geqslant i$. Using (3.10) the above may be expressed as

$$E[\mathscr{L}_S(R_j)|\mathscr{D}^{(i)}] - E[\mathscr{L}_S(R_{j+1})|\mathscr{D}^{(i)}] = \alpha_{j+1}\left\{K \sum_{l_1=0}^{1} \cdots \sum_{l_i=0}^{1} \mathscr{W}(\boldsymbol{l}^i)\frac{\beta}{\beta + S_i(\boldsymbol{l}^i)}\right\}. \tag{4.5}$$

Since α_{j+1} is positive, it follows from the above that the term in the brackets in (4.5) must be positive. Furthermore, since α_j is a decreasing sequence in j and the term inside the brackets is a positive constant in j, it follows that $E[\mathscr{L}_S(R_j)\,|\,\mathscr{D}^{(i)}] - E[\mathscr{L}_S(R_{j+1})\,|\,\mathscr{D}^{(i)}]$ is decreasing in j and $E[\mathscr{L}_S(R_j)\,|\,\mathscr{D}^{(i)}]$ is discrete convex in j.

Proof of (4.4). For $j \geqslant i$, we may write the first difference of expected loss using (4.4) as

$$E[\mathscr{L}_S(R_j)\,|\,\mathscr{D}^{(i)}] - E[\mathscr{L}_S(R_{j+1})\,|\,\mathscr{D}^{(i)}]$$
$$= K\{E[(1-R_j)^2\,|\,\mathscr{D}^{(i)}] - E[(1-R_{j+1})^2\,|\,\mathscr{D}^{(i)}]\}. \tag{4.6}$$

Due to (2.2), the term $E[(1-R_j)^2\,|\,\mathscr{D}^{(i)}]$ is decreasing in j and thus the first differences in expected loss given by (4.6) is positive for all j. Calculating the expectations in (4.6) using (3.5), yields

$$E[\mathscr{L}_S(R_j)\,|\,\mathscr{D}^{(i)}] - E[\mathscr{L}_S(R_{j+1})\,|\,\mathscr{D}^{(i)}]$$
$$= K\sum_{l_1=0}^{1}\cdots\sum_{l_i=0}^{1}\mathscr{W}(\boldsymbol{l}^i)\left\{\frac{\beta^u(\boldsymbol{l}^i)(1-\alpha_j^u(\boldsymbol{l}^i)^*)(\beta^u(\boldsymbol{l}^i)(1-\alpha_j^u(\boldsymbol{l}^i)^*)+1)}{(\beta^u(\boldsymbol{l}^i)+1)\beta^u(\boldsymbol{l}^i)}\right.$$
$$\left.-\frac{\beta^u(\boldsymbol{l}^i)(1-\alpha_{j+1}^u(\boldsymbol{l}^i)^*)(\beta^u(\boldsymbol{l}^i)(1-\alpha_{j+1}^u(\boldsymbol{l}^i)^*)+1)}{(\beta^u(\boldsymbol{l}^i)+1)\beta^u(\boldsymbol{l}^i)}\right\},$$

where $\alpha_j^u(\boldsymbol{l}^i)^* = \sum_{k=i}^{j}\alpha_k^u(\boldsymbol{l}^i)$. Using (3.8) and (3.9), it may be shown that

$$\alpha_j^u(\boldsymbol{l}^i)^* = \frac{S_i(\boldsymbol{l}^i)+\beta\alpha_j^*}{S_i(\boldsymbol{l}^i)+\beta},$$

and the expression for $E[\mathscr{L}_S(R_j)\,|\,\mathscr{D}^{(i)}] - E[\mathscr{L}_S(R_{j+1})\,|\,\mathscr{D}^{(i)}]$ reduces to

$$E[\mathscr{L}_S(R_j)\,|\,\mathscr{D}^{(i)}] - E[\mathscr{L}_S(R_{j+1})\,|\,\mathscr{D}^{(i)}]$$
$$= \{\beta\alpha_{j+1}(\beta(1-\alpha_j^*)+\beta(1-\alpha_{j+1}^*)+1)\}$$
$$\left\{K\sum_{l_1=0}^{1}\cdots\sum_{l_i=0}^{1}\frac{\mathscr{W}(\boldsymbol{l}^i)}{(\beta+S_i(\boldsymbol{l}^i)+1)(\beta+S_i(\boldsymbol{l}^i))}\right\}. \tag{4.7}$$

Given the restrictions on the parameters ($\beta > 0$, $\alpha_j > 0$, $0 < \alpha_j^* < 1$) specified in Section 3.1, the term in the first bracket of (4.7) is positive. Given that this first term is positive and (4.6) is positive, it follows that the term inside the second bracket of (4.7) must be positive. Then since α_j is decreasing in j, the term in the first brackets of (4.7) is decreasing in j. Since the term in the second brackets of (4.7) is a positive constant in j, the first difference $E[\mathscr{L}_S(R_j)\,|\,\mathscr{D}^{(i)}] - E[\mathscr{L}_S(R_{j+1})\,|\,\mathscr{D}^{(i)}]$ is decreasing in j and thus $E[\mathscr{L}_S(R_j)\,|\,\mathscr{D}^{(i)}]$ is discrete convex in j. □

Using the posterior results of Section 3, the additional expected loss associated with testing for δ more stages after TAAF stage i, $L_i^{(\delta)}(\boldsymbol{R})$, can be determined. For $\mathscr{L}_T$ and the linear form of $\mathscr{L}_S$, given by (4.3), $L_i^{(\delta)}(\boldsymbol{R})$ is given by

$$\sum_{l_1=0}^{1} \cdots \sum_{l_i=0}^{1} \mathscr{W}(\boldsymbol{l}^i)\left\{C_T \sum_{k=i+1}^{i+\delta} \frac{\beta^u(\boldsymbol{l}^i)-1}{\beta^u(\boldsymbol{l}^i)(1-\alpha_k^u(\boldsymbol{l}^i)^*)-1} + K(1-\alpha_{i+\delta+1}^u(\boldsymbol{l}^i)^*)\right\}$$

$$= \sum_{l_1=0}^{1} \cdots \sum_{l_i=0}^{1} \mathscr{W}(\boldsymbol{l}^i)\left\{C_T \sum_{k=i+1}^{i+\delta} \frac{\beta+S_i(\boldsymbol{l}^i)-1}{\beta(1-\alpha_k^*)-1} + K\frac{\beta(1-\alpha_{i+\delta+1}^*)}{\beta+S_i(\boldsymbol{l}^i)}\right\}. \tag{4.8}$$

For $\mathscr{L}_T$ and the quadratic form of $\mathscr{L}_S$, given by (4.4), $L_i^{(\delta)}(\boldsymbol{R})$ is given by

$$\sum_{l_1=0}^{1} \cdots \sum_{l_i=0}^{1} \mathscr{W}(\boldsymbol{l}^i)\left\{C_T \sum_{k=i+1}^{i+\delta} \frac{\beta^u(\boldsymbol{l}^i)-1}{\beta^u(\boldsymbol{l}^i)(1-\alpha_k^u(\boldsymbol{l}^i)^*)-1}\right.$$
$$\left. + K\frac{\beta^u(\boldsymbol{l}^i)(1-\alpha_{i+\delta+1}^u(\boldsymbol{l}^i)^*)(\beta^u(\boldsymbol{l}^i)(1-\alpha_{i+\delta+1}^u(\boldsymbol{l}^i)^*)+1)}{\beta^u(\boldsymbol{l}^i)(\beta^u(\boldsymbol{l}^i)+1)}\right\},$$

which reduces to

$$\sum_{l_1=0}^{1} \cdots \sum_{l_i=0}^{1} \mathscr{W}(\boldsymbol{l}^i)\left\{C_T \sum_{k=i+1}^{i+\delta} \frac{\beta+S_i(\boldsymbol{l}^i)-1}{\beta(1-\alpha_k^*)-1}\right.$$
$$\left. + K\frac{\beta(1-\alpha_{i+\delta+1}^*)(\beta(1-\alpha_{i+\delta+1}^*)+1)}{(\beta+S_i(\boldsymbol{l}^i))(\beta+S_i(\boldsymbol{l}^i)+1)}\right\}. \tag{4.9}$$

Note that the resulting expectations are finite when $\beta(1-\alpha_k^*)>1$ for all $k=i+1,\ldots,i+\delta$. When $i=0$ (prior to any testing), (4.8) and (4.9) reduce to

$$C_T \sum_{k=1}^{\delta} \frac{\beta-1}{\beta(1-\alpha_k^*)-1} + K\frac{\beta(1-\alpha_{\delta+1}^*)}{\beta} \tag{4.10}$$

and

$$C_T \sum_{k=1}^{\delta} \frac{\beta-1}{\beta(1-\alpha_k^*)-1} + K\frac{\beta(1-\alpha_{\delta+1}^*)(\beta(1-\alpha_{\delta+1}^*)+1)}{(\beta)(\beta+1)}, \tag{4.11}$$

respectively.

If after TAAF stage i, the optimal decision is to continue testing, then it may be important to estimate the expected time and cost necessary to complete the program. The additional expected loss associated with testing for δ more stages, (specified by (4.8) or (4.9)) is a discrete function over a finite number of integer values ($\delta = 1, \ldots, m-i$) and thus a global minimum can be determined. Once the value δ^* which minimizes this expression is determined, the expected number of additional items tested is given by

$$\sum_{k=i+1}^{i+\delta^*} E[N_k \mid \mathscr{D}^{(i)}] = \sum_{k=i+1}^{i+\delta^*} \frac{\beta+S_i(\boldsymbol{l}^i)-1}{\beta(1-\alpha_k^*)-1}. \tag{4.12}$$

5. Example

Assume that, a priori, the uncertainty for a product's reliability growth over 10 TAAF stages ($m = 10$) is expressed via (3.1) with the following parameters: $\beta = 50$ and

$$\boldsymbol{\alpha} = (0.36,\ 0.34,\ 0.102,\ 0.0985,\ 0.0128,\ 0.0127,\ 0.0126,\ 0.0125,\ 0.0124,\ 0.0123,\ 0.0122,\ 0.0120).$$

Note that the α_j's form a decreasing sequence in j. Suppose that we use the linear form of $\mathscr{L}_s$, given by (4.3). Assuming that the goal of the development testing program is to deliver a batch size of 20 products ($B = 20$) for field use and cost of a field failure is given as $C_F = \$50\,000$/item, then using (4.3), $K = C_F B = \$1\,000\,000$. The testing cost in (4.1) is assumed to be given as $C_T = \$1000$/item.

Based on prior information alone, the $L_0^{(\delta)}$ can be obtained using (4.10). These are presented in the first row of Table 1 for $\delta = 0, 1, \ldots, 10$. Using the stopping rule given by (2.7), $L_0^{(1)}(\boldsymbol{R}) - L_0^{(0)}(\boldsymbol{R}) < 0$ and thus it is determined a priori to initiate testing. Furthermore, using (2.5), $L_0^* = L_0^{(4)}$ (denoted by the asterisk) which implies that testing is expected to terminate after 4 TAAF stages. From the prior information the expected number of items to be tested in 4 stages is given by (4.12) as $22.91 \approx 23$.

Suppose that the actual number of copies tested in the first TAAF stage equals 1, that is, the first item tested, failed. After the completion of the first TAAF stage and revision of the reliability estimates using the development in Section 3.2, the expected additional loss is calculated using (4.8) and given in the second row of Table 1. Again as $L_1^{(1)}(\boldsymbol{R}) - L_1^{(0)}(\boldsymbol{R}) < 0$, and the optimal decision is to continue testing. The testing program is still expected to terminate after the fourth TAAF stage.

We assume that the number of items tested in the subsequent TAAF stages are given in Table 2 and the respective expected additional cost, $L_i^{(j-i)}(\boldsymbol{R})$, are given in Table 1. Note from Table 1, that the optimal decision after TAAF stages 2–7 is to continue testing. It is not until the 8th TAAF stage that $L_8^{(1)}(\boldsymbol{R}) - L_8^{(0)}(\boldsymbol{R}) > 0$

Table 1
Expected additional cost estimates (in \$1000) after each TAAF stage

Loss	TAAF stage (j)										
	0	1	2	3	4	5	6	7	8	9	10
$L_0^{(j)}(\boldsymbol{R})$	640.0	301.6	203.1	110.1	**109.6***	111.6	117.2	128.3	149.8	196.9	418.1
$L_1^{(j-1)}(\boldsymbol{R})$	—	303.3	203.6	109.5	**108.7***	110.4	115.6	126.4	147.4	193.7	412.0
$L_2^{(j-2)}(\boldsymbol{R})$	—	—	209.2	110.3	**108.4***	108.8	112.6	121.7	140.5	183.5	390.4
$L_3^{(j-3)}(\boldsymbol{R})$	—	—	—	113.0	109.2	**107.5***	109.0	115.3	130.7	168.2	356.9
$L_4^{(j-4)}(\boldsymbol{R})$	—	—	—	—	114.6	108.4	**104.8***	105.2	113.2	139.5	290.8
$L_5^{(j-5)}(\boldsymbol{R})$	—	—	—	—	—	109.2	101.8	**98.0***	100.7	119.4	245.8
$L_6^{(j-6)}(\boldsymbol{R})$	—	—	—	—	—	—	99.8	92.2	**90.3***	102.2	207.6
$L_7^{(j-7)}(\boldsymbol{R})$	—	—	—	—	—	—	—	84.3	**79.1***	86.3	176.7
$L_8^{(j-8)}(\boldsymbol{R})$	—	—	—	—	—	—	—	—	**65.4***	68.2	143.9

Table 2
Actual number of product copies tested in each TAAF stage j

TAAF Stage j	1	2	3	4	5	6	7	8
n_j	1	1	1	1	3	4	7	10

Table 3
Prior and posterior growth curve estimates

Reliability	TAAF Stage (j)										
	1	2	3	4	5	6	7	8	9	10	11
$E[R_j \mid \mathscr{D}^{(0)}]$	0.360	0.700	0.802	0.901	0.913	0.926	0.939	0.951	0.964	0.976	0.988
$E[R_j \mid \mathscr{D}^{(4)}]$	0.333	0.657	0.760	0.868	0.885	0.902	0.919	0.935	0.952	0.968	0.984
$E[R_j \mid \mathscr{D}^{(8)}]$	0.324	0.638	0.737	0.840	0.856	0.873	0.892	0.912	0.935	0.957	0.978

implying that the optimal decision is to terminate the testing program. Further note that, as discussed in Section 2, the expected completion time has shifted based on new information obtained from testing.

In addition to the above information, Table 3 gives the reliability estimates after each TAAF stage as obtained using (3.10) and the estimated growth curve can be plotted. Note that the growth curve estimates, $E[R_j \mid \mathscr{D}^{(i)}]$, $j = 1, \ldots, 11$, are decreasingly ordered in i, implying that the specified prior reliability growth was overly optimistic.

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