The Generalized Two-Sided Power Distribution

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Abstract. The Generalized Standard Two-Sided Power (TSP) distribution was mentioned only in passing by van Dorp and Kotz (2004). In this paper we shall further investigate this three-parameter distribution by presenting some novel properties and use its more general form to contrast the chronology of developments of various authors on the two-parameter Two-Sided Power distribution since its initial introduction. GTSP distributions also allow for J-shaped forms of its pdf, whereas TSP distributions are limited to U-shaped and unimodal forms. Hence, GTSP distributions allow for the same three distributional shapes of the classical beta distributions. A novel method and algorithm for the indirect elicitation of the two power parameters of the GTSP distribution is developed. We present a Project Evaluation Review Technique (PERT) example that utilizes this algorithm and demonstrates the benefit of separate powers for the two branches of activity GTSP distributions for project completion time uncertainty estimation.

1. Introduction

Kotz and Van Dorp (2004) briefly mentioned the three-parameter Generalized TSP distribution with the pdf

$$f_Y(y|\Theta) = \frac{mn}{(1-\theta)m + \theta n} \times \begin{cases} \left(\frac{y}{\theta}\right)^{m-1}, & \text{for } 0 < y < \theta \\ \left(\frac{1-y}{1-\theta}\right)^{n-1}, & \text{for } \theta \le y < 1. \end{cases} \tag{1}$$
$$0 \le \theta \le 1, n, m > 0$$

but did not further investigate its properties. The pdf (1) reduces to the two-parameter TSP pdf in Van Dorp and Kotz (2002a, b) with the parameter restriction $m = n$. With the parameter

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restriction $\theta^{m-1} = (1-\theta)^{n-1}$ the pdf (1) reduces to the two-parameter TSP pdf discussed in Nadarajah (1999, 2003, 2005). The family (1) allows for parameter combinations ($m > 1$, $0 < n < 1$) or ($0 < m < 1$, $n > 1$) and as a result allows for J-shaped pdf's extending the unimodal and U-shaped forms modeled by the two-parameter TSP pdf's in Van Dorp and Kotz (2002a, b) and Nadarajah (1999, 2003, 2005). Thus, the three-parameter family (1) exhibits the various distributional shapes that classical beta distribution possesses whereas the two parameter versions in Van Dorp and Kotz (2002a, b) and Nadarajah (1999, 2003, 2005) do not. Van Dorp and Kotz (2002a, b) originally suggested their two-parameter TSP pdf as an alternative to the beta distribution in problems of risk and uncertainty and GTSP distributions can be thought of in the same vain. Given that beta distributions are a member within the Pearson systems of distributions it is interesting to note here that for the GTSP pdf (1) we have:

$$\frac{\frac{d}{dy} f_Y(y|\theta,n)}{f_Y(y|\theta,n)} = \begin{cases} \frac{m-1}{y}, & 0 < y < \theta, \\ \frac{n-1}{y-1}, & \theta < y < 1. \end{cases} \tag{2}$$

Thus, whereas for a pdf $f(\cdot)$ in the Pearson system the quotient $f'(y)/f(y)$ is the ratio of a first and second degree polynomials, for the GTSP pdf this quotient is the ratio of a constant over a first degree polynomial (analogously to Roy's (1971) extension).

The main advantage of the pdf (1) (also shared by its two-parameter versions) over the beta distribution is that it has a closed form cdf and a quantile function express able using only elementary functions:

$$F_Y(y|\Theta) = \begin{cases} \pi(\theta,m,n)\left(\frac{y}{\theta}\right)^m, & \text{for } 0 \le y < \theta \\ 1 - [1-\pi(\theta,m,n)]\left(\frac{1-y}{1-\theta}\right)^n, & \text{for } \theta \le y \le 1, \end{cases} \tag{3}$$

and

$$F_Y^{-1}(z|\Theta) = \begin{cases} \theta\left(\frac{z}{\pi(\theta,m,n)}\right)^{1/m}, & \text{for } 0 \le z < \pi(\theta,m,n) \\ 1-(1-\theta)\left[\frac{1-z}{1-\pi(\theta,m,n)}\right]^{1/n}, & \text{for } \pi(\theta,m,n) \le z \le 1, \end{cases} \tag{4}$$

where we have for the cumulative probability of Y at θ:

$$\pi(\theta, m, n) = \frac{\theta n}{\theta n + (1-\theta)m}. \tag{5}$$

We provide several graphs for selected values of m and n and θ of the pdf (1) in Figure 1. Figures 1A and 1B demonstrate the addition of J-shaped forms. Figures 1D, 1E and 1F contrasts unimodal and U-shaped forms (Figure 1C) previously introduced by Van Dorp and Kotz (2002a, b) and Nadarajah (1999, 2003, 2005). For those distributions defined by Van Dorp and Kotz (2002a, b), the density value at θ (both in the anti-mode and mode case) equals the value of the power parameter n. For $n = 2$, the Van Dorp and Kotz (2002a, b) TSP reduces to an asymmetric triangular distribution (see Figure 1D), whereas the pdf's defined by Nadarajah (1999, 2003, 2005) do not. For $\theta = 0.5$, the pdfs in Van Dorp and Kotz (2002a, b) and Nadarajah (1999, 2003, 2005) coincide (see Figure 1E). Thus we conclude that the unique triangular member of the TSP family of Nadarajah (1999, 2003, 2005) is the symmetric triangular pdf. Finally, from Nadarajah's (1999, 2003, 2005) parameter restriction $\theta^{m-1} = (1-\theta)^{n-1}$ it follows that $0 < \theta < 0.5$, $m > 1$ $[0.5 < \theta < 1$, $m > 1]$ implies $m < n$ $[m > n]$. The effect of this property may be observed in Figures 1D and 1F from the Nadarajah TSP pdf's displayed in these figures.

A five parameters version of (1) follows via the linear scale transformation $X = (b-a)Y + a$ and possesses the pdf

$$f_X(x|\Theta) = \frac{mn(b-a)}{(b-\theta')m + (\theta'-a)n} \times \begin{cases} \left(\frac{x-a}{\theta'-a}\right)^{m-1}, & \text{for } a < x < \theta' \\ \left(\frac{b-x}{b-\theta'}\right)^{n-1}, & \text{for } \theta' \leq x < b \\ 0, & \text{elsewhere,} \end{cases} \tag{6}$$

where as above $\theta' = (b-a)\theta + a$. Now, the pdf (6) may be reparameterized using the parameter transformation

$$\begin{aligned} &a = \alpha - \beta\gamma\sqrt{\epsilon/\delta}, b = \alpha + \beta(1-\gamma)\sqrt{\delta/\epsilon} \\ &\theta = \alpha, m = \delta^{-1}, \, n = \epsilon^{-1}. \end{aligned} \tag{7}$$

resulting into the pdf

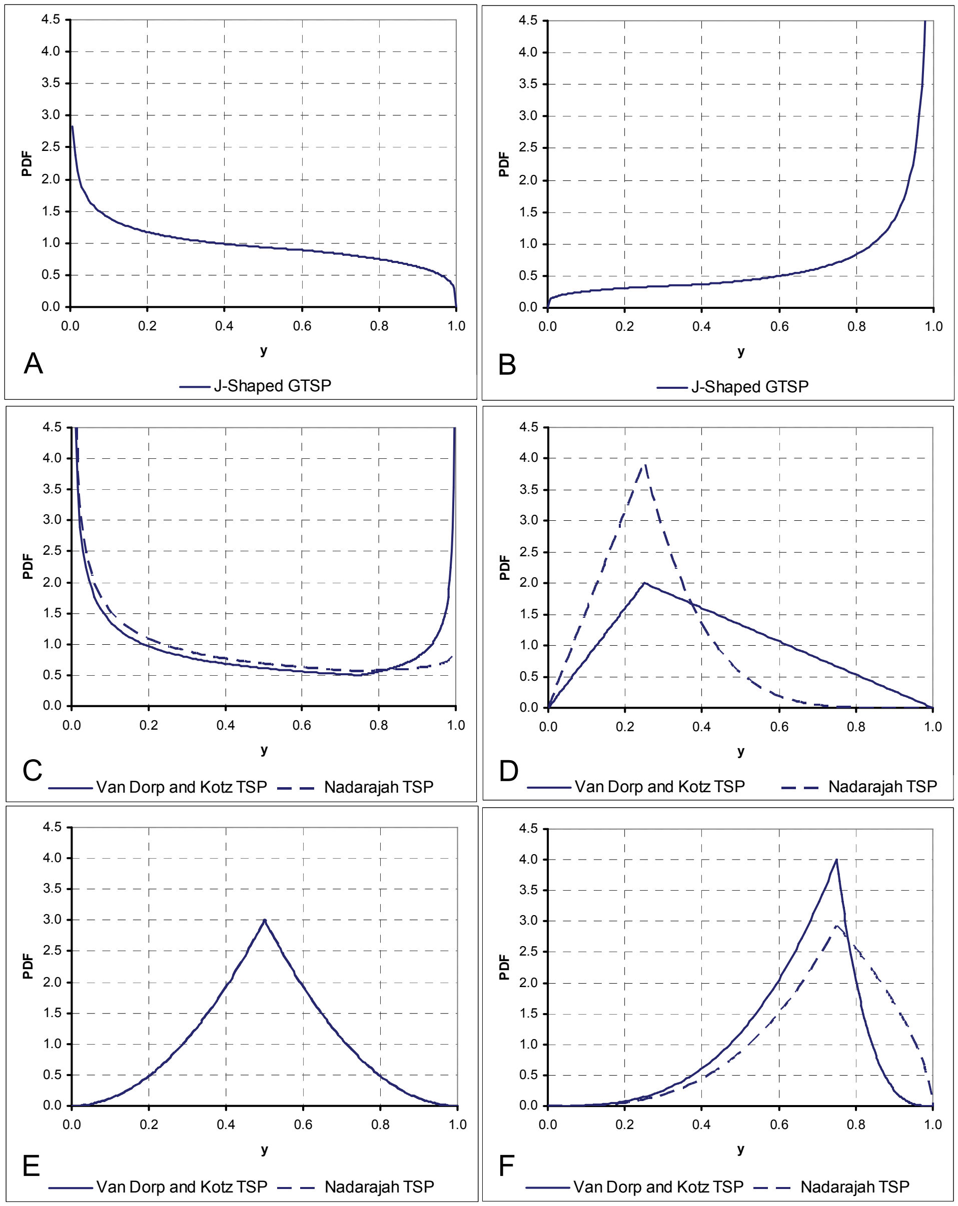

Figure 1. A graphical comparison of GTSP distributions, van Dorp and Kotz (2002a,b) TSP distributions and those defined Nadarajah (1999, 2003, 2005) A: GTSP ($\theta = 0.60, m = 1.25, n = 0.75$) B: GTSP ($\theta = 0.40, m = 0.25, n = 0.75$) C: Van Dorp and Kotz TSP ($\theta = 0.75, n = 0.50$), Nadarajah ($m = 0.50, n \approx 0.90, \theta = 0.75$); D: Van Dorp and Kotz TSP ($\theta = 0.25, n = 2$), Nadarajah ($\theta = 0.25$, $m = 2, n \approx 5.82$); E: Van Dorp and Kotz TSP ($\theta = 0.5, n = 3$), Nadarajah ($\theta = 0.5, m = n = 3$) $-$ the pdf's are identical and are superimposed in the plot; F: Van Dorp and Kotz TSP ($\theta = 0.75, n = 4$), Nadarajah ($\theta = 0.75, m = 4, n \approx 1.62$)

$$
f_X(x|\alpha,\beta,\delta,\epsilon,\gamma) = \left\{\beta\sqrt{\delta\epsilon}\right\}^{-1} \times \tag{8}
$$

$$
\begin{cases} \left[1+\frac{\sqrt{\delta/\epsilon}}{\beta\gamma}(x-\alpha)\right]^{\frac{1-\delta}{\delta}}, & \text{for } a(\alpha,\beta,\delta,\epsilon,\gamma),\ < x \leq \alpha \\ \left[1-\frac{\sqrt{\epsilon/\delta}}{\beta(1-\gamma)}(x-\alpha)\right]^{\frac{1-\epsilon}{\epsilon}}, & \text{for } \alpha \leq x < b(\alpha,\beta,\delta,\epsilon,\gamma) \\ 0, & \text{elsewhere,} \end{cases}
$$

$$
-\infty < \alpha < \infty,\ 0 < \beta < \infty,\ 0 \leq \gamma \leq 1 \text{ and } \delta, \epsilon > 0,
$$

where the boundary points a, b are:

$$
a(\alpha,\beta,\delta,\epsilon,\gamma) = \alpha - \beta\gamma\sqrt{\epsilon/\delta} \text{ and } b(\alpha,\beta,\delta,\epsilon,\gamma) = \alpha + \beta(1-\gamma)\sqrt{\delta/\epsilon}. \tag{9}
$$

The pdf (8) combined with (9) was in fact mentioned by Schmeisser and Lal (1985) in a technical report that only recently came to the attention of Kotz and Van Dorp (2004) through personal communication with these authors and thus far has not been referenced in their monograph or papers on novel distributions with a bounded support. While Schmeiser and Lal (1985), Nadarajah (1999) and Van Dorp and Kotz (2002a) arrived independently at their versions of two-sided power distributions (TSP) independently, evidently Schmeiser and Lal (1985) should be credited with the earliest discovery of even the three-parameter GTSP distribution.

In Section 2, we present general moment expressions for the pdf (1) and demonstrate its flexibility by developing a moment ratio diagram and a mean-μ, standard deviation-σ coverage plot. In Section 3 we develop an indirect elicitation algorithm for the power parameters of GTSP distributions (1) using a lower and upper quantile that could further serve its application in practical problems of risk and uncertainty. In preparation of a PERT example in Section 5, we present in Section 4 a novel comparison between classical PERT mean and variance expressions and those of beta and TSP distributions. We close Section 4 with recently derived closed form expressions for the skewness and kurtosis of TSP distributions. In Section 5, we present a Project Evaluation Review Technique (PERT) example that utilizes the elicitation algorithm of Section 3, exemplifies the effect of the PERT moment expressions in Section 4 and finally demonstrates the benefit of separate powers for the two branches of activity GTSP distributions in this context.

2. Moment expressions for GTSP distribution

Mixing distributions is common practice in dealing with e.g. Phase-Type, Erlang, Poisson and Normal distributions (see, e.g., Johnson and Taaffe (1991) and Karlis and Xekalaki (1999)). The pdf (1) may be expressed as a mixture involving two densities f_{Y_1}, f_{Y_2} with bounded support, such that

$$f_Y(y|\Phi) = \pi_1 f_{Y_2}(y|\,\theta, m) + \pi_2 f_{Y_2}(y|\,\theta, m), \quad \pi_1 + \pi_2 = 1,\ \pi_1, \pi_2 > 0. \tag{10}$$

Here :

$$f_{Y_1}(y|\Phi) = f_{Y_1}(y|a, b, m) = \left(\frac{m}{\theta}\right)\left(\frac{y}{\theta}\right)^{m-1}, \quad 0 \le y < \theta,\ m > 0, \tag{11}$$

$$f_{Y_2}(x|\Phi) = f_{Y_2}(x|c, d, n) = \left(\frac{n}{1-\theta}\right)\left(\frac{1-y}{1-\theta}\right)^{n-1}, \quad \theta \le x < 1,\ n > 0. \tag{12}$$

The mixture probabilities π_i, $i = 1, 2$, are

$$\begin{cases} \pi_1 = \pi(\theta, m, n), \\ \pi_2 = 1 - \pi(\theta, m, n), \end{cases} \tag{13}$$

where $\pi(\theta, m, n)$ is given by (5). Observe that the mixture weight of the first branch π_1 decreases as its tail parameter m increases (keeping the tail parameter of the second stage n fixed). A similar observation can be made for the second stage with obvious modification. We have from (10), (11) and (13)

$$\pi_1 f_{X_1}(\theta|\Phi) = \pi_2 f_{X_2}(\theta|\Phi) = \frac{mn}{(1-\theta)m + \theta n} \tag{14}$$

and thus continuity of the mixture (10) (and consequently the GTSP distribution (1)) at the threshold θ follows from (14) despite separate power parameters for its two branches.

By taking advantage of the mixture structure (10), the following general moment expression may be derived for the pdf (1) :

$$E[X^k] = \pi(\theta,m,n)\left[\frac{m\theta^k}{m+k}\right] + \qquad (15)$$
$$[1-\pi(\theta,m,n)]\left[n\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{(1-\theta)^i}{n+i}\right], k=1,2,\ldots$$

From (15) we obtain the following expressions for the mean and variance of GTSP distributions:

$$E[X] = m\frac{n(n+1)\theta^2 + (m+1)(1-\theta)(n\theta+1)}{(m+1)(n+1)\{(1-\theta)m+\theta n\}}, \qquad (16)$$

$$Var[X] = \qquad (17)$$
$$\frac{m\{n(n+1)(n-m)\theta^3 + (m+2)n(n-1)\theta^2 + 2(m+2)(n-1)\theta + 2m+4\}}{(m+1)(n+1)(n+2)\{(1-\theta)m+\theta n\}}$$
$$-\frac{m^2\{n(n-m\}\theta^2 + (m+1)(n-1)\theta + m+1\}^2}{(m+2)^2(n+1)^2\{(1-\theta)m+\theta n\}^2}.$$

Expression (15) also allows for a straightforward evaluation of other classical measures such as the skewness and kurtosis $\sqrt{\beta_1}$ and β_2. By substituting $m=n$, expressions (16), (17) reduce to the less cumbersome mean and variance expressions of Van Dorp and Kotz (2002a, b, 2004) TSP distributions:

$$E[X] = \frac{(n-1)\theta+1}{n+1}, \qquad (18)$$

$$Var[X] = \frac{n-2(n-1)\theta(1-\theta)}{(n+2)(n+1)^2} \qquad (19)$$

Comparing (16) and (18) we note that the numerator of (16) ((18)) is a linear (quadratic) function of θ while the denominators are independent of θ (16) or a linear function of θ (18). We challenge the reader to simplify (17) and compare it with (19).

2.1 A graphical comparison of the first four moments

Before presenting a graphical comparison of $\sqrt{\beta_1}$ and β_2 for the GTSP pdf's (1) in a moment ratio diagram, we note that it shares the power distribution [with pdf nx^{n-1}, $0 \le x \le 1$] and the reflected power distribution [with pdf $n(1-x)^{n-1}$, $0 \le x \le 1$]. Moreover, the pdf's (1) have

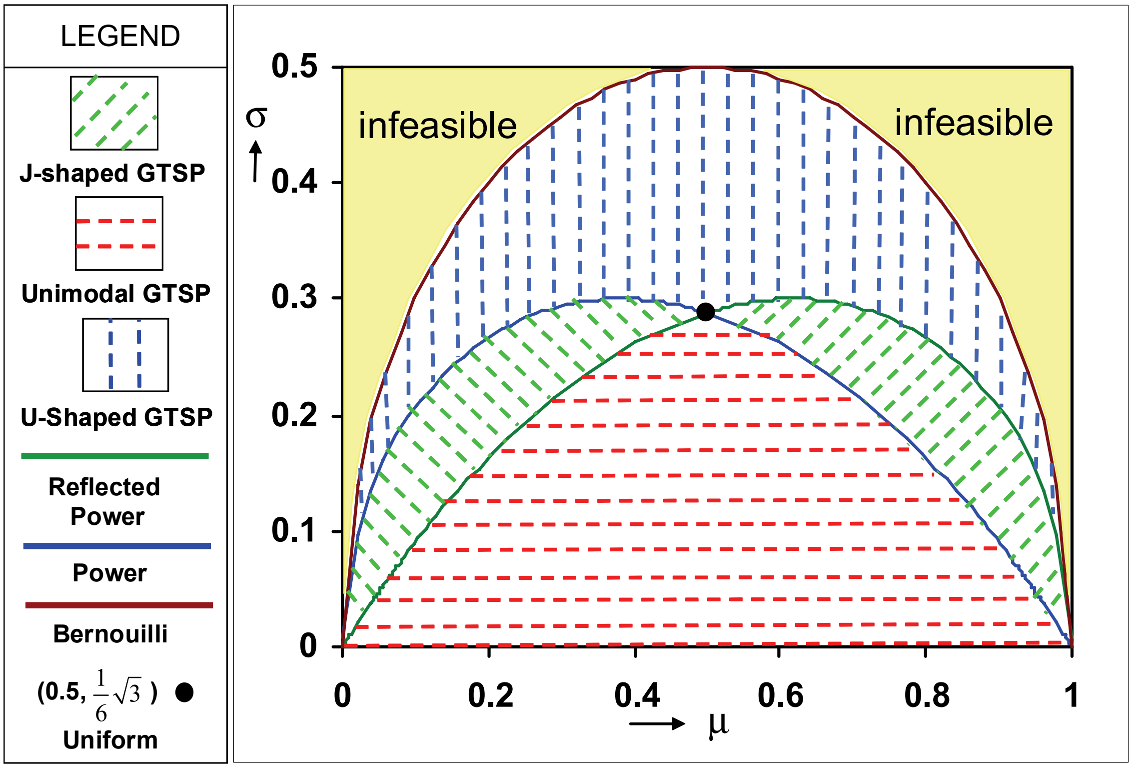

Figure 2. (μ, σ) coverage diagrams for the pdf's (1).

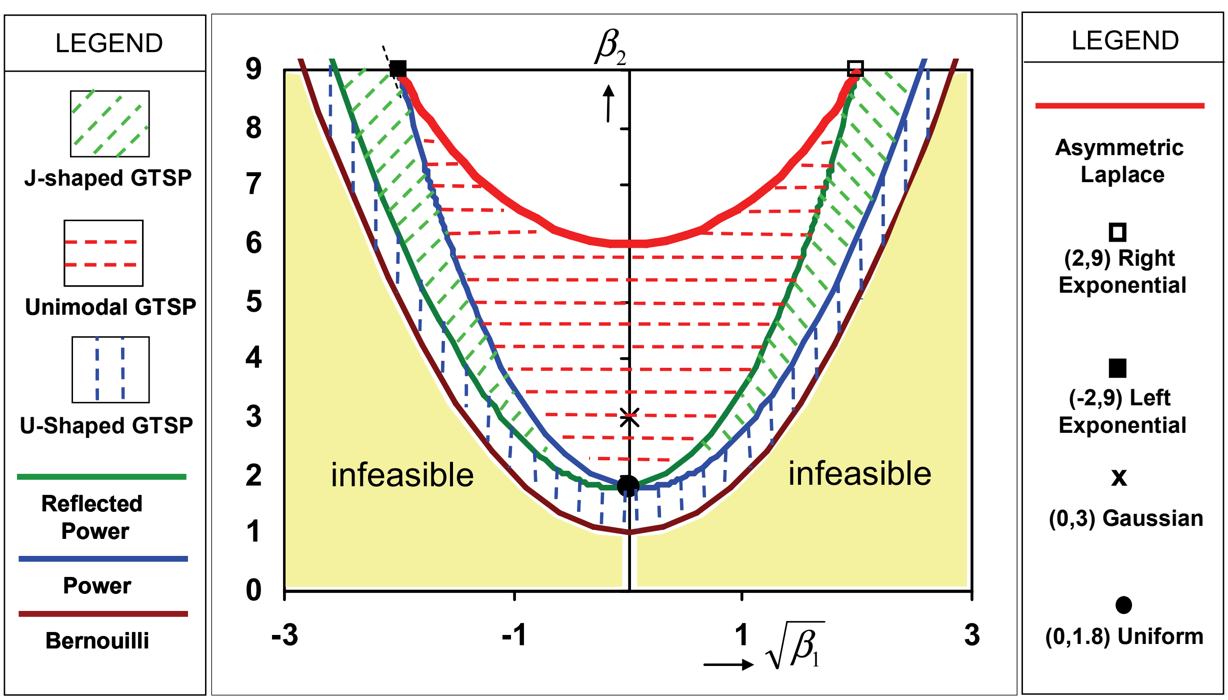

Figure 3. Moment Ratio $(\sqrt{\beta_1}, \beta_2)$ coverage diagrams for the pdf's (1).

Bernoulli(p), asymmetric Laplace and degenerate distributions with single point mass at θ as their limiting distributions. Turning to the diagrams, we present in Figure 2 a (μ, σ)-diagram, where $\mu = E[X]$ and $\sigma = \sqrt{Var[X]}$ and in Figure 3 a moment ratio diagram for the pdf's (1) for their entire parameter ranges (since the boundaries are established by the limiting distributions above). Nadarajah (2005) and Van Dorp and Kotz (2002a, 2004) considered only the parameter ranges $0.1 \leq n, m \leq 25,\ 0 \leq \theta \leq 1$ in their moment ratio diagrams.

Firstly, one observes that GTSP distributions completely cover the feasible ranges of the (μ, σ)-diagram in Figure 2 demonstrating its flexibility. Moreover, we observe that the moment ratio coverage in Figure 3 now also provides coverage in the J-shaped area also covered by the beta distribution (see (25) for its pdf), but not previously covered by either version of the TSP distributions (see, Nadarajah (2005) and Van Dorp and Kotz (2002a, 2004)). Indeed, we have from Nadarajah's (2005) parameter restriction $\theta^{m-1} = (1-\theta)^{n-1}$ that

$$m > 1 \Leftrightarrow n > 1 \quad \wedge \quad 0 < m < 1 \Leftrightarrow 0 < m < 1 \tag{20}$$

and thus the Nadarajah (2005) TSP pdf's similarly to the Van Dorp and Kotz (2002a, 2004)) pdf's only admit for unimodal and U-shaped forms (not J-shaped).

3. An Indirect Elicitation Method for GTSP power parameters.

To facilitate the application of GTSP distributions in problems of risk and uncertainty (such as the PERT method) when data to estimate distributional parameters is not necessarily available, we shall develop a procedure to elicit the power parameters m and n in an indirect manner. In a recent survey paper a leading Bayesian statistician, O'Hagan (2006), explicitly mentions a need for advances in elicitation techniques for prior distributions in Bayesian Analyses, but also acknowledges the importance of their development for those areas where the elicited distribution can not be combined with evidence from data, because the expert opinion is essentially all the available knowledge.

We shall assume here that (similar to the PERT field) that a lower bound a, most likely estimate θ' and upper bound b have been elicited directly from a substantive expert. To elicit the power parameters m and n in an indirect manner we suggest eliciting a lower quantile $x_p < \theta'$ and an upper quantile $x_r > \theta'$. We propose $p = 0.10$ and $r = 0.90$, although our elicitation method works equally well for other popular values, e.g. $p = 0.05$ and $r = 0.95$. In the procedure below, we shall assume that the x_p, x_r and θ' values have been standardized to values y_p, y_r and θ in the domain $(0, 1)$ using the linear transformation $(x - a)/(b - a)$. This allows us to work with the standardized pdf (1). We obtain directly from the cdf (3) the following set of non-linear equations (the quantile constraints)

$$\begin{cases} F(x_p|\theta, m, n) = \pi(\theta, m, n)\left(\frac{\boldsymbol{x_p}}{\theta}\right)^m = p, \\ F(x_r|\theta, m, n) = 1 - [1 - \pi(\theta, m, n)]\left(\frac{1-\boldsymbol{x_r}}{1-\theta}\right)^n = r, \end{cases} \tag{21}$$

from which the parameters m and n need to be solved. From $p < \pi(\theta, m, n)$ given by (5) we immediately obtain the following upper bound for m given a fixed value of $n > 0$ and a specified quantile x_p and quantile level p :

$$m < U_m(n, p, \theta) = n \times \frac{1-p}{p} \times \frac{\theta}{1-\theta}. \tag{22}$$

Moreover, we have that

$$F(x_p|\theta, m, n) \to 1 \text{ when } m \downarrow 0 \tag{23}$$

and since $(x_p/\theta) < 1$ it follows that

$$F(x_p|\theta, U_m(n, p, \theta), n) < p. \tag{24}$$

Hence, from (23), (24) it follows that the first equation in (21) has a unique solution $m^\bullet$ for every fixed value of $n > 0$ and thus it defines an implicit continuous function $\xi(n)$ such that the parameter combination $\{\theta, m^\bullet = \xi(n),\ n\}$ satisfies the first quantile constraint for all $n > 0$. The unique solution $m^\bullet$ may be solved for by employing a standard root finding algorithm such as, for example, GoalSeek in Microsoft Excel. Analogously, the second equation defines an implicit continuous function $\zeta(m)$ such that the parameter combination $(\theta, m,\ n^\bullet = \zeta(m))$ satisfies the second quantile constraint for all $m > 0$. We propose the following direct algorithm solving (21):

Step 1: Set $n^{\bullet} = 1$.

Step 2: Calculate $m^{\bullet} = \xi(n^{\bullet})$ (satisfying for the first quantile constraint in (21)).

Step 3: Calculate $n^{\bullet} = \zeta(m^{\bullet})$ (satisfying for the second quantile constraint in (21)).

Step 4: If $\left|\pi(\theta, m^{\bullet}, n^{\bullet})\left(\frac{x_p}{\theta}\right)^{m^{\bullet}} - p\right| < \epsilon$ Then Stop Else Goto Step 2.

Setting, e.g., $\theta = 0.30$, $y_p = 0.25$, $p = 0.10$, $y_r = 0.85$ and $r = 0.90$ in the algorithm above yields the power parameters $m = 2.884$ and $n = 1.371$. Figures 4A and 4D plot the GTSP pdf and cdf possessing these parameters. (A Microsoft Excel spreadsheet with an implementation of the above algorithm is available from the authors upon request.)

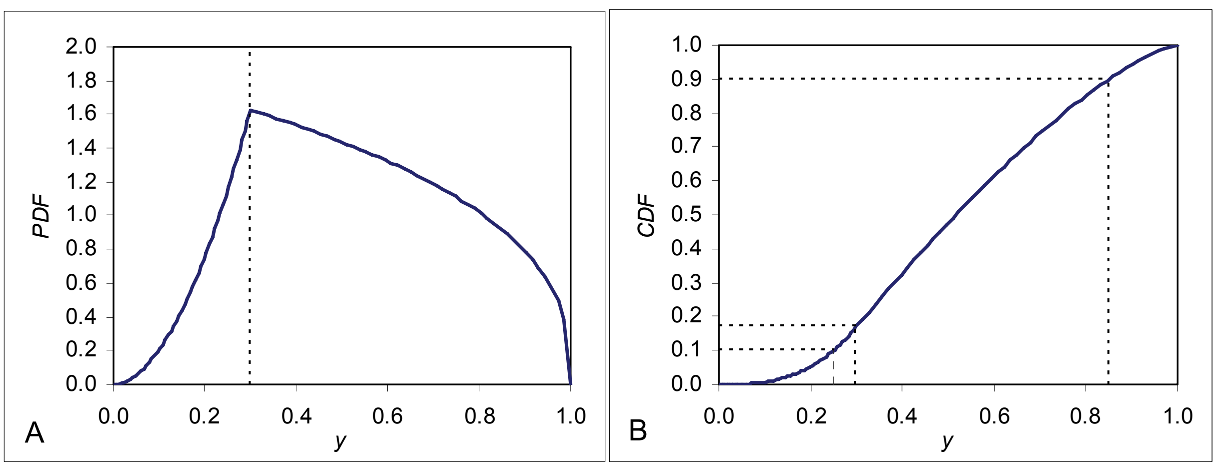

Figure 4. GTSP pdf with parameters $\theta = 0.30$, $m = 2.884$, $n = 1.371$ satisfying the quantile constraints $y_{0.10} = 0.25$ and $y_{0.90} = 0.85$.

We shall utilize the elicitation algorithm above to solve for the power parameters of the GTSP distribution in a PERT example to be discussed in Section 5. We shall demonstrate the additional flexibility of the GTSP distribution by means of a Monte Carlo analysis for the completion time of an activity network (see, e.g., Elmaghraby (1978)) in this example and by contrasting it to earlier suggested methods for distribution parameter specifications given a lower bound, most likely and upper bound for the activities in the activity network. A comparison of these earlier methods is presented in the next section.

4. A comparison of PERT mean and variance expressions of beta and TSP distributions.

The three parameter triangular distribution with lower and upper bounds a and b and most likely value θ' is one of the first continuous distributions on the bounded range proposed back in 1755 by English mathematician Thomas Simpson. It received special attention as late as in the 1960's, in the context of the PERT (see, e.g., Winston (1993)) as an alternative to the four-parameter beta distribution

$$\begin{array}{l} f_T(t|a,b;\alpha,\beta) \;=\; \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{(t-a)^{\alpha-1}(b-t)^{\beta-1}}{(b-a)^{\alpha+\beta-1}}, \\ a \le t \le b,\ \alpha > 0,\ \beta > 0. \end{array} \tag{25}$$

which involves some difficulties regarding the interpretation of its parameters α and β. As a result Malcolm et. al 1959[4] suggested to use the following PERT mean and variance expressions

$$\begin{array}{l} E[T] = \frac{a+4\theta'+b}{6} \\ Var[T] = \frac{1}{36}(b-a)^2, \end{array} \tag{26}$$

where T is a random variable modeling an activities completion time, a and b being the lower and upper bound estimates and θ' is a most like estimate for T. The remaining beta parameters α and β in (25) are obtained from (26) utilizing the method of moments. This somewhat non-rigorous use of (26) has resulted in what we call a 40 year PERT "controversy" (see, e.g., Clark 1962, Grubbs 1962, Moder and Rogers 1968, Elmaghraby 1978, Keefer and Verdini 1993, Kamburowski 1997, Johnson 1997, Lau et. al 1998, Herrerías et. al 2003 and García et. al 2005, among others) in connection with the estimation of the parameters α and β of the beta distribution (25).

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[4]Kamburowski (1997) notes that: "*Despite the criticisms and the abundance of new estimates, the PERT mean and variance* [given by (26) in this paper] *can be found in almost every textbook on OR/MS and P/OM, and are employed in much project management software."*

In response to the criticism of using the approximation (26), Herrerías (1989) suggested substituting

$$\alpha = 1 + s\,\frac{\theta' - a}{b - a},\ \beta = 1 + s\,\frac{b - \theta'}{b - a}, \tag{27}$$

where $s > -1,\ a < \theta' < b$ in the (four-parameter) beta pdf (25). This yields

$$E[T] = \frac{a + s\,\theta' + b}{s + 2} \tag{28}$$

and

$$Var[T] = \frac{(s+1)(b-a)^2 + s^2(b-\theta')(\theta'-a)}{(s+3)(s+2)^2}. \tag{29}$$

Essentially, Herrerías (1989) reparameterizes the beta pdf (25) by managing to express α and β in terms of new parameters θ' and s while retaining the lower and upper bounds a and b. Note that for $s = 4,\ E[T]$ in (28) reduces to the expression for $E[T]$ in (26), but enhances the variance expression in (26) by taking advantage of the mode location in (29). For $s > 0$ $(0 < s < 1)$, the beta pdf (25) is unimodal (U-shaped) and for $s = 0$ it reduces to a uniform distribution. Hence, Herrerías (1989) designated s to be a confidence parameter in the mode location θ' such that higher values of s indicate a higher confidence. Indeed, for $s \to \infty$, the beta pdf converges to a single point mass at θ'.

On the other hand, we obtain for the four parameter TSP pdf the following expressions for the mean and the variance, respectively:

$$E[X] = \frac{a + (n-1)\theta' + b}{n+1}, \tag{30}$$

$$Var[X] = \frac{n(b-a)^2 - 2(n-1)(b-\theta')(\theta'-a)}{(n+2)(n+1)^2}, \tag{31}$$

where $n > 0,\ a < \theta' < b$. These are of course exact values. We immediately observe that by substituting $n = s + 1$ in (30) and (31), the beta mean value (28) and TSP mean value (30) coincide and as above reduce to $E[T]$ in (26) for $s = 4$ or $n = 5,$ respectively. Malcolm *et al.* (1959) were indeed lucky in this respect. However, after some algebraic manipulations we obtain:

$$Var[T] - Var[X] = \frac{(n-1)(b-\theta')(\theta'-a)}{(n+2)(n+1)} = \begin{cases} \leq 0, & 0 \leq n < 1, \\ > 0, & n > 1. \end{cases} \tag{32}$$

Hence, in the unimodal [U-shaped] domains of the TSP $(n > 1)$ and the beta distributions $(s > 0)$ in (25), we have that the variance of the TSP distribution is strictly less [larger] than the PERT variance modification of Herrerías (1989) given by (29) (perhaps adding to the 40-year controversy). This result is consistent with the TSP distributions being more "peaked" than the beta distribution (see, e.g. Kotz and Van Dorp 2004). Unfortunately, Malcolm et al. (1959) were after all not so lucky.

4.1 Some additional results for TSP moments.

We close Section 4 with some novel results regarding TSP moments. From (15) we derive for the Van Dorp and Kotz (2002a, b, 2004) TSP pdf, the following recurrence relationship for $\alpha_k = E[Y^k]$:

$$\alpha_k = \frac{1}{n+k}\left(k\alpha_{k-1} + \theta^k \frac{n(n-1)}{n+k-1}\right). \tag{33}$$

The recurrence relations (33) relies heavily on the $m = n$ property of the Van Dorp and Kotz (2002a, b, 2004) TSP pdf. Taking advantage of this same property, we recently were able to obtain closed form expressions for the skewness and kurtosis for the Van Dorp and Kotz (2002a, b, 2004) TSP distributions of which are given by:

$$\sqrt{\beta_1} = sign\{(\frac{1}{2} - \theta)(n-1)\} \times \frac{2(1-2\theta)(n-1)}{n+3} \times \\ \{n - (n-3)(1-\theta)\theta\} \times \sqrt{(n+2)/\{n-2(n-1)(1-\theta)\theta\}^3}\,, \tag{34}$$

$$\beta_2 = \frac{n+2}{(n+3)(n+4)} \times \frac{1}{\{n-2(n-1)(1-\theta)\theta\}^2} \times \Big\{3(2+n(3n-1)[1+(n-1)\{1-2(1-\theta)\theta\}^2] - 12(n-1)^2(n+4)(1-\theta)^2\theta^2\Big\}. \quad (35)$$

Expressions (33) - (35) have not been mentioned before in any of the previous publications dealing with Van Dorp and Kotz (2002a, b, 2004)) TSP distributions. We have not been able to derive a similar expressions for GTSP distributions nor for the Nadarajah (1999, 2003, 2005) TSP distributions.

5. A PERT Example

Figure 5 shows an 18-activity example project network representing a shipbuilding project from Taggart (1980). Our starting point for our PERT example shall be the parameters a, m and b for the 18 activity durations as provided in Table 1. Typically, these values are elicited from substantive experts knowledgeable about the subject matter, but whom may not necessarily be trained in quantitative methods to translate these activity durations estimates into a completion time distribution for an activity network. This latter task is reserved for what we would like to refer to as a normative expert. Table 1 also provides the parameters α and β for beta distributions (25) estimated from a, m and b via the method of Malcolm et al. (1959) that utilizes equation (26).

Figure 6A provides project network completion time distributions for two different scenarios. These completion time distributions were constructed from 25000 independent samples from the activity duration distributions and subsequently applying the Critical Path Method for each sample. This results for each scenario in 25000 completion time samples for the project network in Figure 5 from which the project completion time distribution for each scenario is constructed. For the first scenario activity durations were sampled from by triangular distributions with parameters a, m and b as specified in Table 1. For the second scenario in Figure 6A they were sampled from beta distributions (25) with parameters a, b, α and β as specified in Table 1.

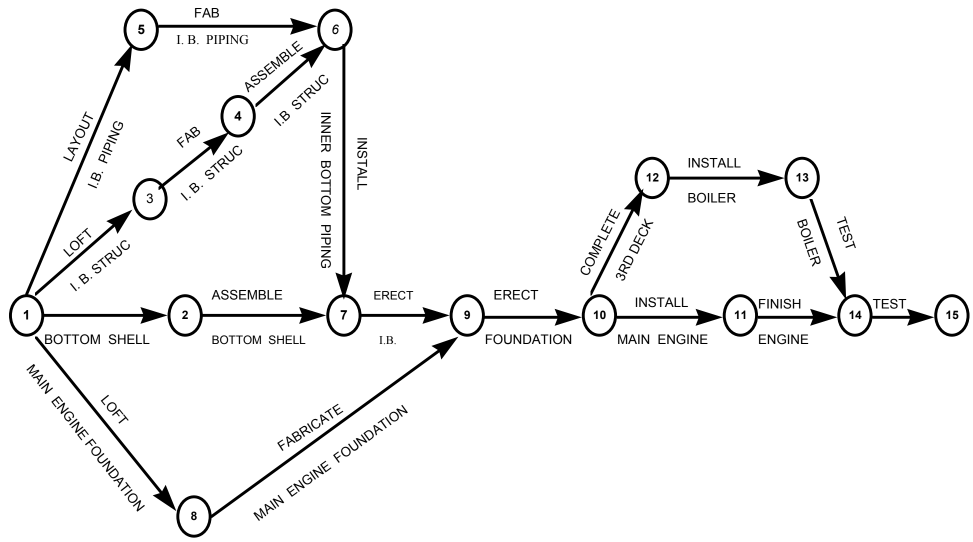

Figure 5. Example project network from Taggart (1980).

Table 1. Data for modeling the uncertainty in activity durations for the project network presented in Figure 5. Beta parameters were determined using Malcolm et al. (1959) expression (26).

ID	Activity Name	a	m	b	α	β
1	Shell: Loft	22	25	31	2.94	4.62
2	Shell: Assemble	35	38	43	3.23	4.52
3	I.B.Piping: Layout	26	27	40	1.08	3.98
4	I.B.Piping: Fab.	6	7	15	1.34	4.24
5	I.B.Structure: Layout	23	24	30	1.56	4.40
6	I.B.Structure: Fab.	14	18	24	3.40	4.44
7	I.B.Structure: Assemb.	9	14	20	3.74	4.22
8	I.B.Structure: Install	5	7	13	2.33	4.67
9	Mach Fdn. Loft	26	28	33	2.59	4.67
10	Mach Fdn. Fabricate	29	30	42	1.12	4.02
11	Erect I.B.	27	30	37	2.70	4.66
12	Erect Foundation	5	7	14	2.13	4.64
13	Complete #rd DK	4	5	9	1.97	4.59
14	Boiler:Install	6	7	12	1.73	4.49
15	Boiler:Test	9	10	16	1.56	4.40
16	Engine: Install	6	7	15	1.34	4.24
17	Engine: Finish	19	20	26	1.56	4.40
18	FINAL Test	13	15	24	1.84	4.54

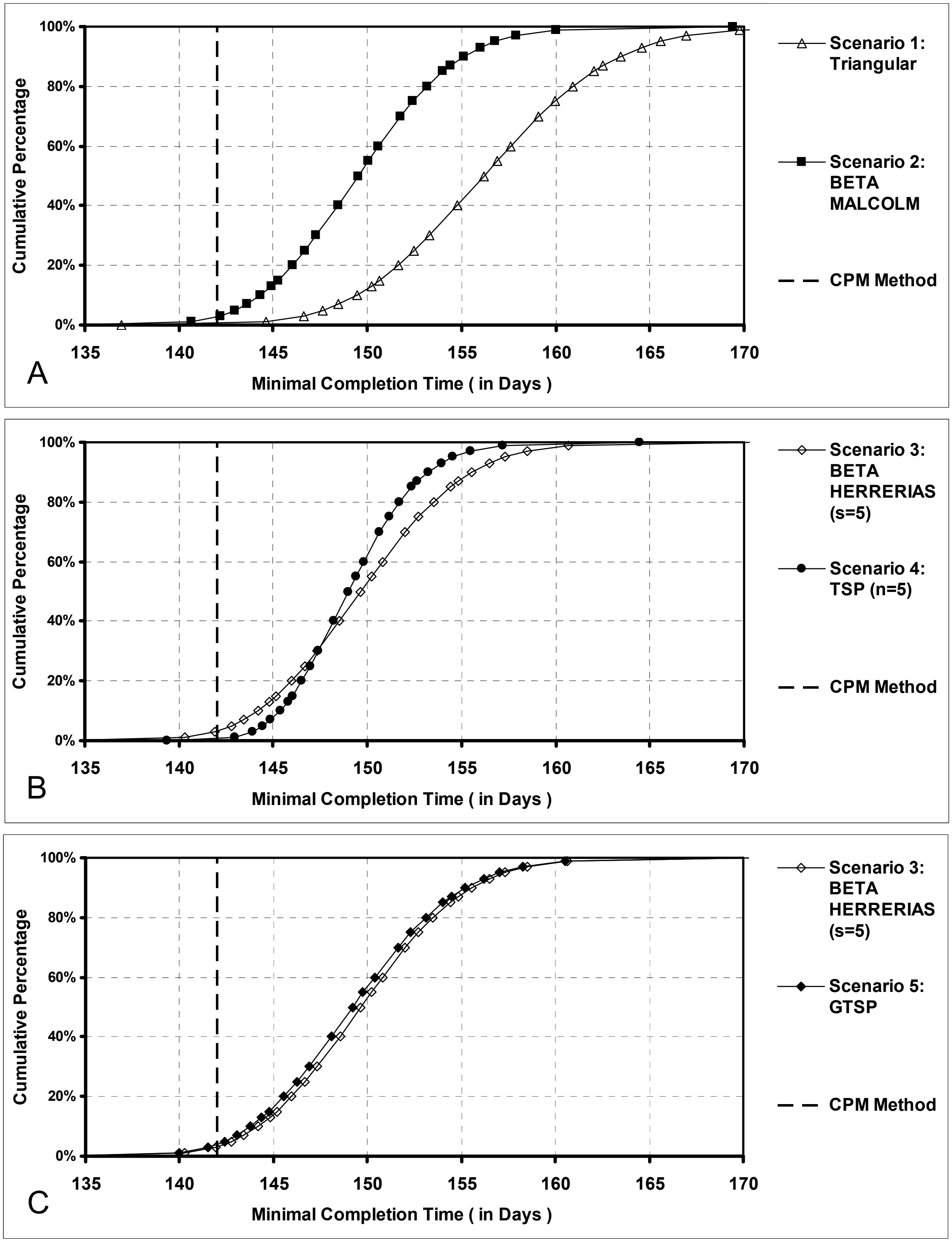

Figure 6. Completion Time Distributions for the project network in Figure 5 for the scenarios: Triangular, Beta (Malcolm), Beta (Herrerías), TSP (n=4) and GTSP.

It is worthwhile noting here that the beta distributions estimated using Malcolm's (1959) method do not share the most likely estimates m in Table 1, whereas the triangular distributions of course do. Figure 6A also provides the completion time of 142 days for the project network in Figure 5 that follows by only using the most likely values m for the activity durations in this project network.

Two important conclusions follow from Figure 6A. Firstly, we immediately observe the importance of modeling uncertainty in the activity durations since for example here we evaluate a probability of less than 3% of making the deadline of 142 that would have followed from using only the most likely values m (and ignoring activity duration uncertainty). It should be noted that this small probability arises in this example since all most likely values of the activity durations in Table 1 are slanted towards the lower bound (in other words the activity durations uncertainties have long right tails). One could argue that this may be prevalent in practice when a, m and b are estimated via the expert judgment method, where "job pressures" to win a contract may in fact bias an expert to provide under estimated most likely estimates m that do tend more towards the lower a than upper bounds b.

Secondly, we observe from Figure 6A the remarkable large difference between the two completion time distributions for this project network. For example, while the Malcolm et al. (1959) scenario here has a approximately a 90% probability of finishing before 155 days, the scenario with triangular distributions has only approximately a 42% probability of finishing before that time. Such a large differences explains at least partially the 40-year controversy surrounding Malcolm's et al. (1959) suggestion provided by equation (25) (see, e.g., Clark 1962, Grubbs 1962, Moder and Rogers 1968, Elmaghraby 1978, Keefer and Verdini 1993, Kamburowski 1997, Johnson 1997, Lau et. al 1998, Herrerías et. al 2003 and García et. al 2005, among others). Indeed, a normative expert ought not be comfortable with such observed large differences that result from the assumption (25) and a subsequent modeling assumption of activity uncertainty via beta distributions.

Figure 6B displays completion time distributions for the project network for the scenario where beta distribution parameters α and β were determined using the proposed method of Herrerías (1989). This method is certainly an improvement over Malcolm's et. al (1959) method since the variance estimation in Herrerías' (1989) equations (27) and (28) takes advantage of the skewness as specified by the estimates a, m and b, but more importantly the parameters s in Herrerías' (1989) method is introduced to be elicited via expert judgement as a measure of confidence in the most likely value compared to the bound parameters, rather than being specified (as is the case in Malcolm's (1959) method). For comparison purposes, we have set $s = 4$ in Figure 6B for all activity durations for the Herrerías' (1989) scenario. The resulting parameters α and β for the beta distributions (24) are specified in Table 2. It should be noted that similar to the Malcolm et al. (1959) scenario that these beta distributions estimated using Herrerías'(1959) method also do not share the most likely estimates m in Table 1.

Table 2. Beta Parameters and GTSP parameters for activity duration distributions that have quantiles a_p and b_{1-p} in common in addition to the lower and upper bound parameters specified in Table 1. Beta parameters were determined using Herrerías (1989) expressions (28) and (29).

ID	α	β	$a_{0.10}$	$b_{0.90}$	m	n
1	2.333	3.667	23.38	27.80	1.86	2.78
2	2.500	3.500	23.57	28.06	1.96	2.64
3	1.286	4.714	22.37	25.93	1.26	4.15
4	1.444	4.556	22.49	26.25	1.36	3.86
5	1.571	4.429	22.60	26.49	1.43	3.63
6	2.600	3.400	23.70	28.21	2.02	2.56
7	2.818	3.182	23.98	28.53	2.16	2.40
8	2.000	4.000	23.01	27.26	1.68	3.09
9	2.143	3.857	23.16	27.49	1.76	2.95
10	1.308	4.692	22.39	25.98	1.28	4.10
11	2.200	3.800	23.23	27.59	1.78	2.90
12	1.889	4.111	22.90	27.06	1.60	3.22
13	1.800	4.200	22.81	26.91	1.56	3.31
14	1.667	4.333	22.68	26.67	1.48	3.49
15	1.571	4.429	22.60	26.49	1.43	3.62
16	1.444	4.556	22.49	26.25	1.35	3.85
17	1.571	4.429	22.60	26.49	1.43	3.63
18	1.727	4.273	22.74	26.78	1.52	3.41

In Figure 6B, we contrast the Herrerías' (1989) scenario to the a scenario where activity durations are modeled via TSP distributions with parameters a, m and b in Table 1 with their single power parameters set to $n = 5$ (The power parameter of a TSP distribution could be elicited in a similar manner as the parameter s in Herrerías' (1989) method). In that case, the mean values of the TSP activity durations agree with those in the Herrerías' (1989) method, but from (31) it follows that the variances for TSP ($n = 5$) distribution are smaller. This then results in a difference in project completion time distributions as displayed in Figure 6B, where the TSP scenario project completion time distribution is more steep and thus exhibits less overall variance. While the difference in project completion time distributions is much less than that exhibited in Figure 6A, it could be considered substantial and could still leave a normative expert with the perhaps uncomfortable choice between beta distributions using the Herrerías' (1989) method or TSP distributions to model activity duration uncertainty.

Finally, in Figure 6C we compare the Herrerías' (1989) scenario project completion time distribution to one where activity duration uncertainties are modeled using the GTSP distribution (1) with two power parameters. Similar to the TSP scenario in Figure 6B, these distributions share the values a, m and b in Table 1. The power parameters m and n for the GTSP distributions are specified in Table 2. They are solved from a lower quantile a_p and upper quantile b_{1-p} where $p = 0.10$ using the algorithm presented in Section 3.2. For comparison, these 10% and 90% quantiles are determined here from the beta distributions in the Herrerías' (1989) as opposed to having been elicited from expert judgement. We immediately observe the small difference between the completion time distributions in Figure 6C as compared to those in Figures 7A and 7B.

Summarizing, we arrive at the overall conclusion that the information a, m and b elicited from substantive expert does not provide a sufficient amount of information for a normative expert to perhaps comfortably select a matching uncertainty distribution. In this paper, we offer the alternative of eliciting additional quantiles a_p and b_{1-p} and modeling activity duration uncertainty in a PERT context via GTSP distributions with two power parameters. Needless to say this method of elicitation and the specification of distributional parameters via expert judgement is not unique to

PERT, but also applies to problems of risk and uncertainty in general that use the Monte Carlo method for propagating uncertainty of input parameters through quantitative models to ultimately evaluate an output parameter's uncertainty. In PERT and our example above the output parameter of interest is the project's completion time.

5. Concluding Remarks

We trust that this paper clarifies the chronology of developments regarding the two-sided power distribution including the original and unpublished note by Schmeisser and Lal (1985). The GTSP pdf enhanced available distributional shapes offered by the Nadarajah (1999, 2003, 2005) and Van Dorp and Kotz (2002a, b, 2004)) versions of TSP distribution by allowing also for J-shaped forms. While GTSP and the Van Dorp and Kotz (2002a, b, 2004)) TSP distributions share both the asymmetric and symmetric triangular distributions as its members (and may thus be viewed as generalizations of triangular distributions), the Nadarajah (1999, 2003, 2005) TSP pdf only shares the symmetric triangular distribution. We hope that the elicitation algorithm presented in this paper for the two power parameters of the GTSP distribution further facilitates its applications of problems involving risk and uncertainty in the absence of data and when distributional parameters need to be estimated using the expert judgement technique.

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