Revisiting the PERT mean and variance

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Abstract: Difficulties with the interpretation of the parameters of the beta distribution let Malcolm et al. (1959) to suggest in the Program Evaluation and Review Technique (PERT) their by now classical expressions for the mean and variance for activity completion for practical applications. In this note, we shall provide an alternative for the PERT variance expression addressing a concern raised by Hahn (2008) regarding the constant PERT variance assumption given the range for an activity's duration, while retaining the original PERT mean expression. Moreover, our approach ensures that an activity's elicited most likely value aligns with the beta distribution's mode. While this was the original intent of Malcolm et al. 1959, their method of selecting beta parameters via the PERT mean and variance is not consistent in this manner.

Keywords: Project scheduling, PERT, beta distribution, expert judgment.

1. INTRODUCTION

The well known beta probability density function (pdf)

$$f(x|a,b,p,q) = \frac{(b-a)^{1-(p+q)}}{\mathcal{B}(p,q)}(x-a)^{p-1}(b-x)^{q-1}. \tag{1}$$

may be characterized by its flexibility, but unfortunately suffers from some difficulty in specifying the parameters p and q when no data is available to estimate them via a statistical means. In that case one has to rely on expert judgment to specify p and q which led Malcolm et al. (1959) to suggest the classical PERT mean and variance

$$E[X|a,m,b] = \frac{a+4m+b}{6},\ V[X|a,m,b] = \frac{(b-a)^2}{36}, \tag{2}$$

to solve for p and q given expert elicited lower and upper bounds estimates a and b and a most likely value m. Despite its long history, the PERT method is still an active research domain perhaps because PERT and the original PERT mean and variance (2) are still prominently displayed in popular operations research college text books (see, e.g., Winston 2004, Hillier and Hillier 2008).

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Originally, the use of (2) was advocated due to similarities of resulting beta distributions to the normal distributions. A further justification along these lines for (2) was provided over 30 years later by Kamburowski (1997) and Herrerías et al. (2003) who demonstrated that the particular asymmetric PERT beta distribution members that have a mode m and match (2), possess a kurtosis value of 3.0 which coincides with the kurtosis of a normal distribution. Recently, Hahn (2008) suggested a uniform-beta mixture as an alternative for the original beta distribution model (1) while calling into question the constant variance assumption in (2) given an activity's support $[a, b]$: "*One of the more notable limitations of the PERT beta involves the variance, the most typically encountered measure of uncertainty. By the third assumption of Littlefield and Randolph (1987) above, the variance in (3) is constant conditional on the range. This may be in direct conflict with reality*". Hahn's concern extends a vigorous debate spanning over 45 years regarding the appropriateness of the somewhat non-rigorous proposition (2) (Grubbs 1962, Kotiah and Wallace 1973, Sasieni 1986, Littlefield and Randolph 1987, Golenko-Ginzburg 1989, Kamburowski 1997, Premachandra 2001 and Herrerías et al. 2003). Moreover, while it was originally the intent of original PERT developers (Malcolm et al., 1959) that the elicited most likely value m matches the beta mode, they do not match in all but three cases when selecting parameters p and q via (2). In principle, normative experts are tasked with specifying distributions that are consistent with the substantive expert's judgment.

Restricting ourselves without loss of generality to beta distributions with support $[0, 1]$ and a mode less than $\frac{1}{2}$, those beta distributions where $1 < p < 3 - \sqrt{2}$ and match the PERT mean and variance (2) have a mode strictly less than the elicited value m (see Figure 1A). Beta distributions satisfying the same properties, but where $3 - \sqrt{2} < p < 4$ have a mode strictly larger than the elicited most likely value m (see Figure 1B). Figure 1C plots the symmetric beta member satisfying (2) with $p = q = 4$ for which the beta mode and elicited value m agree. Finally, Figure 1D displays one of two asymmetric beta distributions satisfying (2) with matching beta mode and elicited most likely value $m < 1/2$ with parameter settings $p = 3 - \sqrt{2}$ and $q = 3 + \sqrt{2}$. The other asymmetric member, has parameter values $p = 3 + \sqrt{2}$ and $q = 3 - \sqrt{2}$ and a mode larger than $1/2$. These $p = 3 \pm \sqrt{2}$, $q = 3 \mp \sqrt{2}$ cases were the ones shown to possess kurtosis 3.0 by Kamburowski (1997) and Herrerías et al. (2003).

Herein we propose an alternative three parameter reparameterization of the four parameter beta distribution involving lower and upper bound estimates a and b and a most likely value m. Most importantly, in Section 2, an adaptation of (2) is proposed such that the beta mode and the elicited most likely value m align for all resulting values of p and q, but which continues to contain the cases $p = 3 \pm \sqrt{2}$, $q = 3 \mp \sqrt{2}$. By including these members, our adaptation inherits the normal distribution similarity as originally proposed. Our reparameterization also retains the original PERT mean expression. The proposed variance expression on the other hand becomes a function of both the elicited most likely value m and an activity's support $[a, b]$ thereby also addressing Hahn's (2008) constant PERT variance concern about (2).

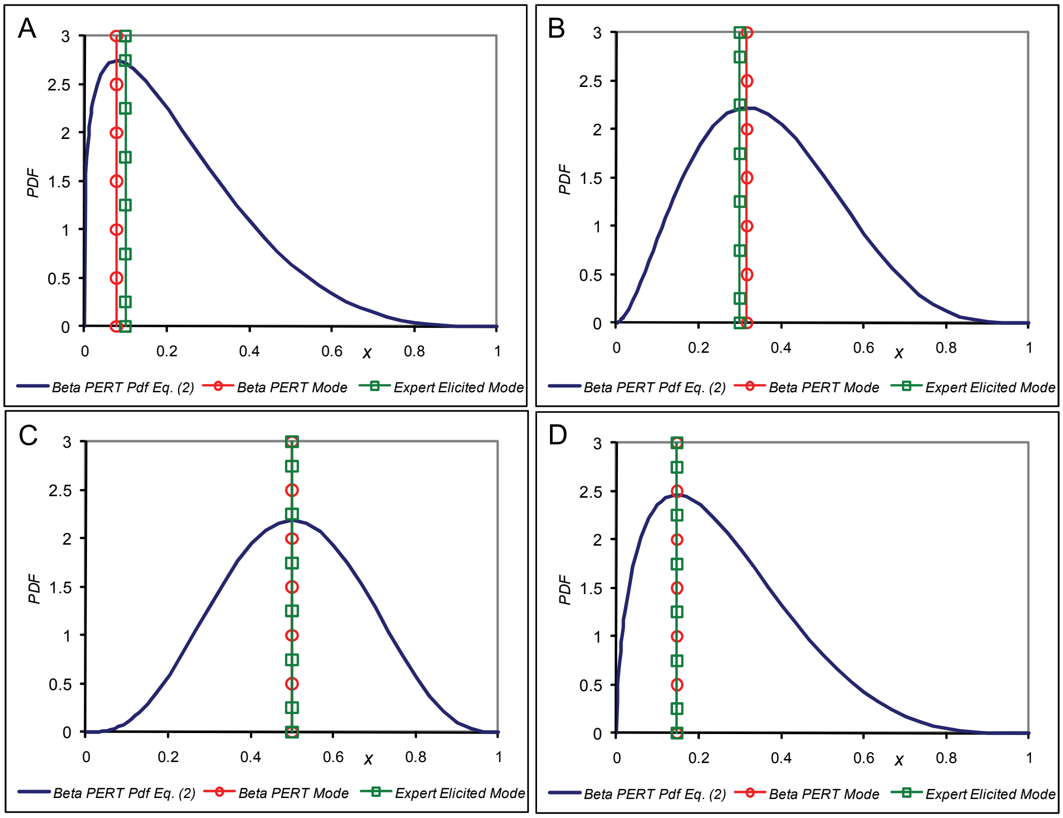

Figure 1. PERT beta distributions satisfying (2), Panels A-B beta mode $\neq m$, Panels C-D beta mode $= m$. A: $m = 0.10$, $1 < p < 3 - \sqrt{2}$, B: $m = 0.25$, $3 - \sqrt{2} < p < 4$, C : $m = 0.50$, $p = q = 4$, D: $m = \frac{1}{2} - \frac{1}{4}\sqrt{2}$, $p = 3 - \sqrt{2}$.

2. A THREE PARAMETER BETA REPARAMETERIZATION

Let $X \sim Beta(a, b, p, q)$ with pdf (1) and $p, q > 1$. For $p, q > 1$ the pdf (1) is unimodal and possesses the following mean $E[X]$ and mode θ as measures of central tendency, respectively:

$$E[X] = \frac{q}{p+q}a + \frac{p}{p+q}b, \theta = \frac{q-1}{p+q-2}a + \frac{p-1}{p+q-2}b. \quad (3)$$

From (3) one derives that: (i) $E[X] = \theta = (a+b)/2 \Leftrightarrow p = q,\ p, q > 1$, and (ii) $E[X]$ for $p \neq q,\ p, q > 1$ may be expressed in terms of the mode θ and the parameters p and q as follows:

$$E[X] = \frac{a + (p+q-2)\theta + b}{p+q}. \quad (4)$$

Expression (4) is reminiscent of the PERT mean in the sense that $E[X]$ is expressed as a weighted linear combination of a, θ and b with weights that sum to one. For a measure of dispersion, one obtains from (1) for the variance of X

$$V[X] = \frac{pq}{(p+q+1)(p+q)^2}(b-a)^2. \tag{5}$$

Suppose now that a lower and upper bound estimates a, b and most likely estimate m have been specified by a substantive expert. To solve for p and q, while allowing for retention of solutions $p = 3 \mp \sqrt{2}, q = 3 \pm \sqrt{2}$ (Figure 1D), we first reparameterize the four parameter pdf (1) into a three parameter one with a mode θ by setting

$$p = 3 - h,\ q = 3 + h, h \in (-2, 2). \tag{6}$$

Introducing δ as the relative distance of the elicited most likely value m to the lower bound a, i.e.

$$\delta = \frac{m-a}{b-a},\ \delta \in [0, 1], \tag{7}$$

we have the identity

$$m = (1-\delta)a + \delta b. \tag{8}$$

Next, utilizing (6) and (3) one obtains

$$\theta = \frac{2+h}{4}a + \frac{2-h}{4}b. \tag{9}$$

We may now solve for h from (8) and (9) by setting $m = \theta$ or equivalently $1 - \delta = (2+h)/4$, yielding

$$h(\delta) = 2 - 4\delta. \tag{10}$$

Please observe from (10) that the modal weight 4 present in the $E[X]$ expression (2) also emerges in $h(\delta)$ as a weight for modal relative distance δ (cf. (7)). Substitution of (10) into (6) and the results subsequently in (4) and (5) yields

$$E[X|a, m, b] = \frac{a + 4m + b}{6},\ V[X|a, m, b] = \mathcal{C}(\delta) \times \frac{(b-a)^2}{36}, \tag{11}$$

$$\mathcal{C}(\delta) = \frac{5}{7} + \frac{16}{7} \times \delta(1-\delta) \in [\frac{5}{7}, \frac{9}{7}], \tag{12}$$

where δ is given by (7). By design, the beta mode and elicited most likely value m match and the mean value expression in (11) is identical to the PERT mean in (2). The variance formula in (11) is reminiscent of the PERT variance in (2). The factor $\mathcal{C}(\delta)$ in (12) may be interpreted as a "PERT variance adjustment factor". Similar to the PERT mean (2), the $V[X]$ expression in (11) is now also a function of assessed asymmetry via the elicited most like value location m relative to the elicited bounds a and b, addressing Hahn's (2008) concern. We too believe this to be more intuitive rather than a constant variance assumption given support $[a, b]$ regardless of the value of m (cf. (2)).

Figure 2 displays the adjustment factor $\mathcal{C}(\delta)$ as a function of $\delta \in [0,1]$. From (11) and (12) it follows that a maximum variance is attained when $\delta = \frac{1}{2} \Leftrightarrow \mathcal{C}(\delta) = 9/7$, i.e. when the most likely value m is centered at $(a+b)/2$. This is consistent with the mode location being least informative in that case. From (10) and (6) we have $p = q = 3$ when $\delta = \frac{1}{2}$. Along this line of thinking, a lesser variance is assessed the closer the most likely value m is to either one of the bounds a or b with a minimum variance when $\delta = \{0,1\} \Leftrightarrow \mathcal{C}(\delta) = 5/7$. Finally, one obtains

$$\mathcal{C}(\delta) = 1 \Leftrightarrow \delta(1-\delta) = \frac{1}{8} \Leftrightarrow \delta = \frac{1}{2} \pm \frac{1}{4}\sqrt{2} \Leftrightarrow p = 3 \mp \sqrt{2}, q = 3 \pm \sqrt{2}. \qquad (13)$$

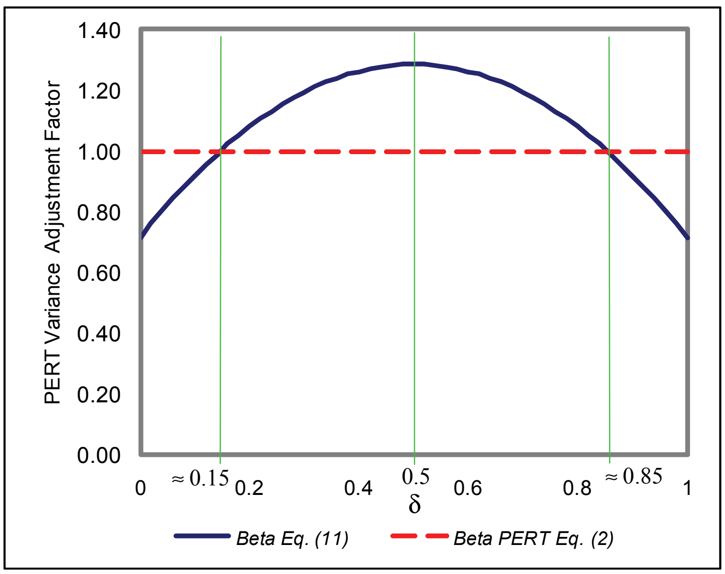

Figure 2. PERT variance adjustment factor $\mathcal{C}(\delta)$ (cf. (12)) as a function the relative distance δ of the most likely value m to the lower bound a given the range $[a, b]$.

Hence for the $p = 3 \mp \sqrt{2}, q = 3 \pm \sqrt{2}$ beta members, the mean and variance in (2) and (11) coincide (cf (13)). Hence, the beta Eq. (11) sub-family inherits the same normal distribution similarity motivation that these specific members provided the original beta PERT sub-family (2) over time (Kamburowski, 1997 and Herrerías et al., 2003). Finally, observe from Figure 2 that within the interval

$$\delta \in [\frac{1}{2} - \frac{1}{4}\sqrt{2}, \frac{1}{2} + \frac{1}{4}\sqrt{2}] \approx [0.15, 0.85], \qquad (14)$$

the variance in (11) is larger than the PERT variance in (2). Thus, within the range (14), which we believe to be a more prevalent domain in practical applications than values of δ outside of it, the procedure (11) is more conservative from an uncertainty analysis perspective than the classical PERT procedure using (2).

Figure 3 compares the beta distributions that follow from (11) and an elicited most likely value $m \in [0, 1]$ with resulting beta distributions displayed in Figure 1 utilizing (2). Observe that, most importantly, in both panels 3A-B the elicited most likely value m now matches the beta mode. Panel 3C demonstrates the effect of the maximum observed discrepancy in variance between (2) and (11) on the beta pdf's when m is centered. Finally, Panel 3D displays that the special case $p = 3 - \sqrt{2}$ and $q = 3 + \sqrt{2}$ is also a member of the beta sub-family of distributions satisfying (11).

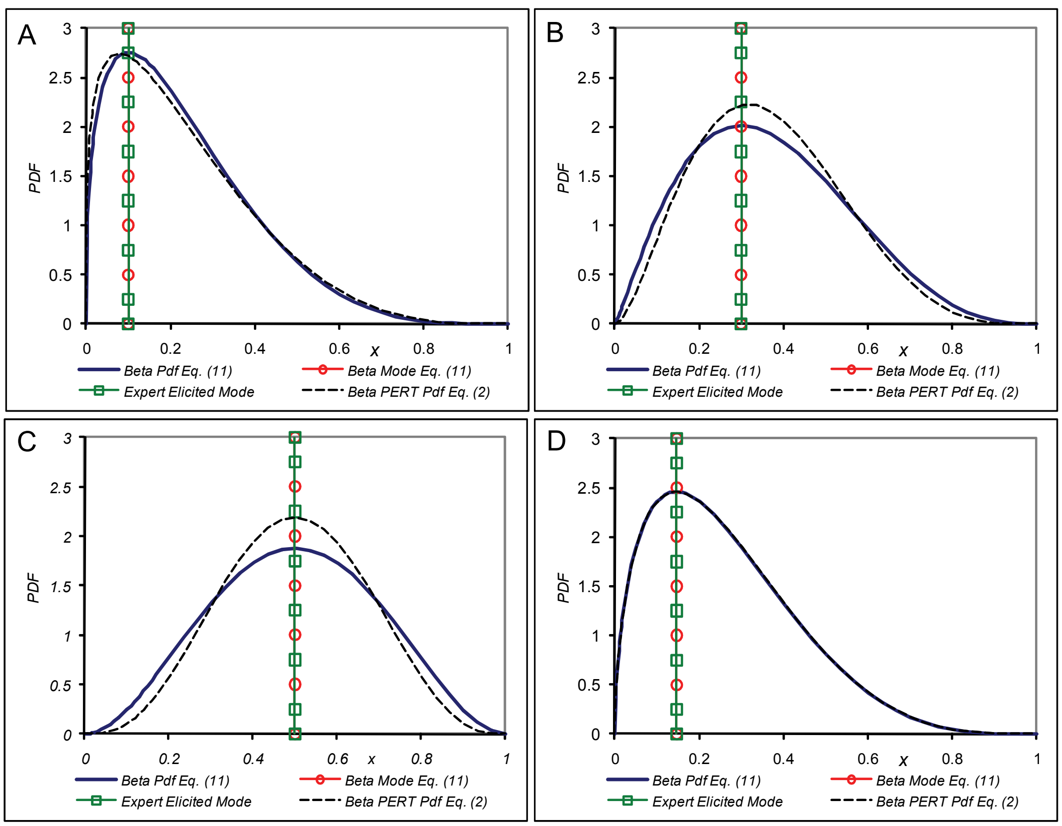

Figure 3. Comparison of beta distributions satisfying (11) and (12) with PERT beta distributions satisfying (2). A: $m = 0.10$, $1 < p < 3 - \sqrt{2}$, B: $m = 0.25$, $3 - \sqrt{2} < p < 3$, C : $m = 0.50$, $p = q = 3$, D: $m = \frac{1}{2} - \frac{1}{4}\sqrt{2}$, $p = 3 - \sqrt{2}$.

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