Lecture on Copulas: Part 2

THE GEORGE WASHINGTON UNIVERSITY

WASHINGTON DC

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OUTLINE

1. **ORDINAL MEASURES OF ASSOCIATION: OVERVIEW**
2. ORDINAL MEASURES OF ASSOCIATION: ARCHIMEDEAN
3. ORDINAL MEASURES OF ASSOCIATION: GDB
4. COPULA PARAMETER ELICITATION: ARCHIMEDEAN
5. COPULA PARAMETER ELICITATION: GDB
6. COPULA SELECTION: AN ENTROPY APROACH
7. A VALUE OF INFORMATION EXAMPLE

Overview

- **Positive (negative) dependence** between $X' \sim G(\cdot)$ and $Y' \sim H(\cdot)$ when **large values** of one go with **large (small) values** of the other.

- In case of positive (negative) dependence, X' and Y' are said to be **concordant (disconcordant)**.

- Classical measures for *the degree of positive or negative dependence*:

  Blomquist's (1950) β , Kendall's (1938) τ and Spearman's (1904) ρ_s.

- All three measures attain **values ranging from -1 to 1**.

- All three are *ordinally invariant*. Hence,

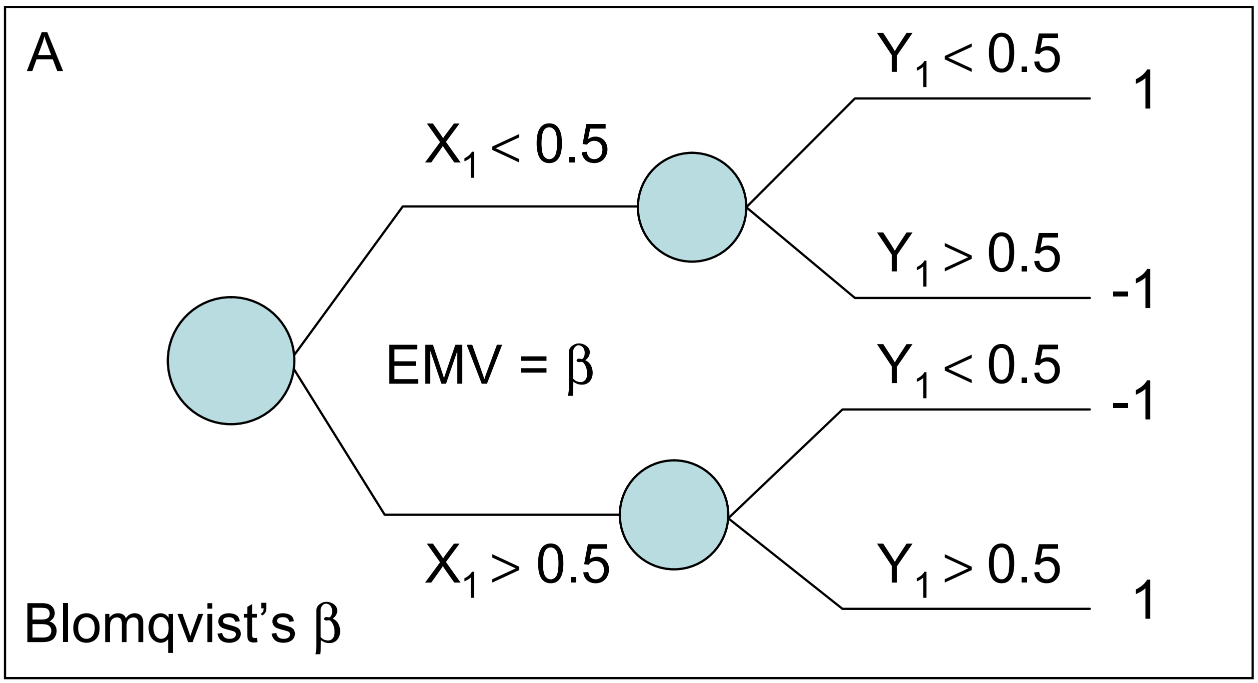
  $$\rho_s(X', Y') = \rho_s(X, Y), \text{ where } (X, Y) = \{G(X'), H(Y')\}, \text{ etc.}$$

- Recall, X and $Y \sim U[0, 1]$ and thus the joint pdf of (X, Y) is a copula.

Blomquist's β

- **Excellent review** of classical measures β, τ and ρ_s is given by Kruskal (1958).

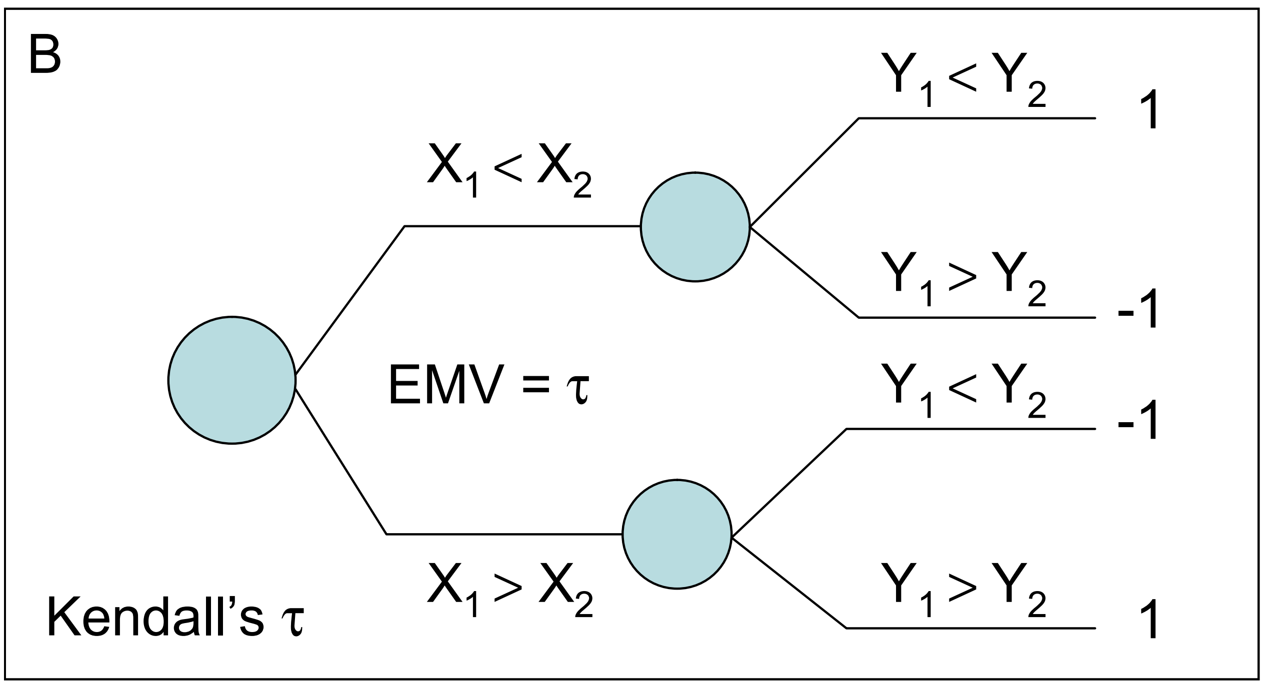
$$\boldsymbol{\beta(X,Y) = 4C(\tfrac{1}{2},\tfrac{1}{2}) - 1}, \text{where } C(\,\cdot\,,\,\cdot\,) \text{ is copula cdf}$$



Kendalls's τ

- Let $c(\cdot\,,\,\cdot), C(\cdot\,,\,\cdot)$ be the copula pdf and cdf and let $(X_i, Y_i) \sim C(\cdot\,,\,\cdot)$, $i = 1, 2$ be two independent bivariate samples from the copula.

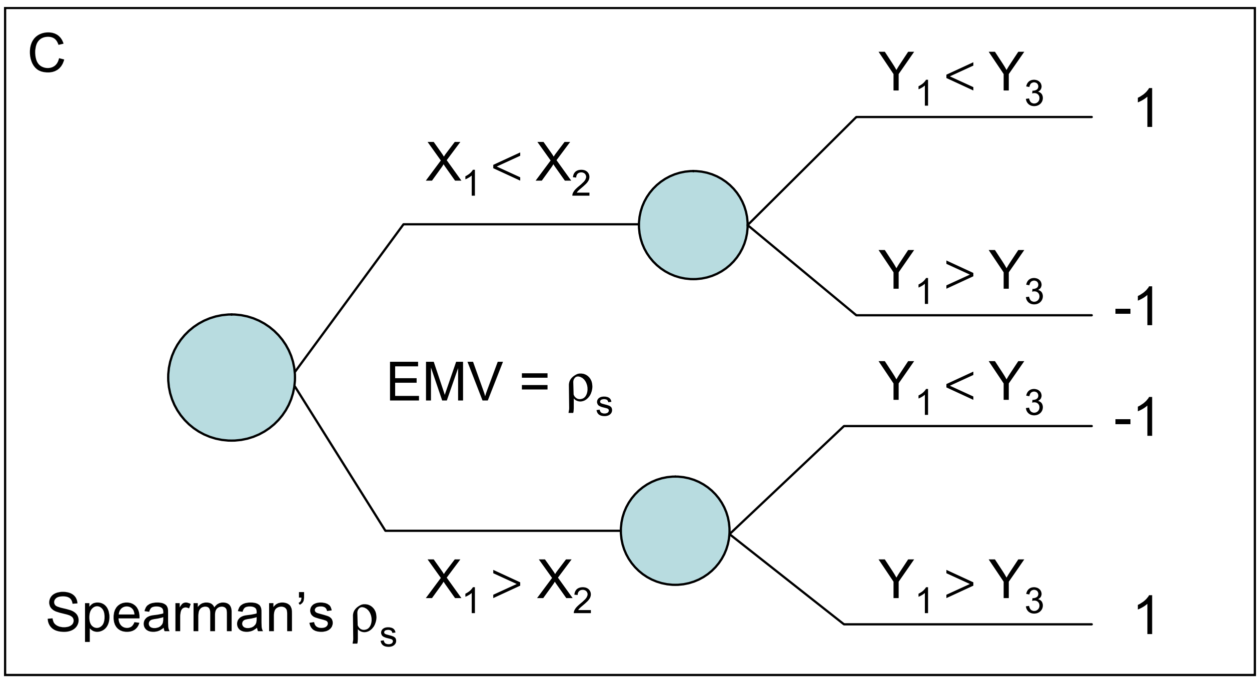
$$\tau(X,Y) = 4\int_0^1\int_0^1 C(x,y)c(x,y)dxdy - 1.$$



Spearman's ρ_s

- Let $c(\cdot,\cdot), C(\cdot,\cdot)$ be the copula pdf let $(X_i, Y_i) \sim C(\cdot,\cdot), i = 1,2,3$ be three independent bivariate samples from the copula.

$$\rho_s(X,Y) = 12\int_0^1\int_0^1 xyc(x,y)dxdy - 3.$$



Tail Dependence

- **Lower and upper tail dependence measures are in vogue**, particularly in problem contexts dealing with modeling **the joint occurrence of extreme events**, such as insurance and modeling of default risk in finance.

- Recent **burst of attention** to the copula approach may be credited to **the Gaussian copula** which has been widely adopted by the "financial quants" .

- Embrechts (2008) even refers to this attention as **"the copula craze".**

- Unfortunately, some (see, e.g., Salmon, 2009) **blamed** Gaussian copula for the **2008 financial crash**, in part due to **lack of** lower and upper tail dependence.

- These measures too are **ordinal measures of association**, although they focus primarily on modeling positive dependence and not negative dependence.

Tail Dependence

- $X' \sim G(\cdot), Y' \sim H(\cdot)$, **Lower tail dependence λ_L**:

$$\lambda_L = \lim_{x \downarrow 0} Pr\{Y' \leq H^{-1}(x) | X' \leq G^{-1}(x)\}$$
$$= \lim_{x \downarrow 0} Pr(Y \leq x | X \leq x) = \lim_{x \downarrow 0} \frac{C(x,x)}{x},$$

- $X' \sim G(\cdot), Y' \sim H(\cdot)$, **Upper tail dependence λ_U**:

$$\lambda_U = \lim_{x \uparrow 1} Pr\{Y' > H^{-1}(x) | X' > G^{-1}(x)\}$$
$$= \lim_{x \uparrow 1} Pr(Y > x | X > x) = \lim_{x \uparrow 1} \frac{1 - 2x + C(x,x)}{1-x}.$$

- Clayton and Gumbel copulae exhibit lower and upper tail dependence, respectively.

Tail Dependence

- **For GDB copula cdf $C\{x, y|p(\cdot|\Psi)\}$ we have $\lambda_L = \lambda_U = 0$,** similar to the Gaussian copulae (Embrechts et. al, 2002).

- Blomquist's β, Kendall's τ and Spearman's ρ_S are more applicable in contexts dealing with **the modeling of joint events in general, not extremes per se.**

- Blomquist's β, Kendall's τ and Spearman's ρ_S pertain to full copula support and not just to their asymptotic extreme values.

- **Caution** to those who believe that the Clayton and Gumbel copulae could serve as the panacea as an alternative to the Gaussian Copula.

- Heteroscedastic behavior of financial processes **suggests** their dependence cannot be modelled using a copula with **a constant correlation over time**, **regardless** of a copula displaying tail dependence or not.

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- **Population expressions for $\beta, \tau, \rho_s, \lambda_L$ and λ_U are:**

$$\begin{cases} \beta(X,Y) = 4C(\tfrac{1}{2},\tfrac{1}{2}) - 1, \\ \tau(X,Y) = 4\int_0^1\int_0^1 C(x,y)c(x,y)dxdy - 1, \\ \rho_s(X,Y) = 12\int_0^1\int_0^1 xyc(x,y)dxdy - 3, \end{cases} \qquad \begin{cases} \lambda_L = \lim\limits_{x \downarrow 0} \frac{C(x,x)}{x} \\ \lambda_U = \lim\limits_{x \uparrow 1} \frac{1-2x+C(x,x)}{1-x} \end{cases}$$

- Archimedean copula with **generator function pdf $\varphi(\cdot)$ (e.g Genest and Mackay, 1986):**

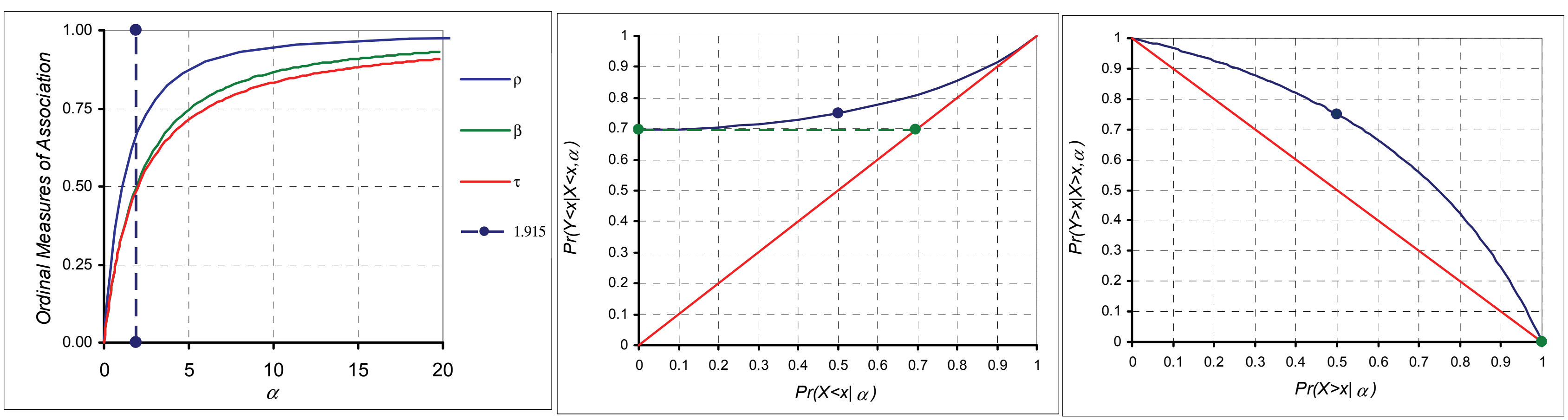
$$\begin{cases} \beta\{X,Y|\alpha\} = 4\varphi^{-1}\{\varphi(\tfrac{1}{2}) + \varphi(\tfrac{1}{2})\} - 1, \\ \tau\{X,Y|\alpha\} = 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}dt + 1, \\ \rho_s\{X,Y|\alpha\}\text{: No expression available.} \end{cases} \qquad \begin{cases} \lambda_L = \lim\limits_{x \downarrow 0} \frac{\varphi^{-1}\{2\varphi(x)\}}{x} \\ \lambda_U = \lim\limits_{x \uparrow 1} \frac{1-2x+\varphi^{-1}\{2\varphi(x)\}}{1-x} \end{cases}$$

Archimedean Copula

- **Clayton Copula:** $\varphi(t) = t^{-\alpha} - 1,\ \alpha \in (0, \infty)$

$$\begin{cases} \beta(X,Y) = 4\left[2^{\alpha+1} - 1\right]^{-1/\alpha} - 1 \in [0,1], \\ \tau(X,Y) = \frac{\alpha}{\alpha+2} \in [0,1], \\ \rho_s(X,Y)\text{: No expression available,} \\ \lambda_L = 2^{\alpha}, \lambda_U = 0 \end{cases}$$

$$\boldsymbol{\alpha \approx 1.915 \Rightarrow Pr(Y \leq 0.5 | X \leq 0.5) = Pr(Y > 0.5 | X > 0.5) = 0.75}$$



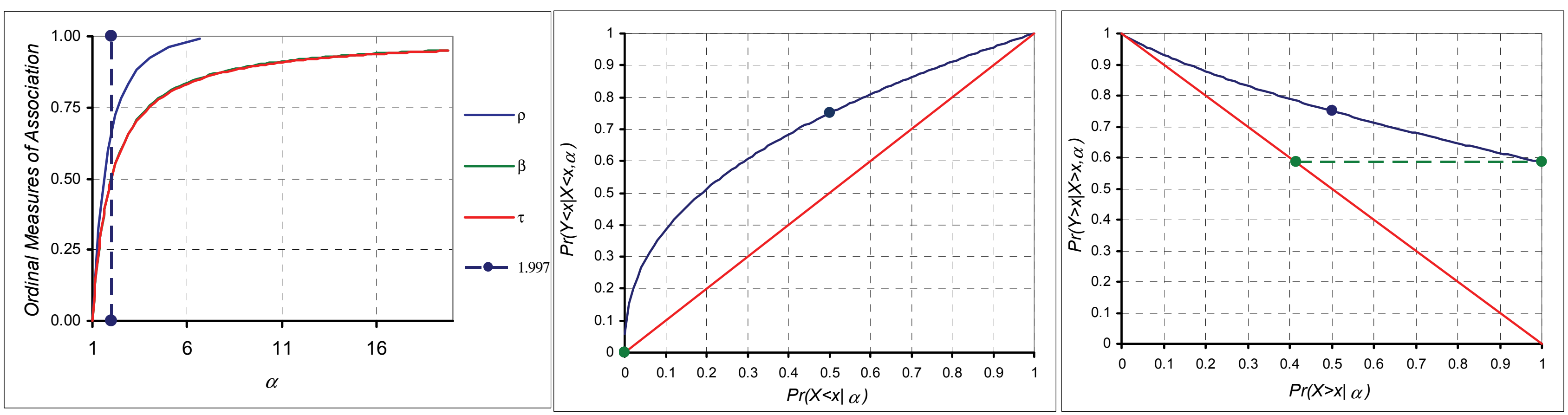
2. ORD. MEAS. OF ASSOC...

Archimedean Copula

- **Gumbel Copula:** $\varphi(t) = t^{-\alpha} - 1,\ \alpha \in (0, \infty)$

$$\begin{cases} \beta(X,Y) = 4 \times (1/2)^{(2^{1/\alpha})} - 1 \in [0,1], \\ \tau(X,Y) = \frac{\alpha - 1}{\alpha} \in [0,1], \\ \rho_s(X,Y)\text{: No expression available,} \\ \lambda_L = 0, \lambda_U = 2 - 2^{\alpha} \end{cases}$$

$$\boldsymbol{\alpha \approx 1.997 \Rightarrow Pr(Y \leq 0.5 | X \leq 0.5) = Pr(Y > 0.5 | X > 0.5) = 0.75}$$

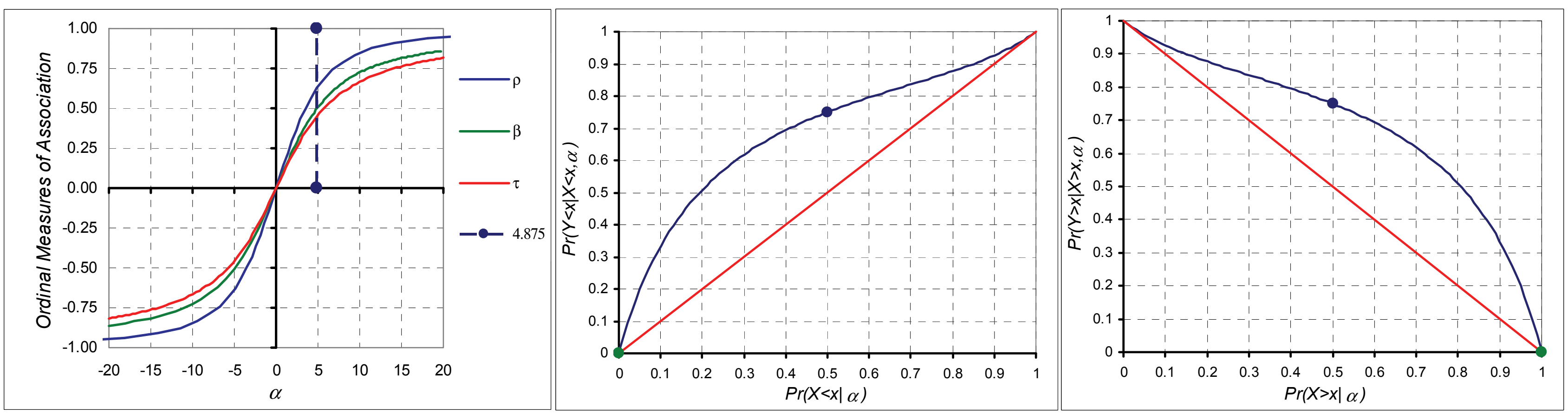


Archimedean Copula

- **Frank Copula:** $\varphi(t) = \ln\frac{e^{\alpha t}-1}{e^{\alpha}-1},\ \alpha \in \mathbb{R}\backslash\{0\}$

$$\begin{cases} \beta(X,Y) = -\frac{4}{\alpha}\ln\left\{1+\frac{(e^{-\frac{\alpha}{2}}-1)^2}{e^{-\alpha}-1}\right\}-1 \in [-1,1], \\ \tau(X,Y) = 1-\frac{4}{\alpha}[1-D(\alpha)] \in [-1,1],\ D(\alpha) = \frac{1}{\alpha}\int_0^{\alpha}\frac{t}{e^t-1}dt, \\ \rho_s(X,Y)\text{: No expression available,} \\ \lambda_L = 0, \lambda_U = 0. \end{cases}$$

$$\alpha \approx 4.875 \Rightarrow Pr(Y \leq 0.5|X \leq 0.5) = Pr(Y > 0.5|X > 0.5) = 0.75$$



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- **Population expressions for β, τ and ρ_s ar**

$$\begin{cases} \beta(X,Y) = 4C(\frac{1}{2},\frac{1}{2}) - 1, \\ \tau(X,Y) = 4\int_0^1\int_0^1 C(x,y)c(x,y)dxdy - 1, \\ \rho_s(X,Y) = 12\int_0^1\int_0^1 xyc(x,y)dxdy - 3, \end{cases}$$

$$\begin{cases} \lambda_L = \lim\limits_{x \downarrow 0} \frac{C(x,x)}{x} \\ \lambda_U = \lim\limits_{x \uparrow 1} \frac{1-2x+C(x,x)}{1-x} \end{cases}$$

- We have for GDB copula with **TS pdf with generating pdf $p(\cdot|\Psi)$ and $Z \sim P(\cdot|\Psi)$, $P(\cdot|\Psi)$ the generating cdf:**

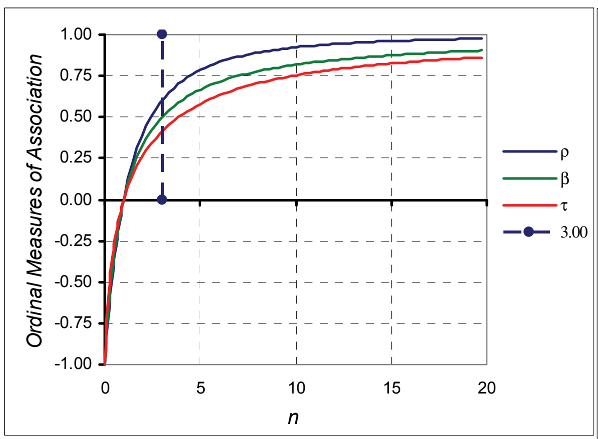
$$\begin{cases} \beta\{X,Y|p(\cdot|\Psi)\} = 2E[Z|\Psi] - 1, \\ \tau\{X,Y|p(\cdot|\Psi)\} = 2E[Z^2|\Psi] - 2\int_0^1 P^2(s|\Psi)ds + 4\int_0^1 sP^2(s|\Psi)ds - 1, \\ \rho_s\{X,Y|p(\cdot|\Psi)\} = -4E[Z^3|\Psi] + 6E[Z^2|\Psi] - 1. \\ \lambda_L = 0, \lambda_U = 0 \end{cases}$$

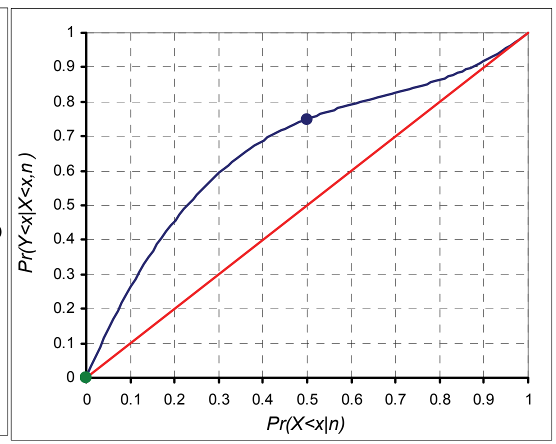
GDB Copula with TS Gen. PDF

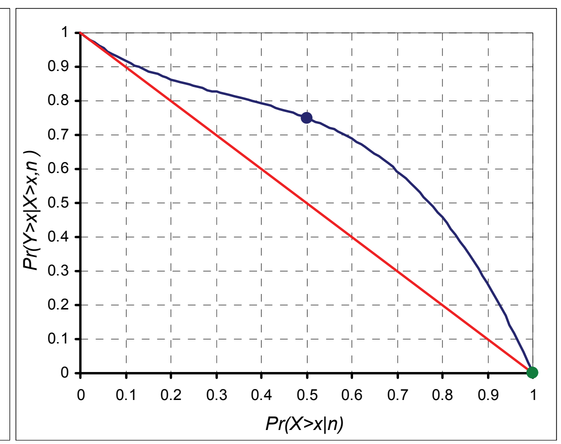
- **Power pdf :** $p(z|n) = nz^{n-1},\ n > 0,$

$$\begin{cases} \beta\{X,Y|p(\cdot|n)\} = \frac{n-1}{n+1}, & \in[-1,1], \\ \tau\{X,Y|p(\cdot|n)\} = \frac{n-1}{n+2} + \frac{n-1}{(n+1)(n+2)(2n+1)}, & \in[-1,1], \\ \rho_s\{X,Y|p(\cdot|n)\} = \frac{(n-1)(n+6)}{(n+2)(n+3)}, & \in[-1,1]. \\ \lambda_L = 0, \lambda_U = 0 & \end{cases}$$

$$n = 3 \Rightarrow Pr(Y \leq 0.5|X \leq 0.5) = Pr(Y > 0.5|X > 0.5) = 0.75$$





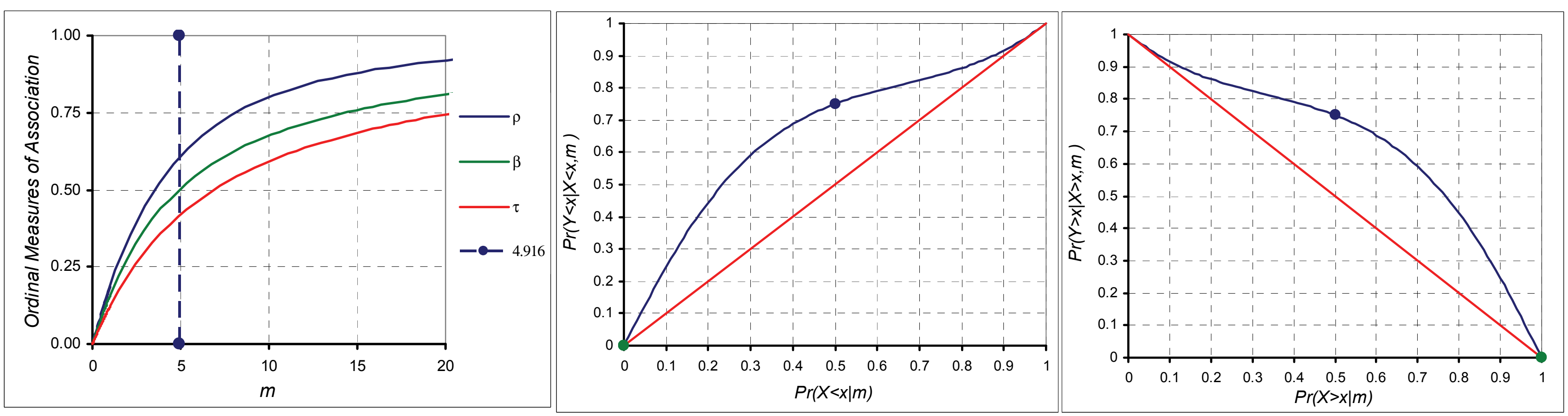


GDB Copula with TS Gen. PDF

- **Ogive pdf:** $p(z|m) = \frac{m+2}{3m+4}\{2(m+1)\sqrt{z^m} - mz^{m+1}\}, m > 0,$

$$\begin{cases} \beta\{X,Y|p(\cdot|m)\} = \frac{m(m+1)(3m+8)}{(m+3)(m+4)(3m+4)}, & \in [0,1], \\ \tau\{X,Y|p(\cdot|m)\} = \frac{m(m+1)(162m^6+2643m^5+18132m^4+66108m^3+140032m+58880)}{(m+3)(m+4)(m+6)(2m+5)(3m+4)^2(3m+8)(3m+10)}, & \in [0,1], \\ \rho_s\{X,Y|p(\cdot|m)\} = \frac{m(m+1)(3m^3+70m^2+424m+736)}{(m+4)(m+5)(m+6)(m+8)(3m+4)}, & \in [0,1]. \\ \lambda_L = 0, \lambda_U = 0 \end{cases}$$

$$m = 4.916 \Rightarrow Pr(Y \le 0.5|X \le 0.5) = Pr(Y > 0.5|X > 0.5) = 0.75$$

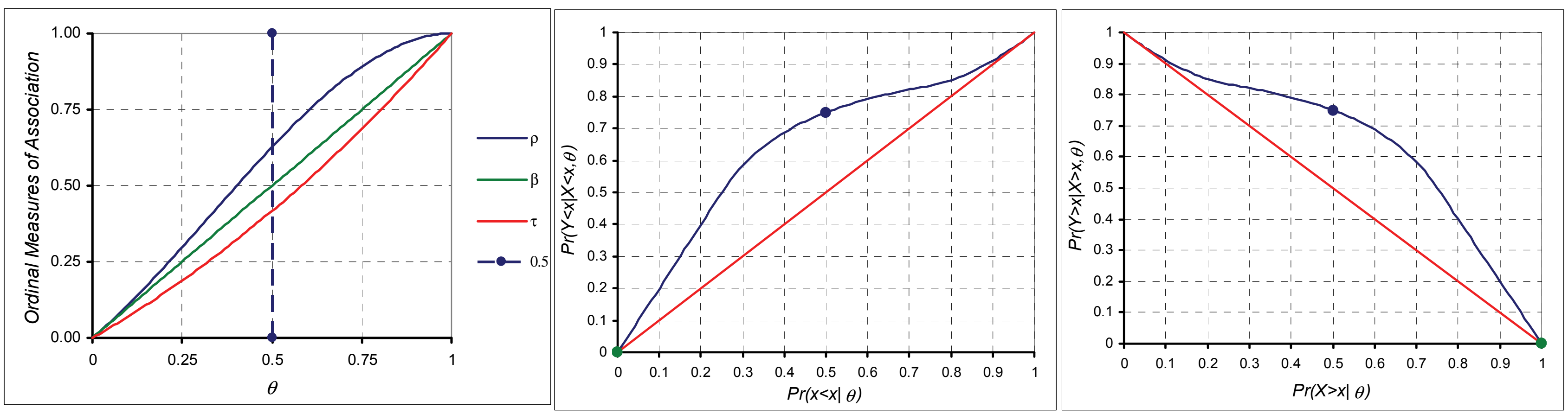


GDB Copula with TS Gen. PDF

- $\text{U}[\theta,1]\ \text{pdf}: p(z|\theta)=\dfrac{1}{1-\theta},\ \theta\le z\le 1,\ 0\le\theta\le 1,$

$$\begin{cases}\beta\{X,Y|p(\cdot|\theta)\}=\theta, & \in[0,1],\\ \tau\{X,Y|p(\cdot|\theta)\}=\theta(\theta+2)/3, & \in[0,1],\\ \rho_s\{X,Y|p(\cdot|\theta)\}=\theta(1+\theta-\theta^2), & \in[0,1].\\ \lambda_L=0,\lambda_U=0 & \end{cases}$$

$$\theta=0.5\Rightarrow Pr(Y\le 0.5|X\le 0.5)=Pr(Y>0.5|X>0.5)=0.75$$

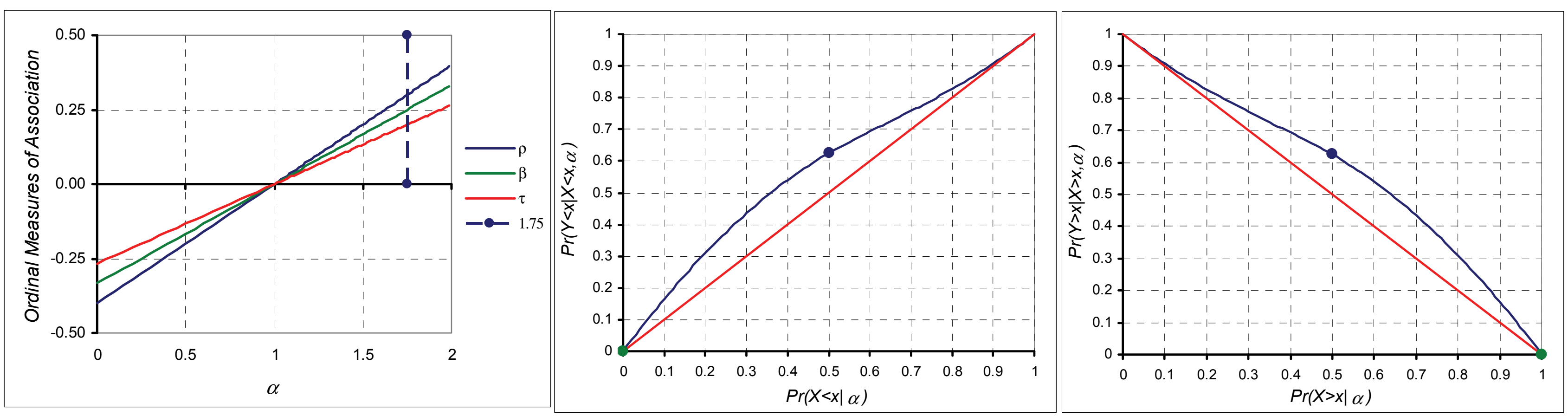


GDB Copula with TS Gen. PDF

- **Slope pdf :** $p(z|\alpha) = 2 - \alpha + 2(\alpha - 1)z,\ 0 \le \alpha \le 2,$

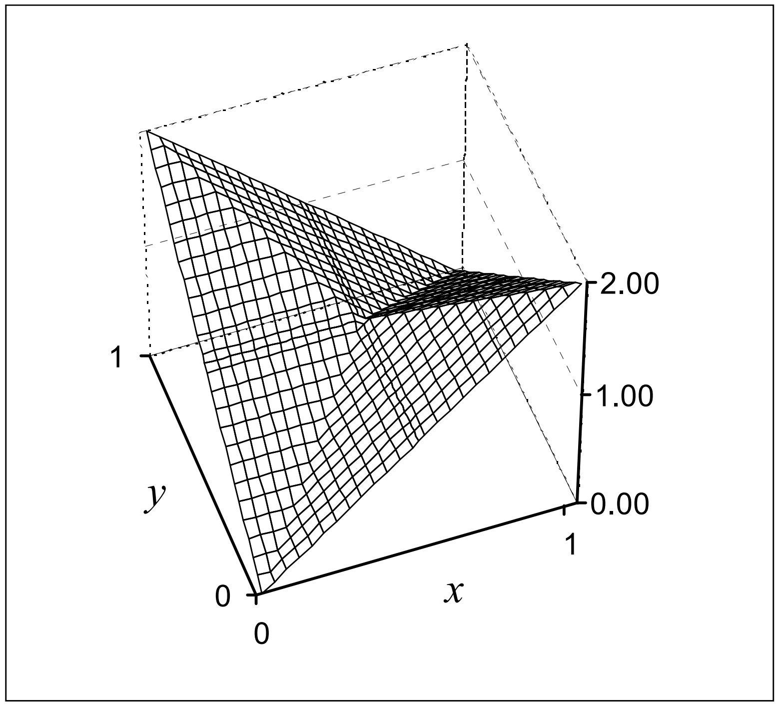
$$\begin{cases} \beta\{X,Y|p(\cdot|\alpha)\} = -\frac{1}{3} + \frac{1}{3}\alpha, & \in [-\frac{1}{3}, \frac{1}{3}], \\ \tau\{X,Y|p(\cdot|\alpha)\} = -\frac{4}{15} + \frac{4}{15}\alpha, & \in [-\frac{4}{15}, \frac{4}{15}], \\ \rho_s\{X,Y|p(\cdot|\alpha) = -\frac{2}{5} + \frac{2}{5}\alpha, & \in [-\frac{2}{5}, \frac{2}{5}]. \\ \lambda_L = 0, \lambda_U = 0 \end{cases}$$

$$\alpha = 1.75 \Rightarrow Pr(Y \le 0.5|X \le 0.5) = Pr(Y > 0.5|X > 0.5) = 0.625$$



Reflection Property

- Let $q(z|\Psi)$ be pdf $Z' = 1 - Z, Z \sim p(z|\Psi) \Rightarrow$ **$q(z|\Psi) = p(1-z|\Psi)$.**
- $c\{x, y|q(z|\Psi)\} = c\{x, y|p(1-z|\Psi)\}$ obtained via **a right angle rotation.**



Graph of rotated copula using $p(1-z|\Psi) = 2(1-z)$.

Reflection Property

- We have for GDB copula with **TS gen. pdf $p(\cdot|\Psi)$ and $Z \sim p(\cdot|\Psi)$:**

$$\begin{cases} \beta\{X,Y|p(\cdot|\Psi)\} = 2E[Z|\Psi]-1, \\ \tau\{X,Y|p(\cdot|\Psi)\} = 2E[Z^2]-2\int_0^1 P^2(s|\Psi)ds + 4\int_0^1 sP^2(s|\Psi)ds - 1, \\ \rho_s\{X,Y|p(\cdot|\Psi)\} = -4E[Z^3|\Psi]+6E[Z^2|\Psi]-1. \end{cases}$$

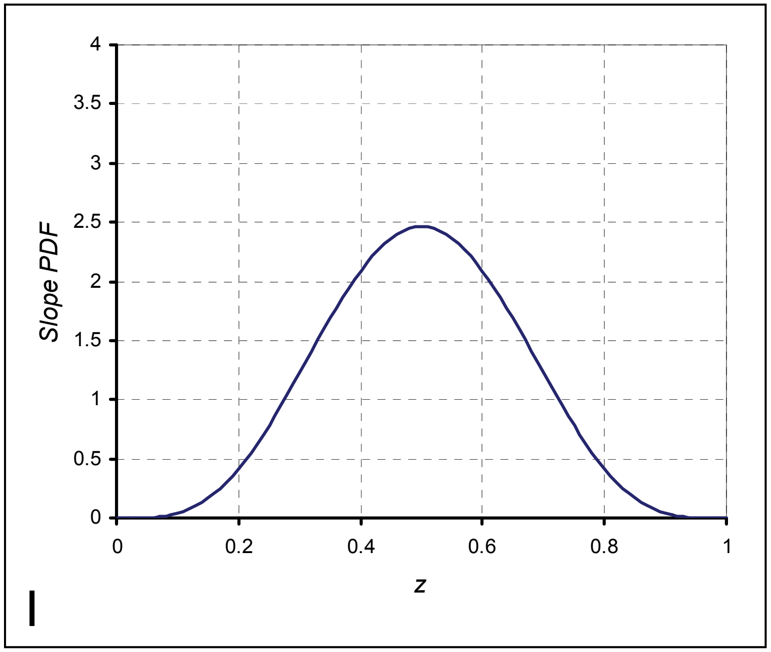
- **Let $q(z|\Psi)$ be pdf $Z' = 1 - Z, Z \sim p(z|\Psi)$** $\Rightarrow$

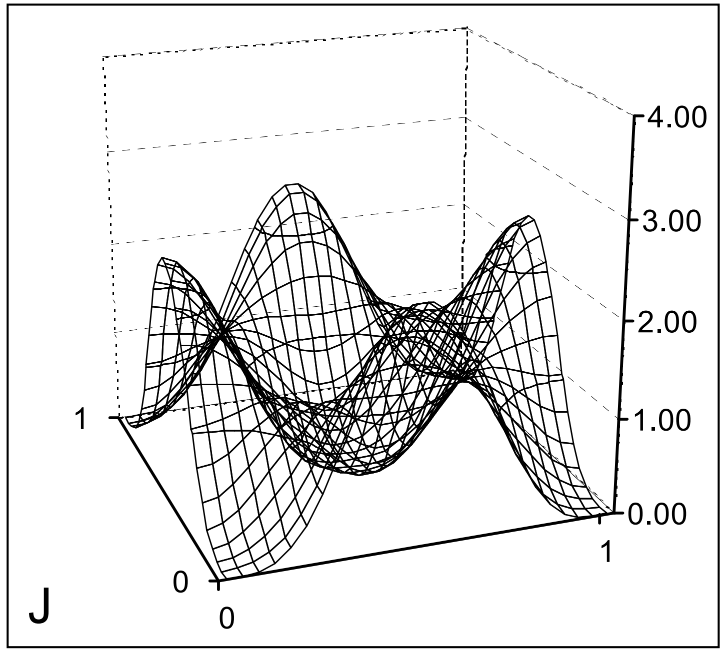
$$\begin{cases} \beta\{X,Y|q(z|\Psi)\} = \beta\{X,Y|p(1-z|\Psi)\} = -\beta\{X,Y|p(z|\Psi)\}, \\ \tau\{X,Y|q(z|\Psi)\} = \tau\{X,Y|p(1-z|\Psi)\} = -\tau\{X,Y|p(z|\Psi)\}, \\ \rho_s\{X,Y|q(z|\Psi)\} = \rho_s\{X,Y|p(1-z|\Psi)\} = -\rho_s\{X,Y|p(z|\Psi)\}. \end{cases}$$

- $p(z|\Psi)$ **symmetric** on $[0,1] \Rightarrow$ **$p(1-z|\Psi) = p(z|\Psi) \Rightarrow \beta, \tau, \rho_s \equiv 0$**

Reflection Property

$$p(z|a) = \frac{\Gamma(2a)}{\Gamma(a)\Gamma(a)} x^{a-1}(1-x)^{a-1}, a > 0 \Rightarrow \beta,\ \tau,\ \rho_s \equiv 0, \forall\, a > 0$$





G: Beta generating pdf; H: GDB Copula with TS Gen. PDF in A.

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Procedure

- $X' \sim G(\cdot), \; Y' \sim H(\cdot), \{G(X'), H(Y')\} = (X, Y) \sim C\{x, y | p(\cdot | \Psi)\}$

- Elicit: $\boldsymbol{Pr(Y' \leq y'_{0.5} | X' \leq x'_{0.5}) = Pr(Y \leq 0.5 | X \leq 0.5) = \pi.}$

- This *elicitation procedure* falls within **the conditional fractile estimation method for eliciting degree of dependence** - Clemen and Reilly (1999).

- We have for Blomquist's β

$$\beta\{X, Y\} = 2Pr(Y \leq 0.5 | X \leq 0.5) - 1 = 2\pi - 1$$

- Hence, elicitation of π is equivalent to **an indirect elicitation of Blomquist's $\boldsymbol{\beta}$** which has a more straightforward interpretation as τ and ρ_s.

- **General expression for Archimean Copulas with generator $\varphi(\cdot|\alpha)$:**

$$\pi(X,Y|\alpha) = Pr(Y \le 0.5 | X \le 0.5) = 2C(\frac{1}{2}, \frac{1}{2}|\alpha)$$
$$= 2 \times \varphi^{-1}\{\varphi(\frac{1}{2}) + \varphi(\frac{1}{2})\}$$

- **For different Archimean Copulas with generator $\varphi(\cdot|\alpha)$ herein :**

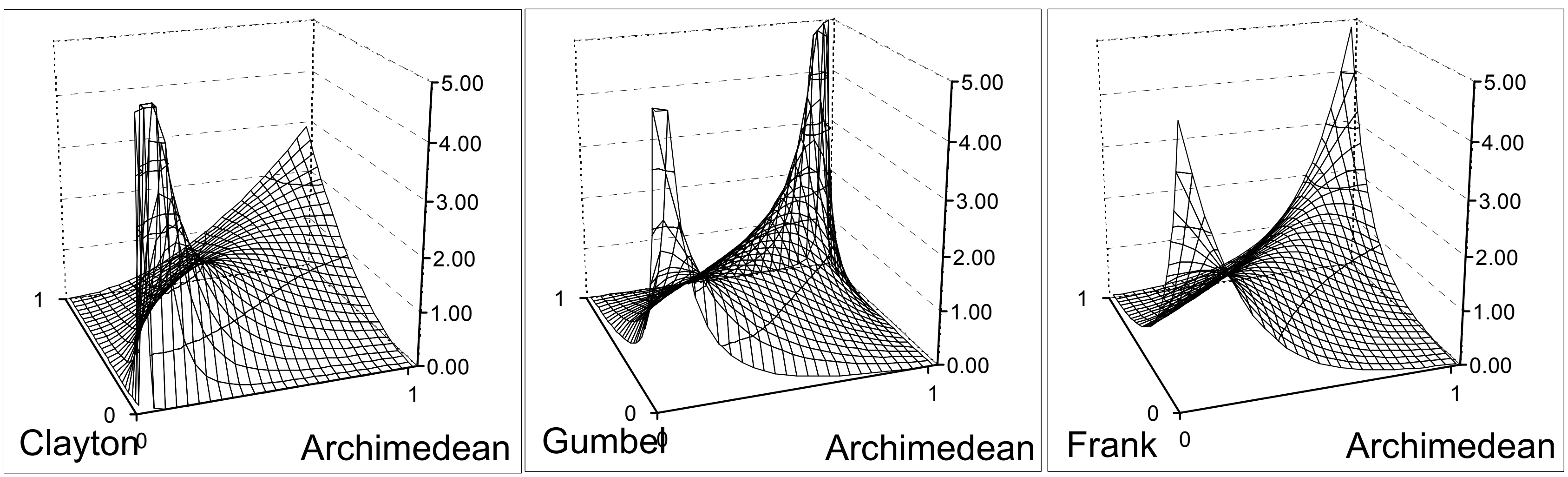
$$\pi(X,Y|\alpha) = \begin{cases} (2^{\alpha+1}-1)^{-1/\alpha} \in [0.5,1], & \text{Clayton Copula,} \\ (1/2)^{2^{1/\alpha}} \in [0.5,1], & \text{Gumbel Copula,} \\ -\frac{1}{\alpha}\ln\left\{1 + \frac{(e^{-\alpha/2}-1)^2}{e^{-\alpha}-1}\right\} \in [0.5,1], & \text{Frank Copula,} \end{cases}$$

- Having elicited π one solves for α using a numerical produres, except in case of Gumbel copula.

- **Assume that an expert has assessed a value :**

$$\pi\{X, Y|p(\cdot|\Psi)\} = Pr(Y \leq 0.5|X \leq 0.5) = 0.75 \Rightarrow$$

$$\begin{cases} \text{Clayton Copula,} & \alpha \approx 1.915, \\ \text{Gumbel Copula,} & \alpha \approx 1.997, \\ \text{Frank Copula,} & \alpha \approx 4.875. \end{cases}$$



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$$C\{x,y|p(\cdot|\Psi)\} = \begin{cases} x - \frac{1}{2}\int_{1-x-y}^{1+x-y} P(z|\Psi)dz, & (x,y) \in A_1, \\ y - \frac{1}{2}\int_{1-x-y}^{1-x+y} P(z|\Psi)\}dz, & (x,y) \in A_2, \\ x - \frac{1}{2}\int_{x+y-1}^{1+x-y} P(z|\Psi)dz, & (x,y) \in A_3, \\ y - \frac{1}{2}\int_{x+y-1}^{1-x+y} P(z|\Psi)dz, & (x,y) \in A_4. \end{cases} \Rightarrow$$

$$\frac{1}{2}\pi = Pr(Y \leq 0.5, X \leq 0.5) = C\{\frac{1}{2}, \frac{1}{2}|p(\cdot|\Psi)\} = \frac{1}{2}\int_0^1 \{1 - P(z|\Psi)\}dz$$

- With $Z \sim P(z|\Psi)$, $E[Z|\Psi] = \int_0^1 \{1 - P(z|\Psi)\}dz$, thus we have:

$$\pi\{X,Y|p(\cdot|\Psi)\} = E[Z|\Psi]. \tag{12}$$

- Thus, having elicited π one solves for ψ using **the method of moments**.

- **For different GDB Copulas with generating pdf's $p(\cdot \mid \cdot)$ herein**

$$\pi\{X, Y \mid p(\cdot \mid \Psi)\} =$$
$$\begin{cases} (2+\alpha)/6 \in [\frac{1}{3}, \frac{2}{3}], & p(z|\alpha), \quad \text{Slope pdf,} \\ n/(n+1) \in [0,1], & p(z|n), \quad \text{Power pdf,} \\ \frac{(m+2)^2}{3m+4}\left[\frac{3m+6}{(m+4)(m+3)}\right] \in [0.5, 1], & p(z|m), \quad \text{Ogive pdf,} \\ (\theta+1)/2 \in [0.5, 1], & p(z|\theta), \quad \text{U}[\theta, 1] \text{ pdf.} \end{cases}$$

- Having elicited π one solves for α, n and θ in closed form for GDB copula with slope, power or U$[\theta, 1]$ generating pdf.

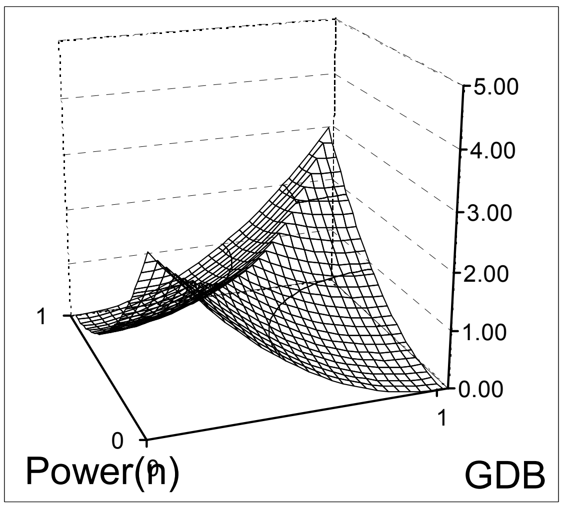
- Having elicited π one solves for m using a numerical procedure in case of GDB copula with Ogive generating pdf.

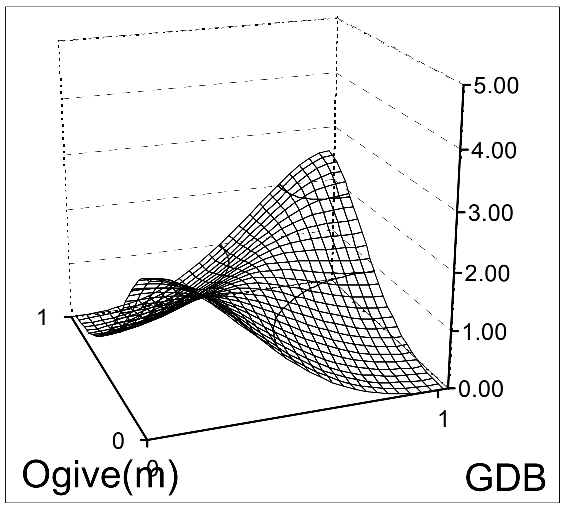
GDB Copulas

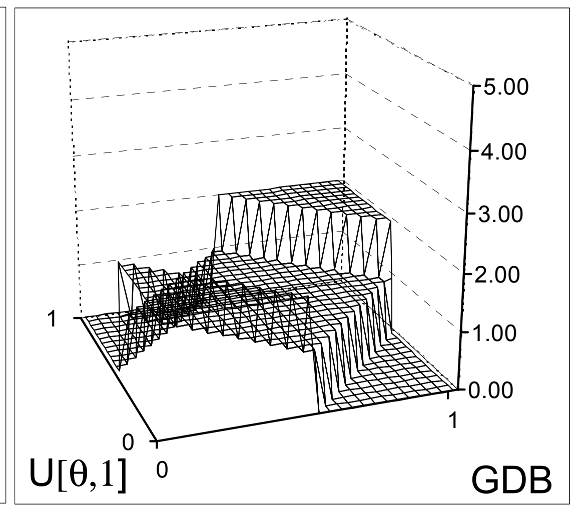
- **Assume that an expert has assessed a value :**

$$\pi\{X, Y|p(\cdot|\Psi)\} = Pr(Y \leq 0.5|X \leq 0.5) = 0.75 \Rightarrow$$

$$\begin{cases} \text{GDB - Power Gen. Density,} & n = 3, \\ \text{GDB - Ogive Gen. Density,} & m \approx 4.916, \\ \text{GDB - U}[\theta, 1] \text{ Gen. Density,} & \theta = 0.5. \end{cases}$$







OUTLINE

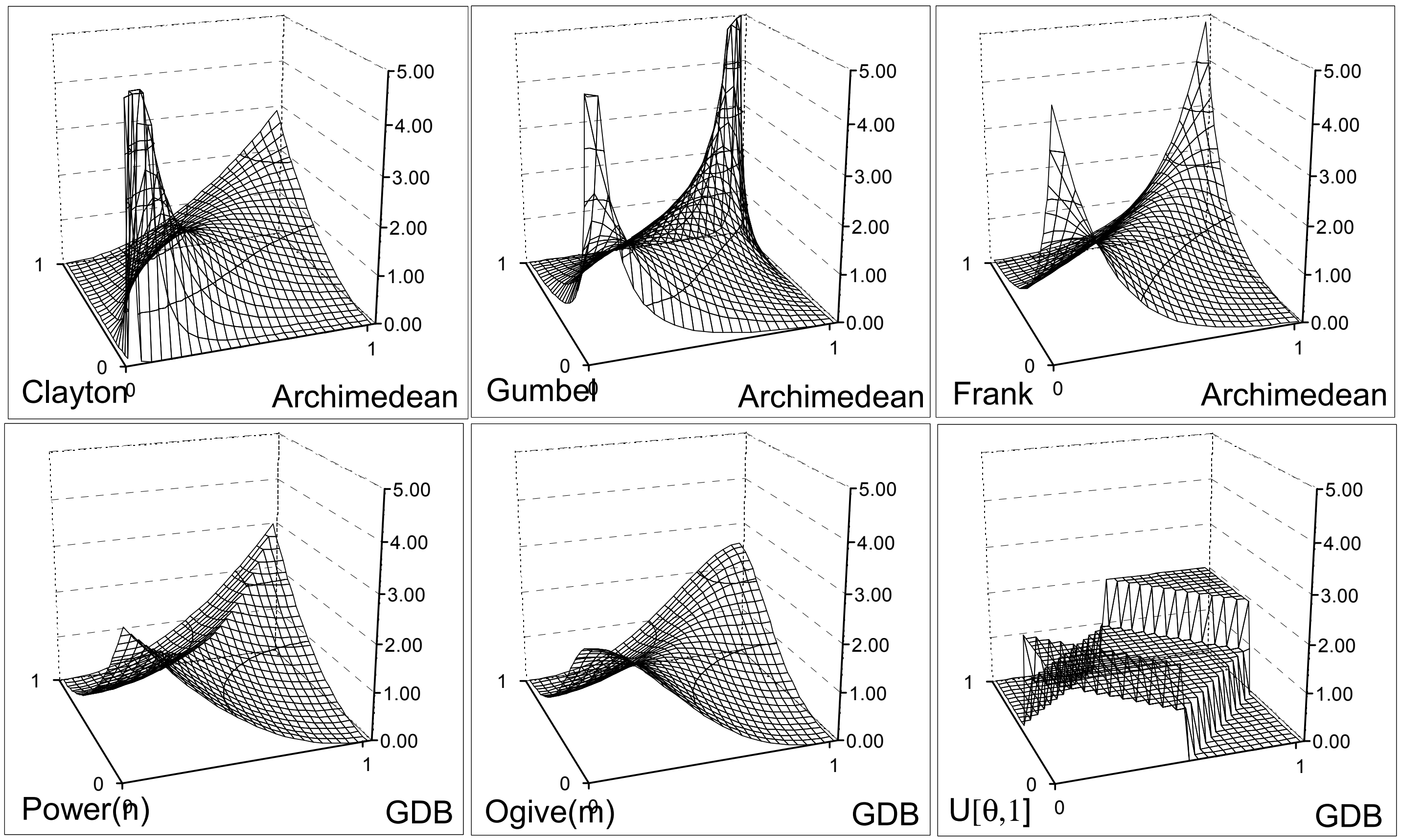
1. ORDINAL MEASURES OF ASSOCIATION: OVERVIEW
2. ORDINAL MEASURES OF ASSOCIATION: ARCHIMEDEAN
3. ORDINAL MEASURES OF ASSOCIATION: GDB
4. COPULA PARAMETER ELICITATION: ARCHIMEDEAN
5. COPULA PARAMETER ELICITATION: GDB
6. **COPULA SELECTION: AN ENTROPY APROACH**
7. A VALUE OF INFORMATION EXAMPLE

6. COPULA SELECTION...

How to pick one?

- Assume that an expert has assessed a value

$$\pi\{X,Y|p(\cdot|\Psi)\} = Pr(Y \leq 0.5|X \leq 0.5) = 0.75.$$



6. COPULA SELECTION...

How to pick one?

- **All six copulas above match the constraint:**

$$\pi\{X, Y|p(\,\cdot\,|\Psi)\} = Pr\{Y < 0.5|X < 0.5\} = 0.75$$

- Lower tail dependence $\Rightarrow$ Pick Clayton Copula

- Upper tail dependence $\Rightarrow$ Pick Gumbel Copula

- **If neither is required, pick one that is uniform as possible?**

$$\begin{aligned}
&\text{Clayton}: \alpha \approx 1.915 \Rightarrow \rho_s \approx 0.6625, \beta = 0.5, \tau \approx 0.489 \\
&\text{Gumbel}: \alpha \approx\ 1.997 \Rightarrow \rho_s \approx 0.6616, \beta = 0.5, \tau = 0.5 \\
&\text{Frank}:\ \alpha \approx 4.875 \Rightarrow \rho_s \approx 0.6244, \beta = 0.5, \tau \approx 0.448 \\
&\text{Power}(n):\ n = 3 \Rightarrow \rho_s(n) = 0.6000, \beta = 0.5, \tau = 0.414, \\
&\text{Ogive}(m): m \approx 4.916 \Rightarrow\ \rho_s(m) = 0.6059, \beta = 0.5, \tau = 0.415, \\
&DB(\theta):\ \theta = 0.5 \Rightarrow \rho_s(\theta) = 0.6250,\ \beta = 0.5, \tau = 0.417,
\end{aligned}$$

Smallest ρ_s?

- **Can we select the one with the smallest rank correlation in general?**
- Pdf $p(\cdot|\Psi)$ symmetric on $[0,1] \Rightarrow E[Z|\psi] = \pi\{X,Y|p(\cdot|\Psi)\} = \frac{1}{2}$.
- Pdf $p(\cdot|\Psi)$ symmetric on $[0,1] \Rightarrow \rho_s \equiv 0, \beta \equiv 0, \tau \equiv 0$.
- When elicited $\pi\{X,Y|p(\cdot|\Psi)\} = Pr(Y \leq 0.5|X \leq 0.5) = \frac{1}{2}$ it seems to intuitive to select copula with independent uniform marginals.
- **Hence, answer is: No.**

- Select copula that **minimizes distance** between it and copula with independent uniform marginals.

- **Kullback-Liebler distance** measures **the relative information** of one candidate pdf $f(x,y)$ with respect to pdf $g(x,y)$ given by

$$I(f|g) = \int\int f(x,y)\ln\{f(x,y)/g(x,y)\}dxdy.$$

- Setting $f(x,y) = c\{x,y\}$ and $g(x,y) = u(x,y) = 1,$ yields:

$$I(c|u) = \int\int_{S_c} c(x,y)\ln\{c(x,y)\}dxdy, \qquad (40)$$

- The quantity $E = -I(c|u) \geq$ is known as **the entropy** of the pdf $c(x,y)$.

- Bedford and Meeuwissen (1997) constructed **maximum entropy copulae given a correlation constraint** that are *minimally informative*.

6. COPULA SELECTION...

Max. Entropy

- Bedford and Meeuwissen's (1997) **maximum entropy copulae given a correlation constraint**, do not possess closed form pdf and cdf.

- Select amongst a set of GDB copula that matches a specified constraint that one that is **minimally informative** (or has maximum entropy).

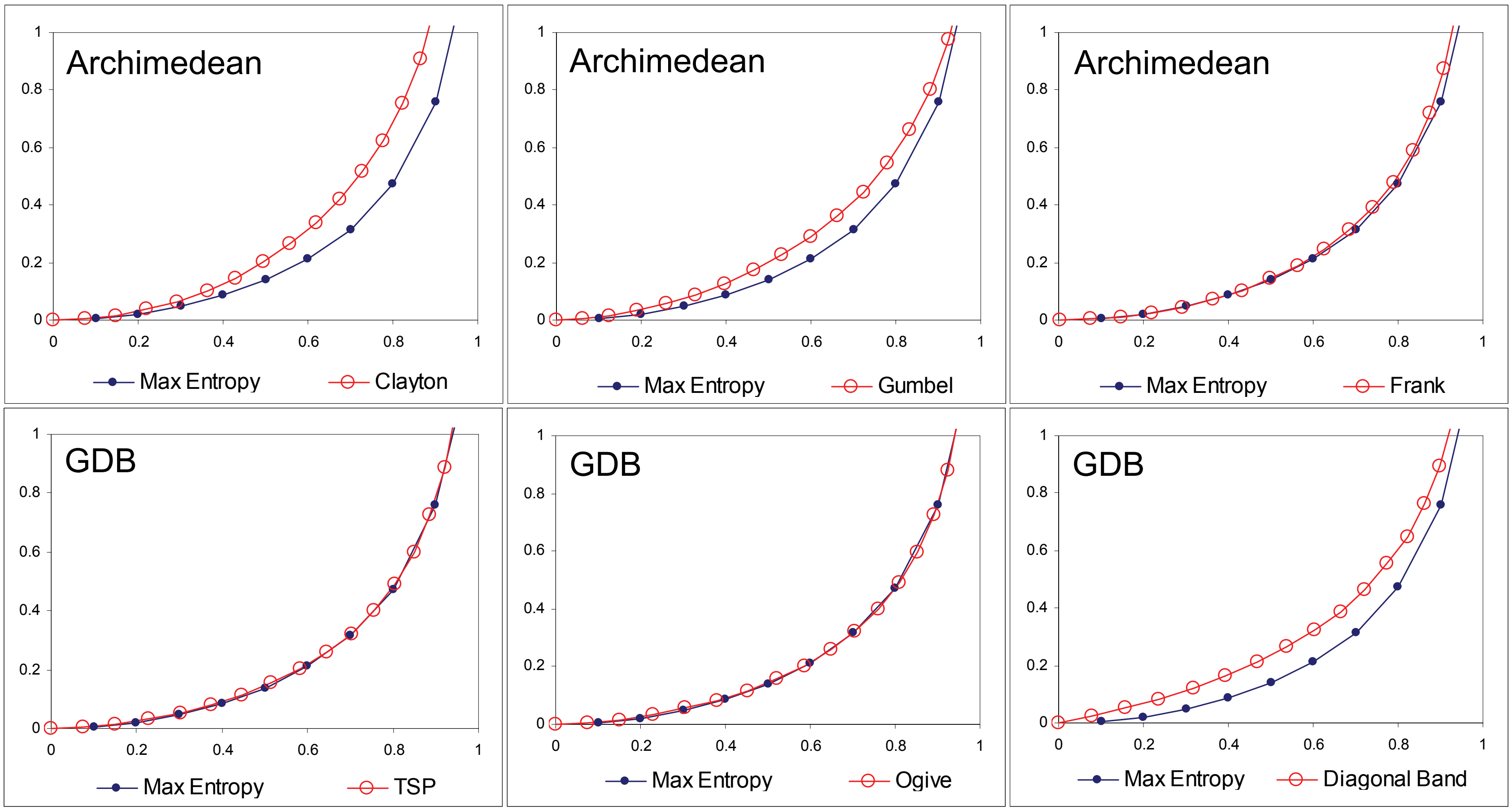
- Utilizing **numerical integration** over a 100 by 100 grid over $[0, 1]^2$, we have

$$I\{c(x,y)|p(\cdot|\psi)\} = \begin{cases} 0.2136, & p(z|n), n = 3, \quad \text{Power pdf,} \\ 0.2222, & p(z|m), m = 4.916, \quad \text{Ogive pdf} \\ 0.3400, & p(z|\theta), \theta = 0.5. \quad \text{U}[\theta, 1] \text{ pdf} \end{cases}$$

- Summarizing, given *the constraint* set by $\pi = Pr(Y \leq 0.5|X \leq 0.5) = 0.75$, the relative information approach above would suggest to **use the GDB copula with the power(n) generating pdf (26) with $n = 3$**.

6. COPULA SELECTION...

Max. Entropy



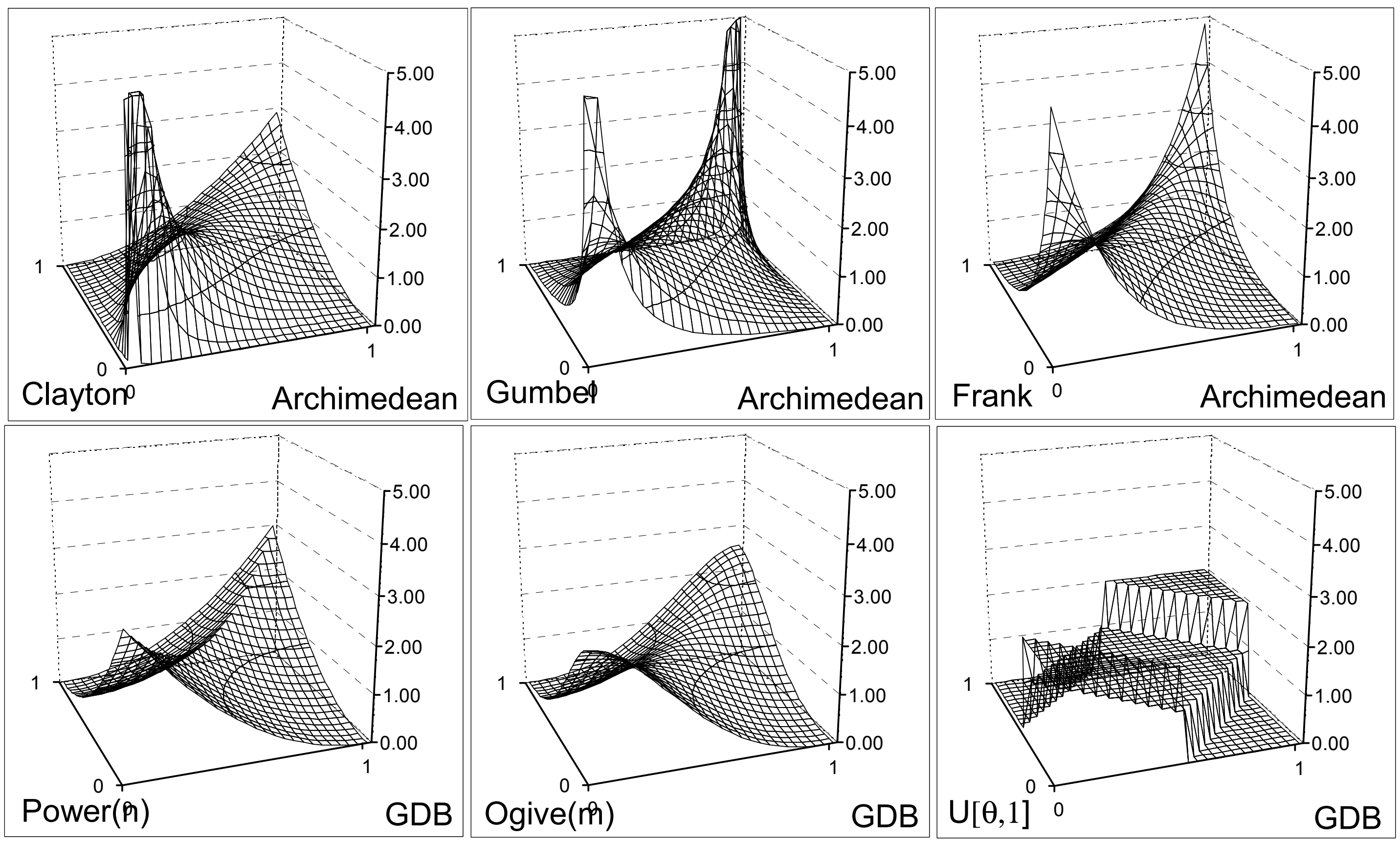
X-Axis: Rank Correlation **Y-Axis: Relative Information = – Entropy**

6. COPULA SELECTION…

How to pick one?

- **Assume that an expert has assessed a value**

$$\pi\{X, Y|p(\,\cdot\,|\Psi)\} = Pr(Y \leq 0.5|X \leq 0.5) = 0.75.$$



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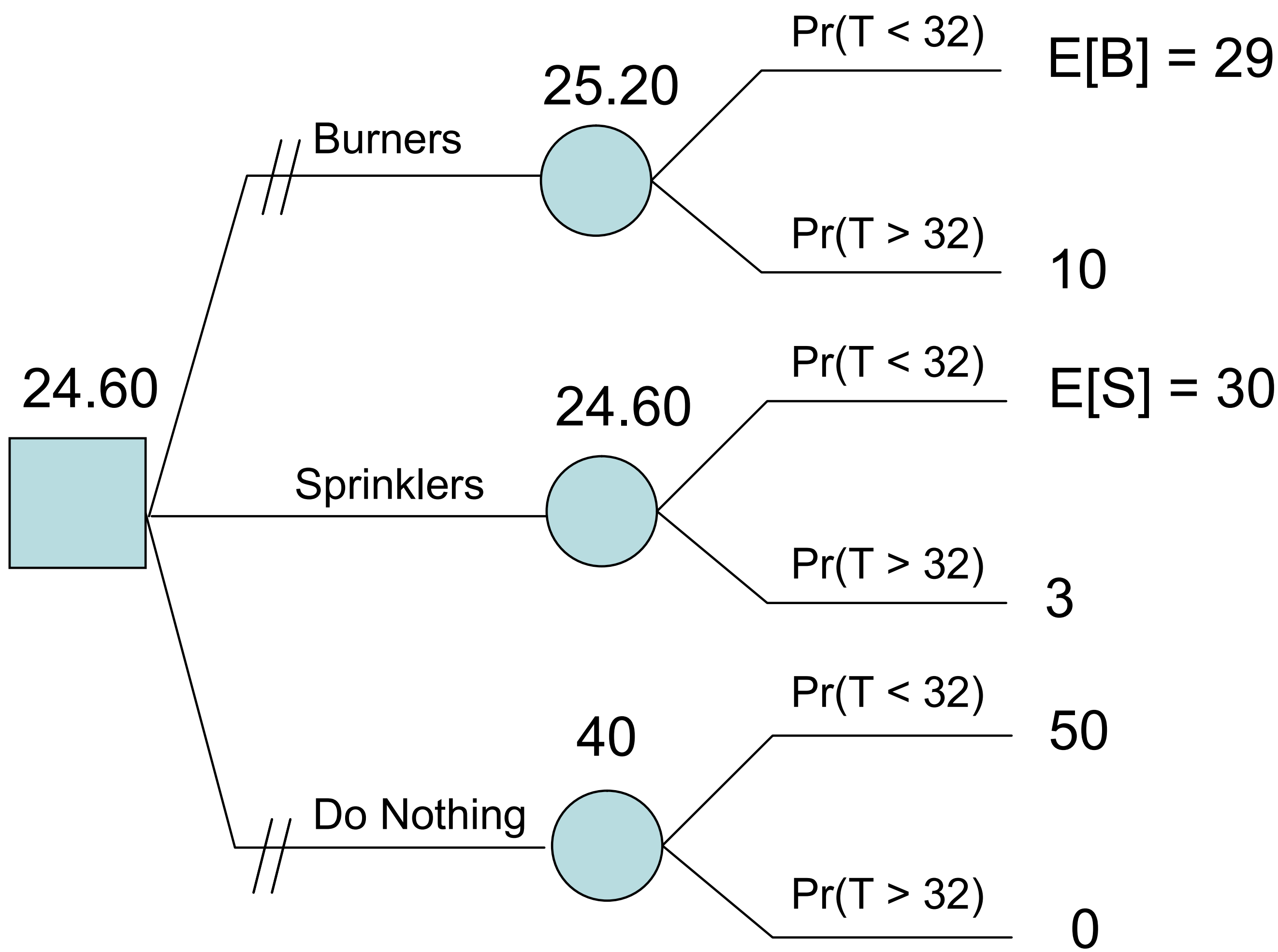
7. A VALUE OF INFORMATION EXAMPLE...

Description

- The farmer needs to **protecting his/her crop of oranges** (with a total worth of $50,000) against freezing weather with the objective of minimizing losses.
- **Temperature T** (in Fahrenheit) $\sim U[24, 34]$ that night.
- $T < 32$ (below freezing) $\Rightarrow$ Farmer looses entire crop **without protection.**
- Two protection alternatives: **Burners** or **Sprinklers** with $\$10,000$ or $\$3,000$, respectively, in mobilization cost.
- Effectiveness of both is uncertain. Farmer assesses **all-in loss** $B\,(S)$ to vary between $a =$ $25000 ($28,000) and $b =$ $35,000 ($33,000) with a most likely value of $m =$ $27,000 ($29,000) **if it freezes**.
- Assume B and S to be **triangular distributed** $\Rightarrow E[B] =$ $29,000, $E[S] =$ $30,000.

7. A VALUE OF INFORMATION EXAMPLE…

Decision Tree



- **Effectiveness sprinkler** option is based on an **insular layer of freezing water** on the oranges.

- **Effectiveness burner** option is based on **gas usage.**

- The farmer assesses a 90% chance (60% chance) that the burning loss B (sprinkler loss S) is **above it median value** $b_{0.5}$ ($s_{0.5}$) when the temperature T is **below its median value** $29F$. Hence, we have:

$$Pr(B > b_{0.5} | T < 29) = 0.9, Pr(S > s_{0.5} | T < 29) = 0.6,$$

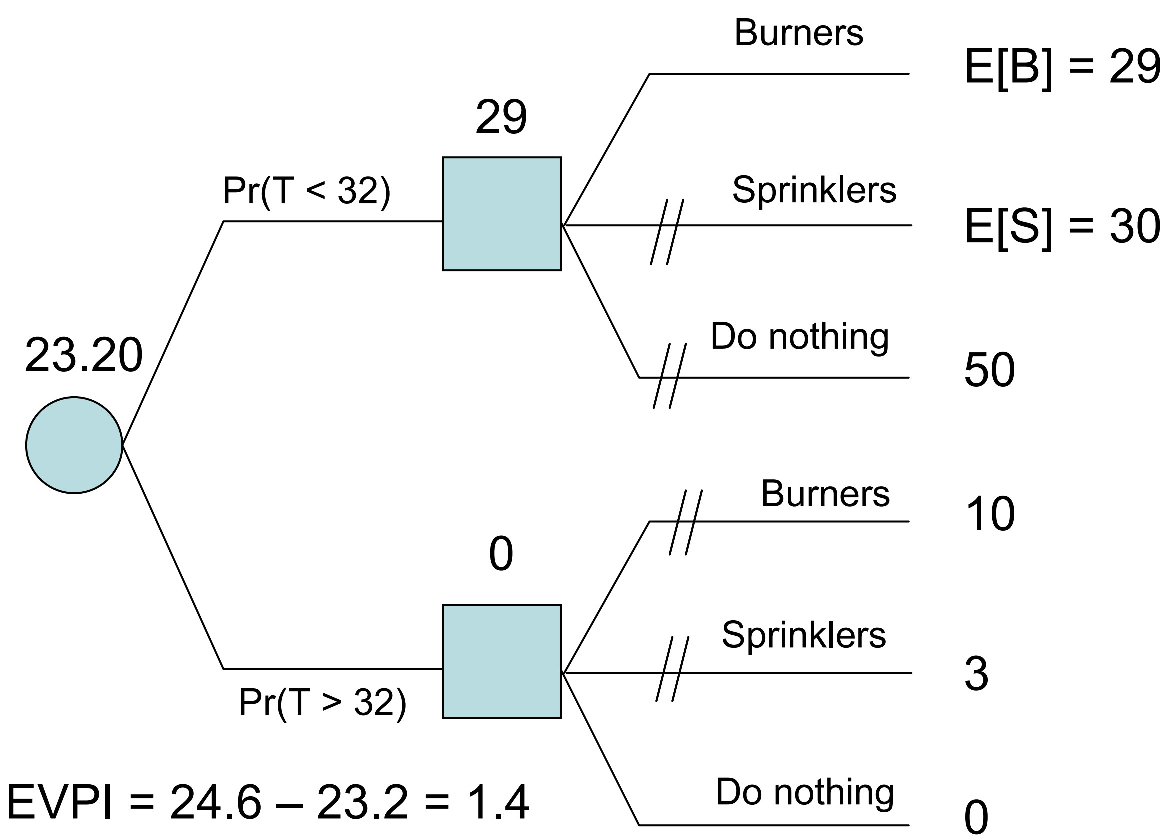
where $b_{0.5} \approx$ \$28,675 and $s_{0.5} \approx$ \$29,838.

- **Model dependence** between B (S) and T using a GDB copula with a power (slope) generating density with $n = 1/11$ ($\alpha = 0.4$).

7. A VALUE OF INFORMATION EXAMPLE…

EVPI Freezing

- To reduce losses further, the farmer considers consulting either *a clairvoyant Expert A* on **"Freezing"** or *a clairvoyant Expert B* on **the temperature *T*.**



EVPI Temperature T

- EVPI on the temperature T from Expert B is **more complicated** since it requires **evaluation of $E[B|t]$ and $E[S|t]$**.

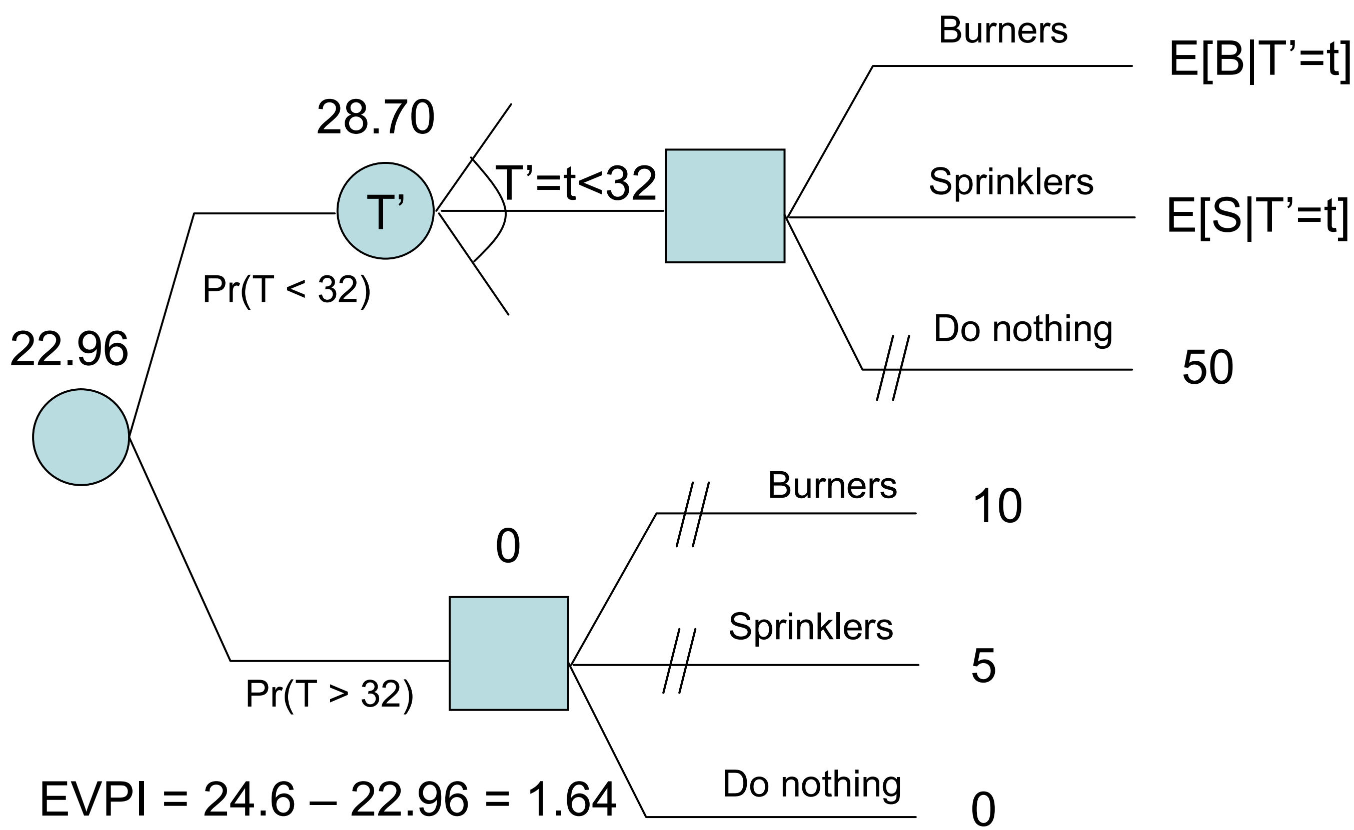
- Given t, we evaluate $E[B|t]$ using $s = 2500$ realizations using the steps:

  Step 1: $x = \frac{t-24}{34-24}$ (Recall, $T \sim Uniform[24, 34]$)
  Step 2: **Sample quantile levels y_i**, $i = 1, \ldots, s$ from GDB(X, Y) copula with power(n) generating density for B, $n = 1/11$.
  Step 3: $E[B|t] = \frac{1}{s}\sum_{i=1}^{s} H^{-1}(y_i)$,

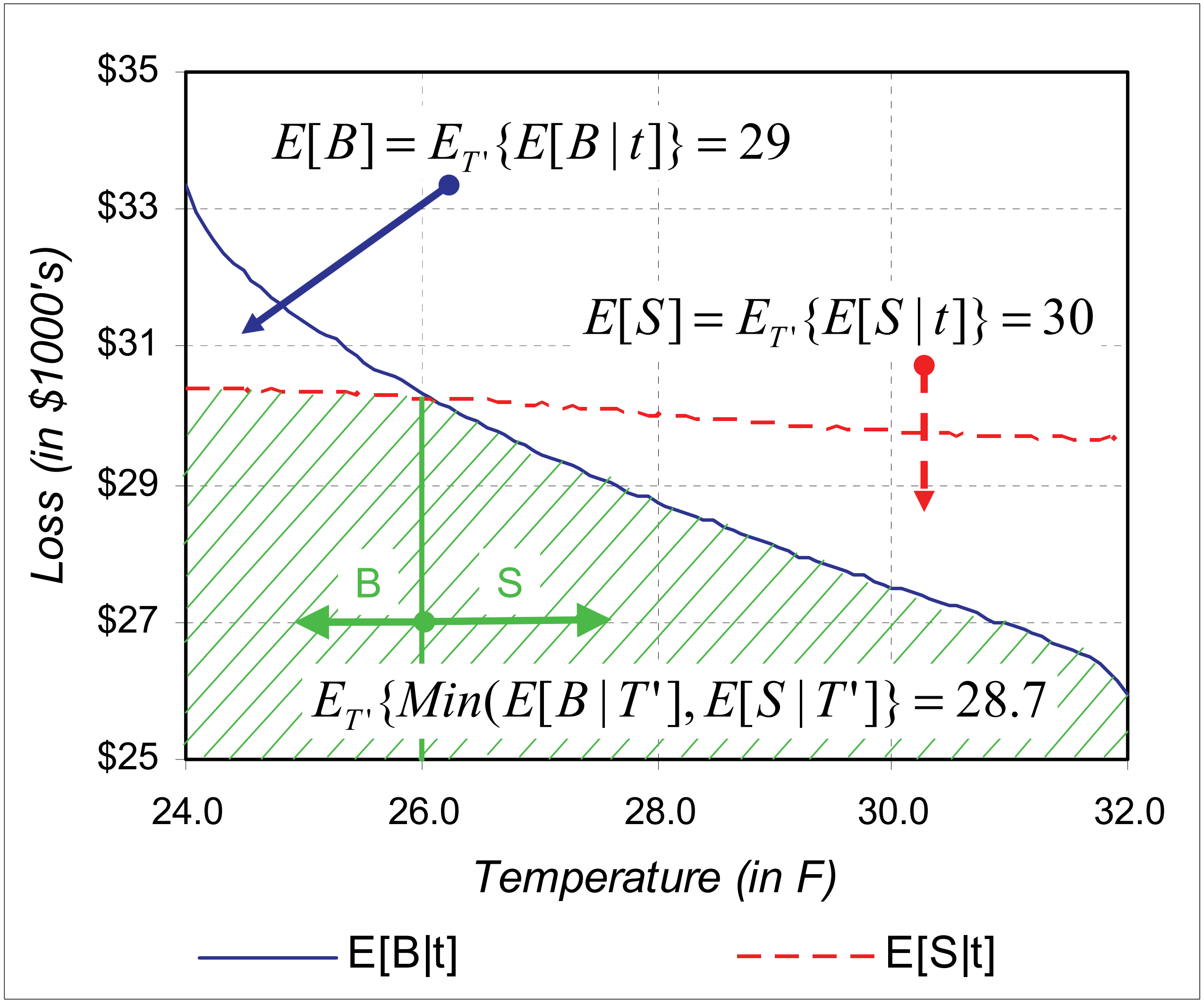
- $B \sim$ Triang(\$25,000; \$27,000; \$35,000), $H^{-1}(\cdot)$ is the inverse cdf or quantile function of B. $S \sim Triang$(\$28,000; \$29,000; \$33,000). Evaluation of $E[S|t]$ is analogous.

7. A VALUE OF INFORMATION EXAMPLE...
EVPI Temperature T

- $T' = (T|T < 32) \sim \mathrm{U}[24, 32]$ since $T \sim \mathrm{U}[24, 34]$.



7. A VALUE OF INFORMATION EXAMPLE…
EVPI Temperature T



7. A VALUE OF INFORMATION EXAMPLE…
EVPI Temperature T

- EVPI "freezing" $\approx$ \$1,400, EVPI "freezing" $\approx$ \$1,640

- Summarizing, the farmer is willing to pay \$240 dollars more for perfect information on the temperature T.

- Optimal decision switches to Sprinkler option when *Expert A* provides "Freezing" information.

- Optimal decision switches to Sprinkler option when *Expert B* provides "temperature t" information, where $26 < t < 32$.

- When *Expert B* provides "temperature t" information, where $24 < t < 26$, the optimal decision remains the Burner option.

QUESTIONS?

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