Lecture on Copulas: Part 1

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OUTLINE

1. **INTRODUCTION**
2. COPULA CONSTRUCTION - ARCHIMEDEAN
3. ARCHIMEDEAN EXAMPLES
4. COPULA CONSTRUCTION - GENERALIZED DIAGONAL BAND
5. GENERALIZED DIAGONAL BAND EXAMPLES
6. SAMPLING PROCEDURE - ARCHIMEDEAN COPULA
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CDF Theorem

Theorem: Let X be a continuous random variable with distribution funnction $F(\cdot)$. Let Y be a transformation of X such that $Y = F(X)$. The distribution of Y is uniform on $[0,1]$.

Proof: For a uniform random variable U on $[0,1]$ we have

$$Pr(U \leq u) = u, \forall u \in [0,1]$$

Hence, we need to show that $Pr(Y \leq y) = y, \forall y \in [0,1]$. Since we have that $F(x) = Pr(X \leq x) \in [0,1]$ for all values of x it follows that $Y = F(X)$ has support $[0,1]$.

$$\begin{aligned} Pr(Y \leq y) &= Pr[F(X) \leq y] = Pr\{F^{-1}[F(X)] \leq F^{-1}(y)\} \\ &= Pr[X \leq F^{-1}(y)] = F[F^{-1}(y)] = y. \end{aligned} \qquad \square$$

Bivariate Normal PDF

- Probability density function of a bivariate normal distribution:

$$\boldsymbol{X} = \binom{X_1}{X_2} \sim MVN(\mu, \Sigma), \text{ Mean Vector } \boldsymbol{\mu} = \binom{\mu_1}{\mu_2},$$

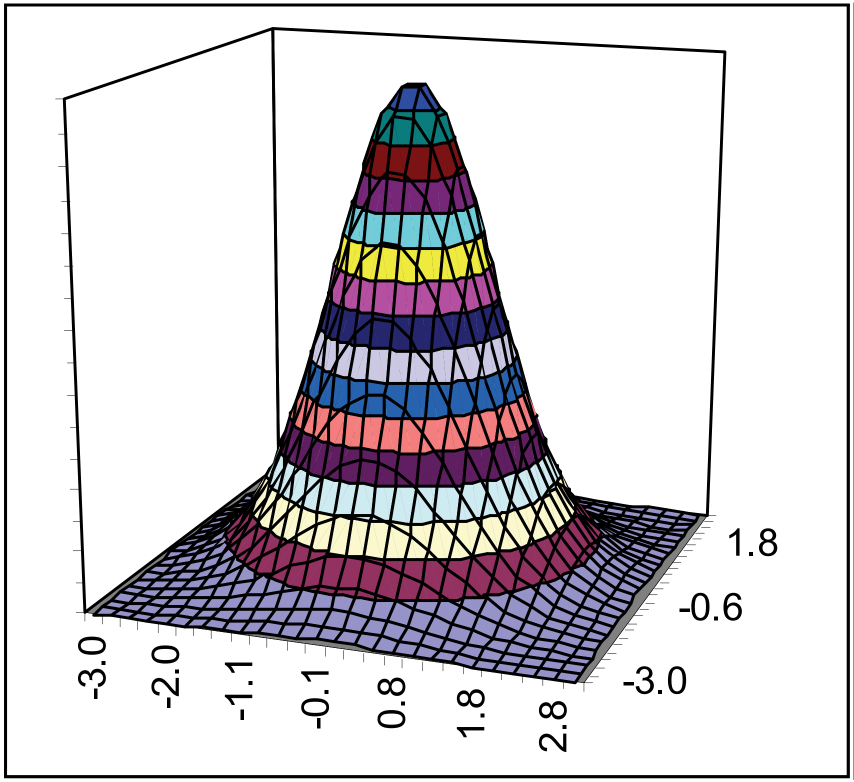
$$\text{Covariance Matrix } \Sigma = \begin{pmatrix} \sigma_1^2 & Cov(X_1, X_2) \\ Cov(X_1, X_2) & \sigma_2^2 \end{pmatrix}$$

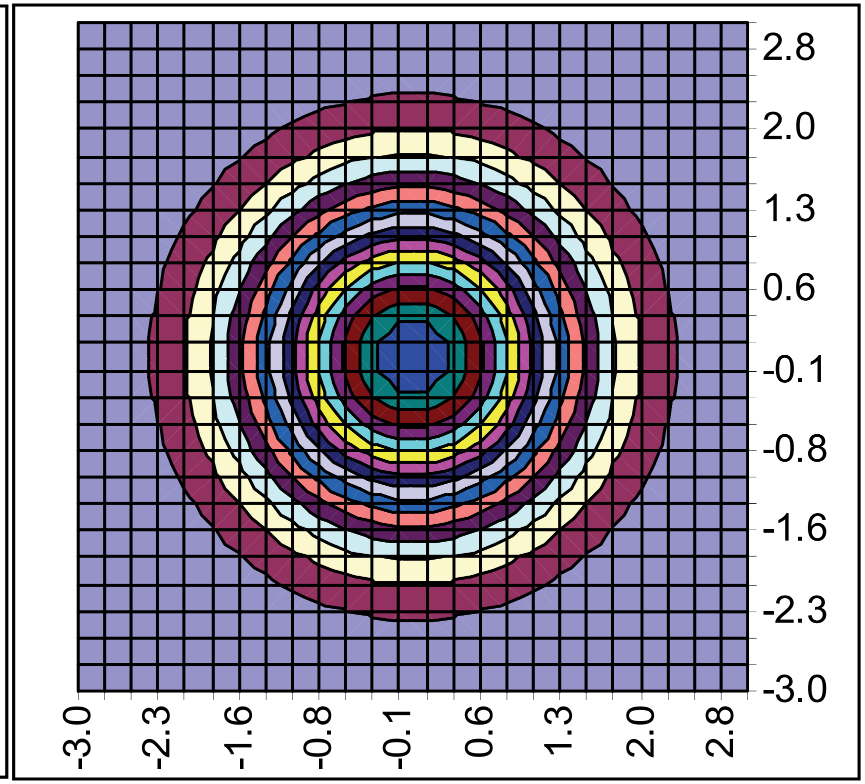
$$f(x, y) = \frac{1}{\sqrt{2\pi|\boldsymbol{\Sigma}|}} exp\Big[(\boldsymbol{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu})\Big]$$

- Independence in case of the bivariate normal distribution implies (and vice versa):

$$\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}, \Sigma^{-1} = \begin{pmatrix} 1/\sigma_1^2 & 0 \\ 0 & 1/\sigma_2^2 \end{pmatrix}$$

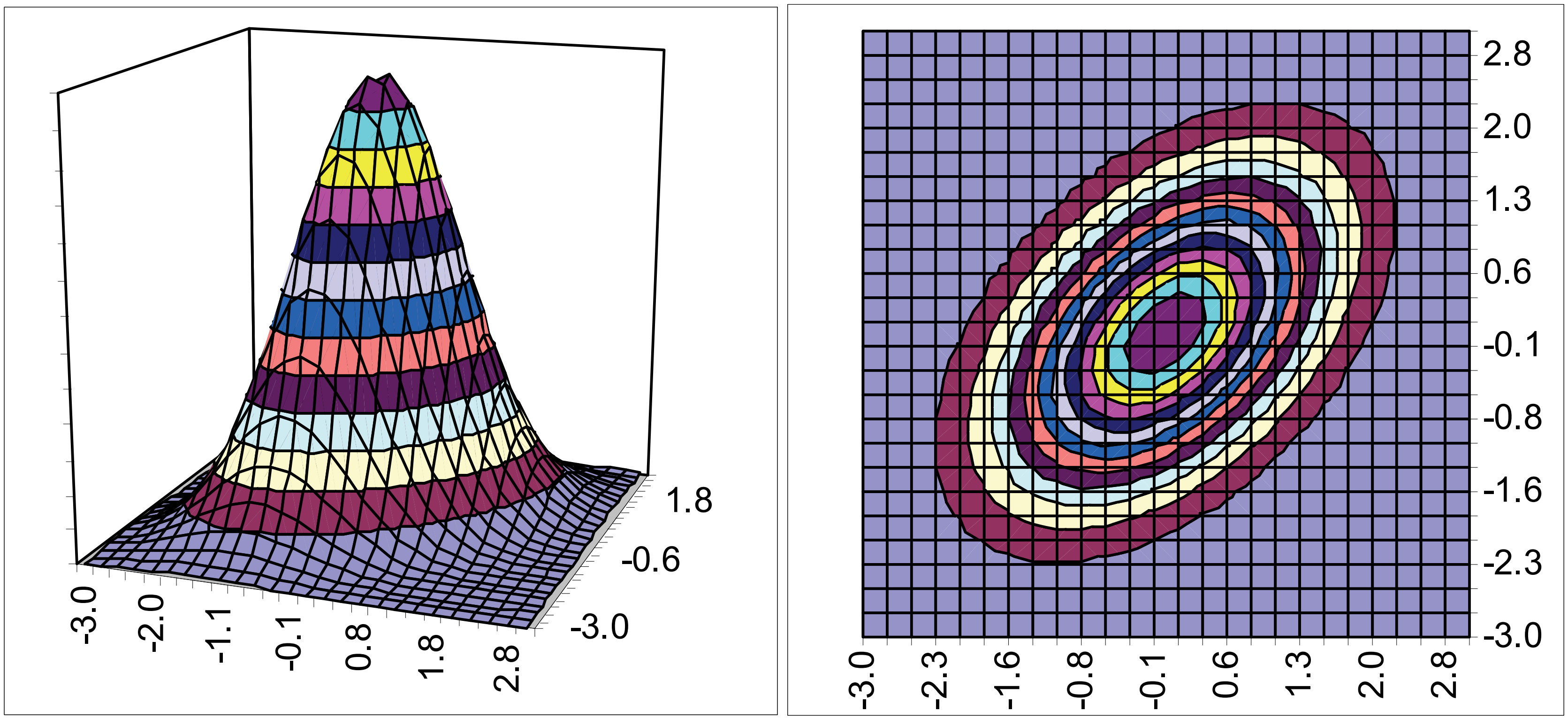
BVN PDF - Independence





$$\boldsymbol{\mu} = \binom{0}{0}, \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \rho = 0.0,\ Pr(Y \leq 0 | X \leq 0) \approx 0.5$$

BVN PDF - Dependence



$$\boldsymbol{\mu} = \binom{0}{0}, \Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \rho \approx 0.483, \ Pr(Y \leq 0 | X \leq 0) \approx 0.75$$

Sklar's (1959) Theorem

Sklar's Theorem (1959). Given a joint CDF $F(x_1,\ldots,x_n)$ for random variables $X_1,\ldots,X_n$ with marginal CDFs $F_{X_1}(\cdot),\ldots\ F_{X_w}(\cdot)$. Then $F(x_1,\ldots,x_n)$ can be written as a function of its marginals:

$$F(x_1,\ldots,x_n) = C\{F_{X_1}(x_1),\ldots,F_{X_n}(x_n)\}$$

where $C(u_1,\ldots,u_n)$ is a joint distribution function with uniform$[0,1]$ marginals. Moreover, if each $F_{X_i}(x_i)$ is continuous, then $C(u_1,\ldots,u_n)$ is unique, and if each $F_{X_i}(x_i)$ is discrete, then C is unique on

$$Ran[F_{X_1}(\cdot)] \times \ldots \times Ran[F_{X_n}(\cdot)]$$

where $Ran[F_{X_n}(\cdot)]$ is the range $F_{X_n}(\cdot)$.

For $F_{X_1}(\cdot)$ and $C(u_1,\ldots,u_n)$ continuous and differentiable case one has:

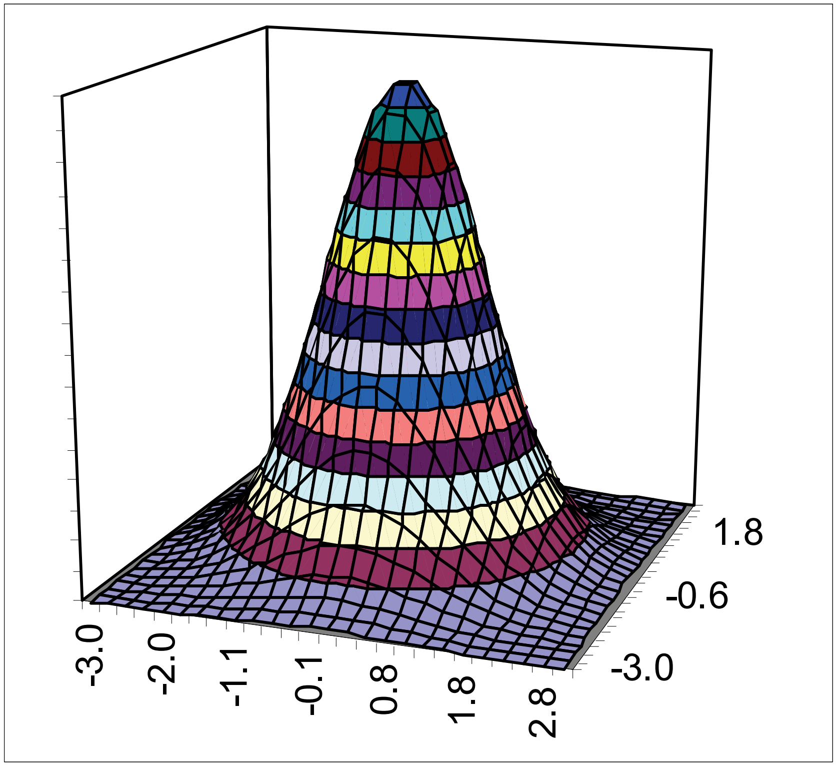
$$f(x_1,\ldots,x_n) = f_{X_1}(x_1) \times \ldots \times f_{X_n}(x_n) \times c\{F_{X_1}(x_1),\ldots\ F_{X_n}(x_n)\}$$

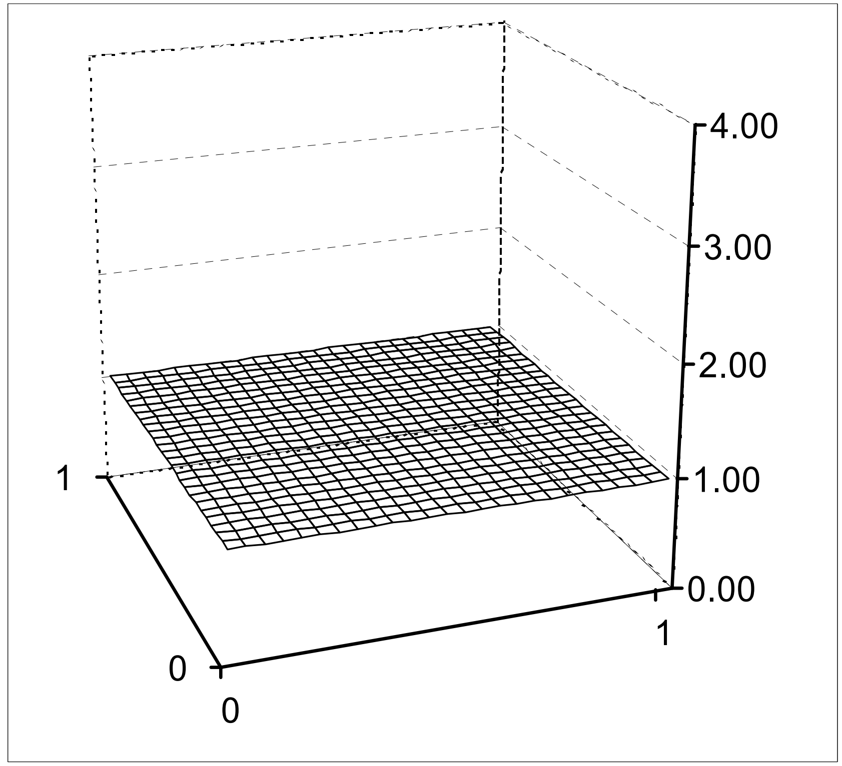
where $c(u_1,\ldots,u_n)$ is copula pdf or *dependence function.*

Sklar's (1959)- The Bivariate Case

- X', Y': **Continuous random variables** such that $X' \sim G(\cdot)$, $Y' \sim H(\cdot)$
- $G(\cdot)$, H(·): **Cumulative distribution functions** - cdf's.
- The mapping $X' \rightarrow X = G(X') \Rightarrow \boldsymbol{X \sim}$ **Uniform[0, 1]** is called the *probability integral transformation* e.g. Nelsen (1999).
- Any bivariate joint distribution of (X', Y') can be transformed to a bivariate copula $(X, Y) = \{G(X'), H(Y')\}$ - Sklar (1959).
- Thus, a bivariate copula is **a bivariate distribution with uniform marginals.**
- As such, many authors studied copulae **indirectly.**
- Gaussian and Student-t Copulae (of this construct) were studied **explicitly.**

BVN Normal Copula

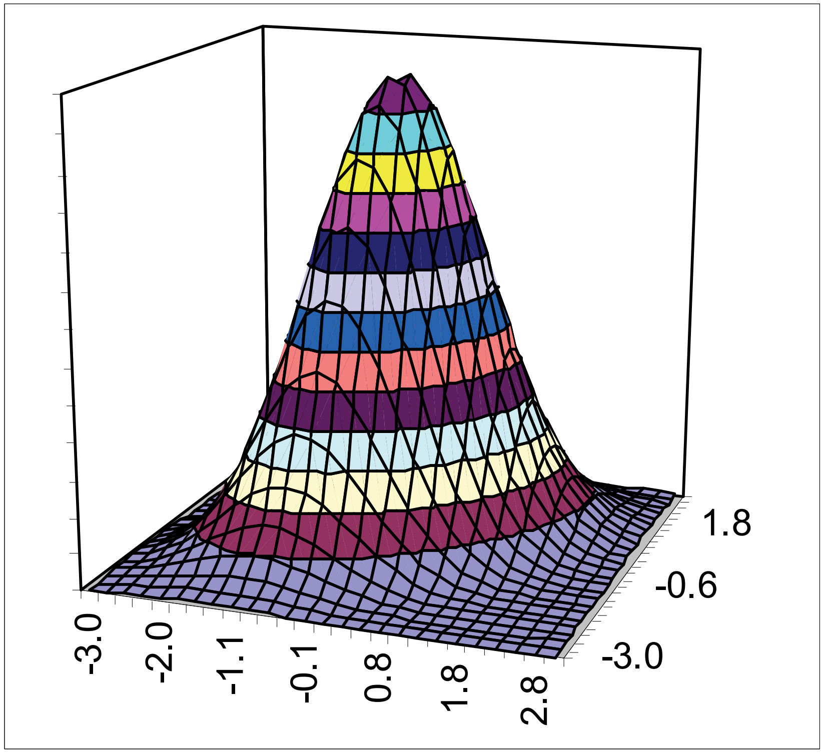


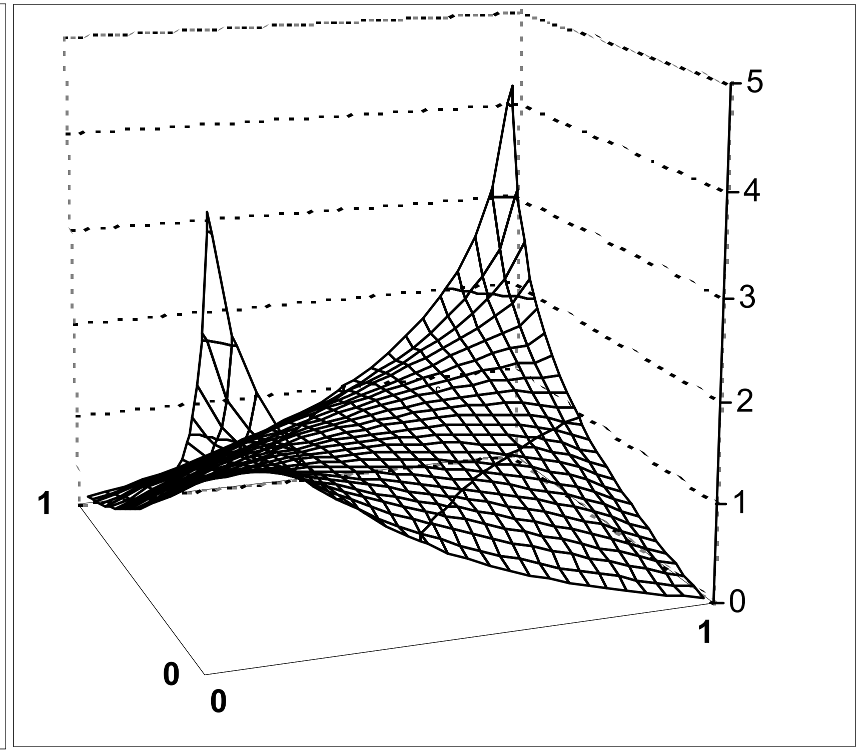


BVN (X', Y') - Independence BVN Copula (X, Y) - Independence

$$X \sim U[0,1], Y \sim U[0,1],\ (X,Y) \sim c(u_1,u_2) = 1, \forall (u_1,u_2) \in [0,1]^2$$

BVN Normal Copula





BVN (X', Y') - Dependence BVN Copula (X, Y) - Dependence

$$X \sim U[0,1], Y \sim U[0,1], \ (X,Y) \sim c(u_1,u_2), (u_1,u_2) \in [0,1]^2$$

Summary

- The dependence relationship between two random variables X', Y' is *obscured* by the marginal densities of X' and Y'.

- One can think of the copula density as the densities that filters or extracts the marginal information from the joint distribution of X' and Y'.

- To describe, study and measure statistical dependence between random X', Y' variables one may study the copula densities.

- Two random vectors (X_1, Y_1) and (X_2, Y_2) share the same dependence relationship when their copula densities are the same.

Inverse CDF Theorem

- Vice versa, to build a joint distribution between two random variables $X' \sim G(\cdot)$ and $Y' \sim H(\cdot)$, one may construct first the copula on $[0,1]^2$ and utilize the inverse transformation $G^{-1}(\cdot)$ and $H^{-1}(\cdot)$.

Theorem: Let X be a continuous random variable with distribution funnction $F(\cdot)$. Let Y be a transformation of $U \sim [0,1]$ such that $Y = F^{-1}(U)$. Then Y also has distribution function $F(\cdot)$.

Proof: For a uniform random variable U on $[0,1]$ we have

$$Pr(U \leq u) = u, \forall u \in [0,1]$$

Hence, we need to show that $Pr(Y \leq y) = F(y)$.

$$\begin{aligned} Pr(Y \leq y) &= Pr[F^{-1}(U) \leq y] = Pr\{F[F^{-1}(U)] \leq F(y)\} \\ &= Pr[U \leq F(y)] = F(y). \end{aligned} \qquad \square$$

Risk Management?

Example 1: Let S_i be the value of Stock i. Let $R = \sum_{i=1}^{n} w_i S_i$, $\sum_{i=1}^{n} w_i = 1$, $w_i > 0$.

"5% Value-at-Risk" of a Portfolio is defined as follows:

$$Pr(R < VAR) = 0.05$$

Gaussian Copulas have been used to model dependence between $(S_1, \ldots, S_n)$

Example 2: Four satellite are needed in orbit to make three dimensional observations of the earth magnetosphere. The orbit each selected so that each islocated at the corner point of a predetermined tethahedron, when crossing regions of interest within the magnetosphere.

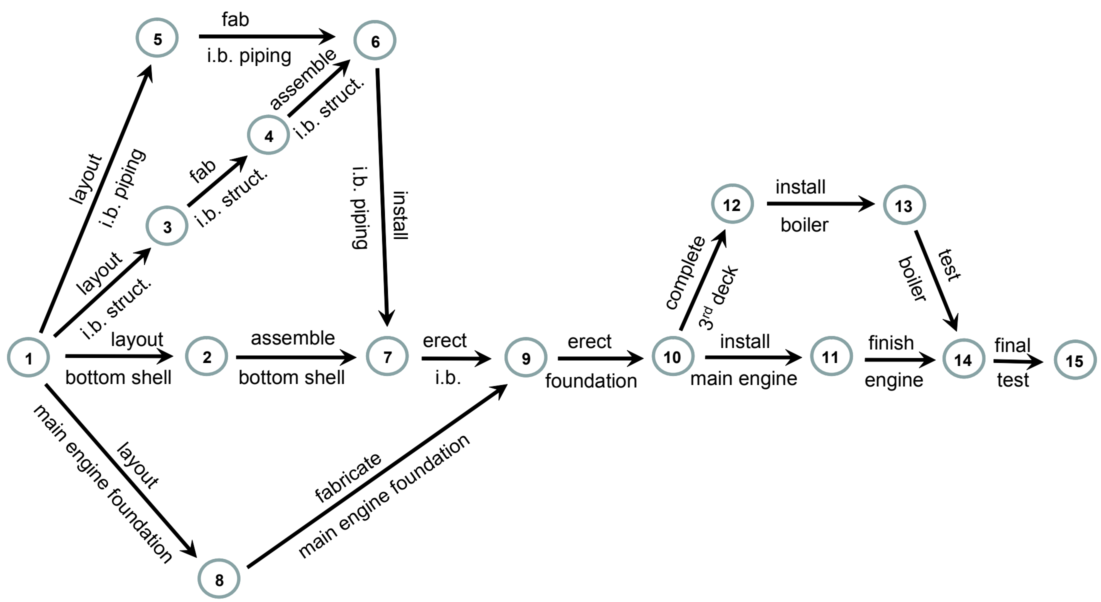
Risk Management?

Lifetime of the indivdual satelites are random but dependent. Let $X_{4,S}$ be the lifetime of the system if four satellites are put into orbit and $X_{5,S}$ if a fifth redundant satelite if put into orbit. Copulas can be used to study the difference between $X_{4,S}$ and $X_{5,S}$. Is reliability improvement of sending a fifth satelite into orbit worth its cost when taking dependence into account?

$$X_{4,S} = Min(X_1, X_2, X_3, X_4)$$

$$X_{5,S} = Max\Big[Min\{X_1, X_2, X_3, X_4\}, Min\{X_1, X_2, X_3, X_5\}, \\ Min\{X_1, X_2, X_4, X_5\}, Min\{X_1, X_3, X_4, X_5\}, \\ Min\{X_2, X_3, X_4, X_5\}\Big]$$

Example 3: Consider the following project network representating the constuction of a ship. The activity completion times are dependent random variables. Copulas can be used to construct the dependence between these random variables and evaluate the project completion time distribution.



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2. COPULA CONSTRUCTION…

Archimedean Copulas

- Genest and Mackay (1986) used **an algabraic method** for copula construction.

- $\varphi : (0,1] \rightarrow [0,\infty)$, *a convex decreasing function* with $\varphi(1) = 0$ - **The generator function.**

- They possess **joint cdf and probability density function (pdf)**:

$$C\{x,y|\varphi(\cdot)\} = \begin{cases} \varphi^{-1}\{\varphi(x)+\varphi(y)\} & \varphi(x)+\varphi(y) \leq 0 \\ 0 & \text{elsewhere} \end{cases} \qquad (1)$$

$$c\{x,y|\varphi(\cdot)\} = -\frac{\varphi''\{C(x,y)\}\varphi'(x)\varphi'(y)}{[\varphi'\{C(x,y)\}]^3} \qquad (2)$$

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Clayton Copula

- **Generator Function:**

$$\varphi(t) = t^{-\alpha} - 1,\ \varphi^{-1}(s) = (1+s)^{-1/\alpha},\ \alpha \geq 0.$$

- **Cumulative Distribution Function:**

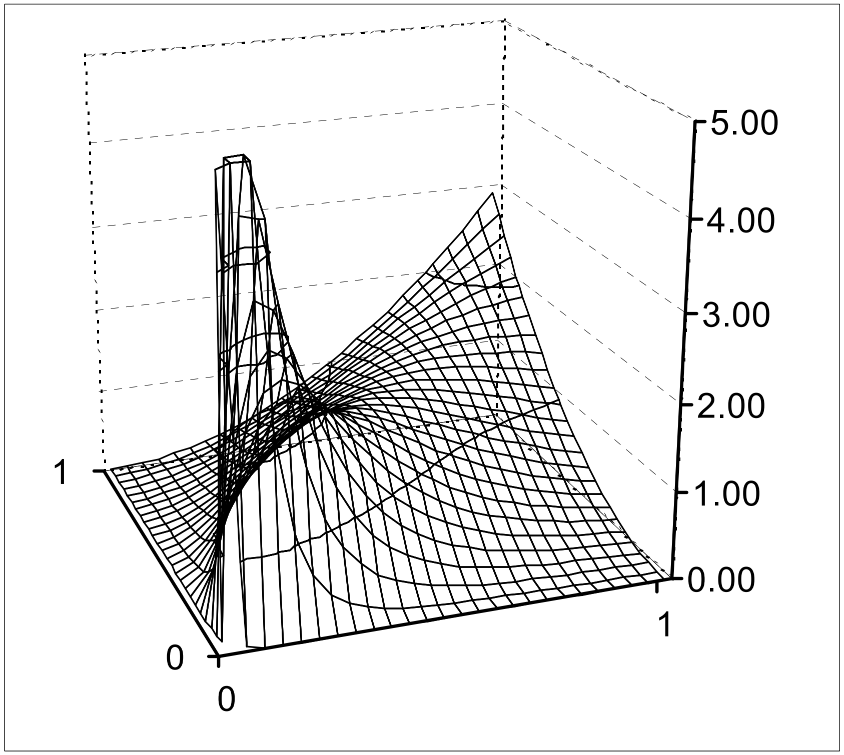
$$C(x, y|\alpha) = \Big[(x)^{-\alpha} + (y)^{-\alpha} - 1\Big]^{-1/\alpha},\ \alpha \geq 0.$$

- **Probability Density Function:**

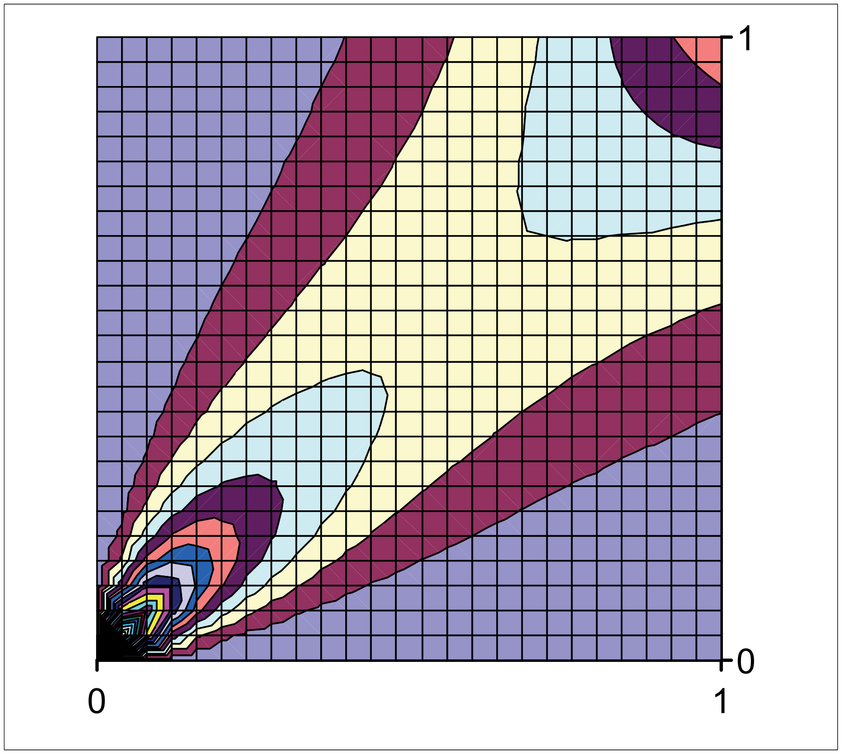
$$c(x, y|\alpha) = \frac{(1+\alpha)}{(xy)^{\alpha+1}}\left[\frac{(xy)^{\alpha}}{x^{\alpha} + y^{\alpha} - (xy)^{\alpha}}\right]^{\frac{1+2\alpha}{\alpha}},\ \alpha \geq 0.$$

Clayton Copula

$$Pr(Y \leq 0.5 | X \leq 0.5) = 0.75,\ \alpha \approx 1.915$$



Joint Probability Density Function



Contour Plot PDF

Gumbel Copula

- **Generator Function:**

$$\varphi(t) = (-\ln t)^{\alpha},\ \varphi^{-1}(s) = exp(-s^{1/\alpha}),\ \alpha \geq 1.$$

- **Cumulative Distribution Function:**

$$C(x, y|\alpha) = Exp\Big\{-\Big[(-\ln x)^{\alpha} + (-\ln y)^{\alpha}\Big]^{1/\alpha}\Big\},\ \alpha \geq 1.$$

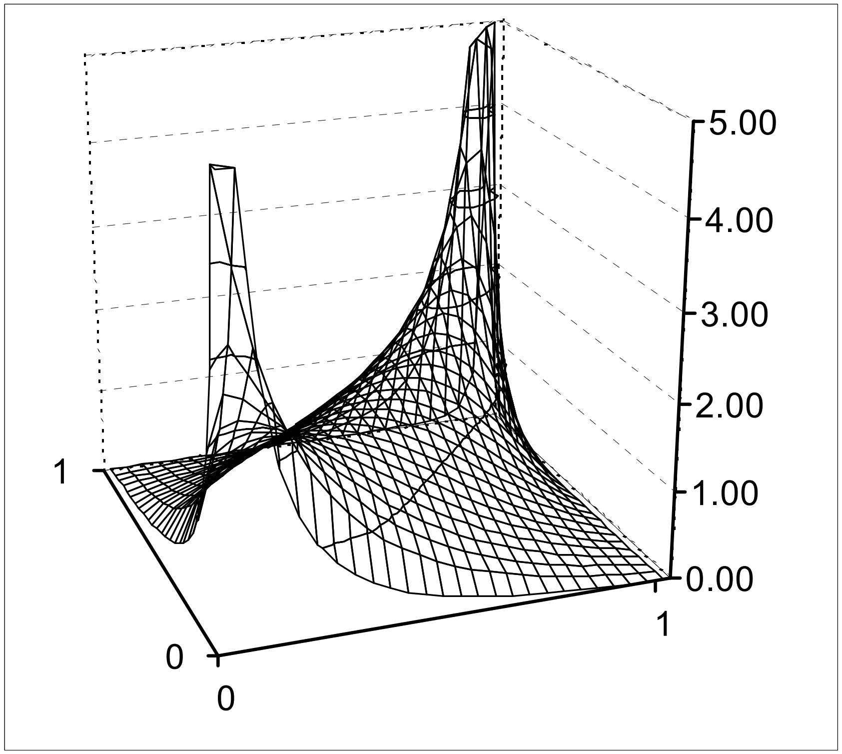
- **Probability Density Function:**

$$c(x, y|\alpha) = Exp\Big\{-\Big[(-\ln x)^{\alpha} + (-\ln y)^{\alpha}\Big]^{1/\alpha}\Big\}\frac{(-\ln x)^{\alpha-1}}{x}\frac{(-\ln y)^{\alpha-1}}{y} \times$$
$$\Big\{\Big[(-\ln x)^{\alpha} + (-\ln y)^{\alpha}\Big]^{2/\alpha-2} + (\alpha-1)\Big[(-\ln x)^{\alpha} + (-\ln y)^{\alpha}\Big]^{1/\alpha-2}\Big\},\ \alpha \geq 1.$$

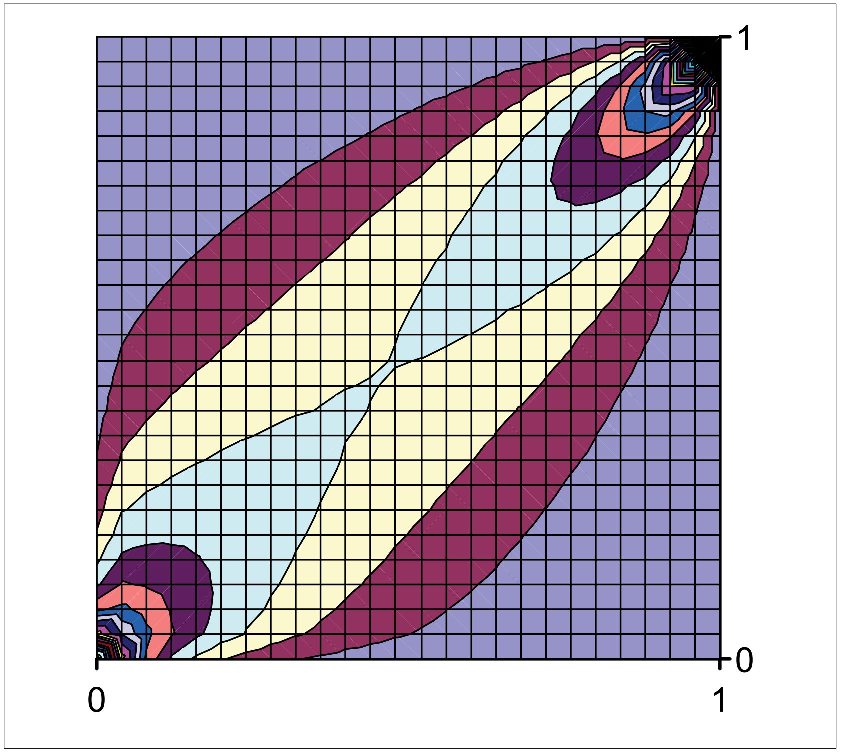
3. ARCHIMEDEAN EXAMPLES...

Gumbel Copula

$$Pr(Y \leq 0.5|X \leq 0.5) = 0.75,\ \alpha \approx 1.997$$



Joint Probability Density Function



Contour Plot PDF

3. ARCHIMEDEAN EXAMPLES...

Frank Copula

- **Generator Function:**

$$\varphi(t) = \ln\frac{e^{\alpha t}-1}{e^{\alpha}-1},\ \varphi^{-1}(s) = \alpha^{-1}\ln[1+e^{s}(e^{\alpha}-1)],\ \alpha \in \mathbb{R}\backslash\{0\}$$

- **Cumulative Distribution Function:**

$$-\frac{1}{\alpha}\ln\left\{1+\frac{(e^{-\alpha x}-1)(e^{-\alpha x}-1)}{e^{-\alpha}-1}\right\},\ \alpha \in \mathbb{R}\backslash\{0\}$$

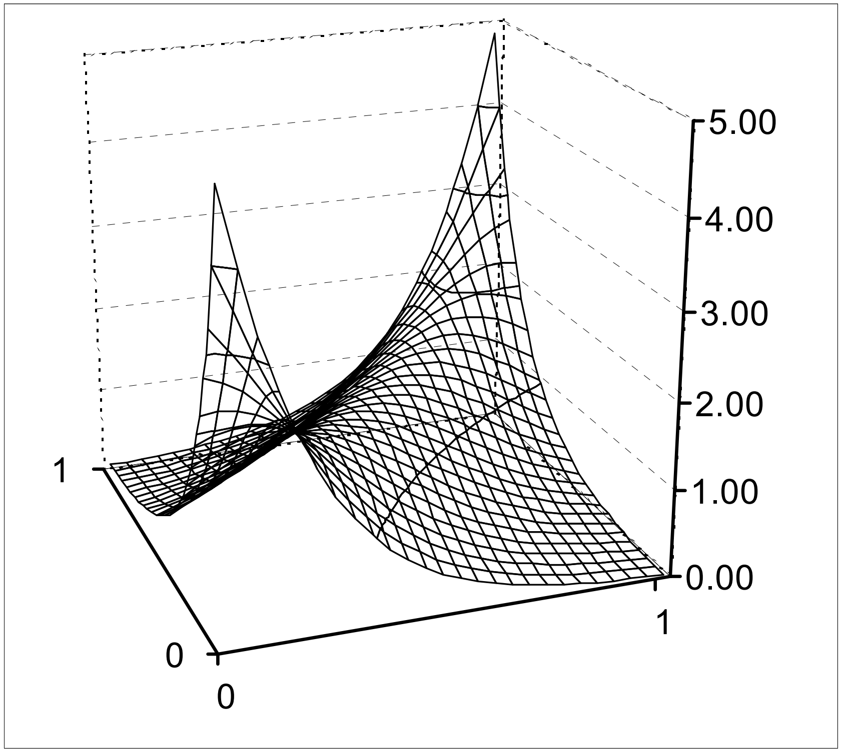
- **Probability Density Function:**

$$c(x,y|\alpha) = \frac{\alpha e^{-\alpha x}e^{-\alpha y}}{e^{-\alpha}-1}\left\{1+\frac{(e^{-\alpha x}-1)(e^{-\alpha x}-1)}{e^{-\alpha}-1}\right\}^{-1} \times$$
$$\left[\frac{(e^{-\alpha x}-1)(e^{-\alpha x}-1)}{e^{-\alpha}-1}\left\{1+\frac{(e^{-\alpha x}-1)(e^{-\alpha x}-1)}{e^{-\alpha}-1}\right\}^{-1}-1\right],\ \alpha \in \mathbb{R}\backslash\{0\}$$

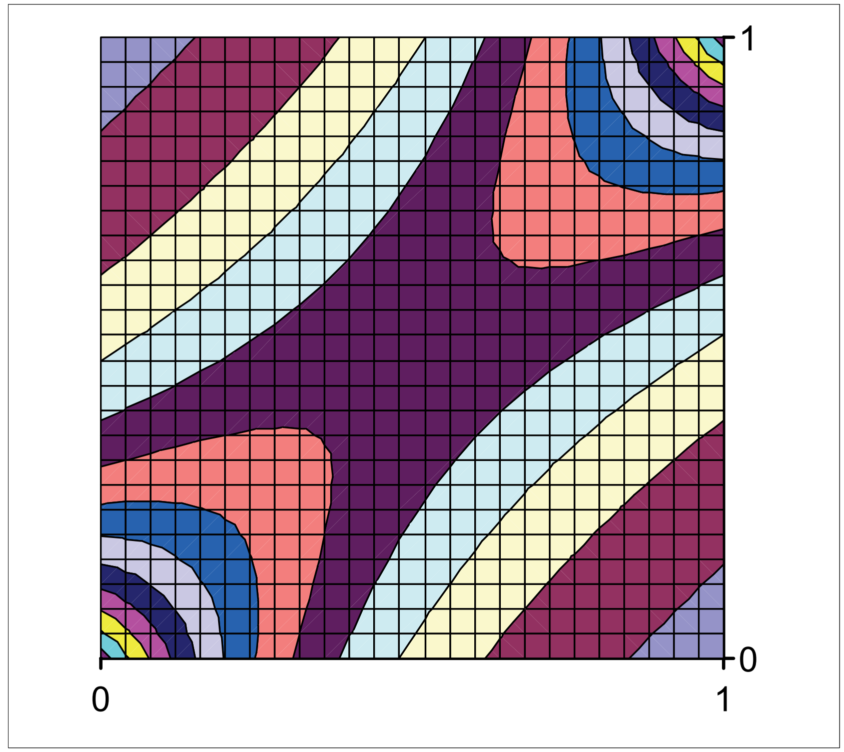
3. ARCHIMEDEAN EXAMPLES…

Frank Copula

$$Pr(Y \leq 0.5 | X \leq 0.5) = 0.75,\ \alpha \approx 4.875$$



Joint Probability Density Function



Contour Plot PDF

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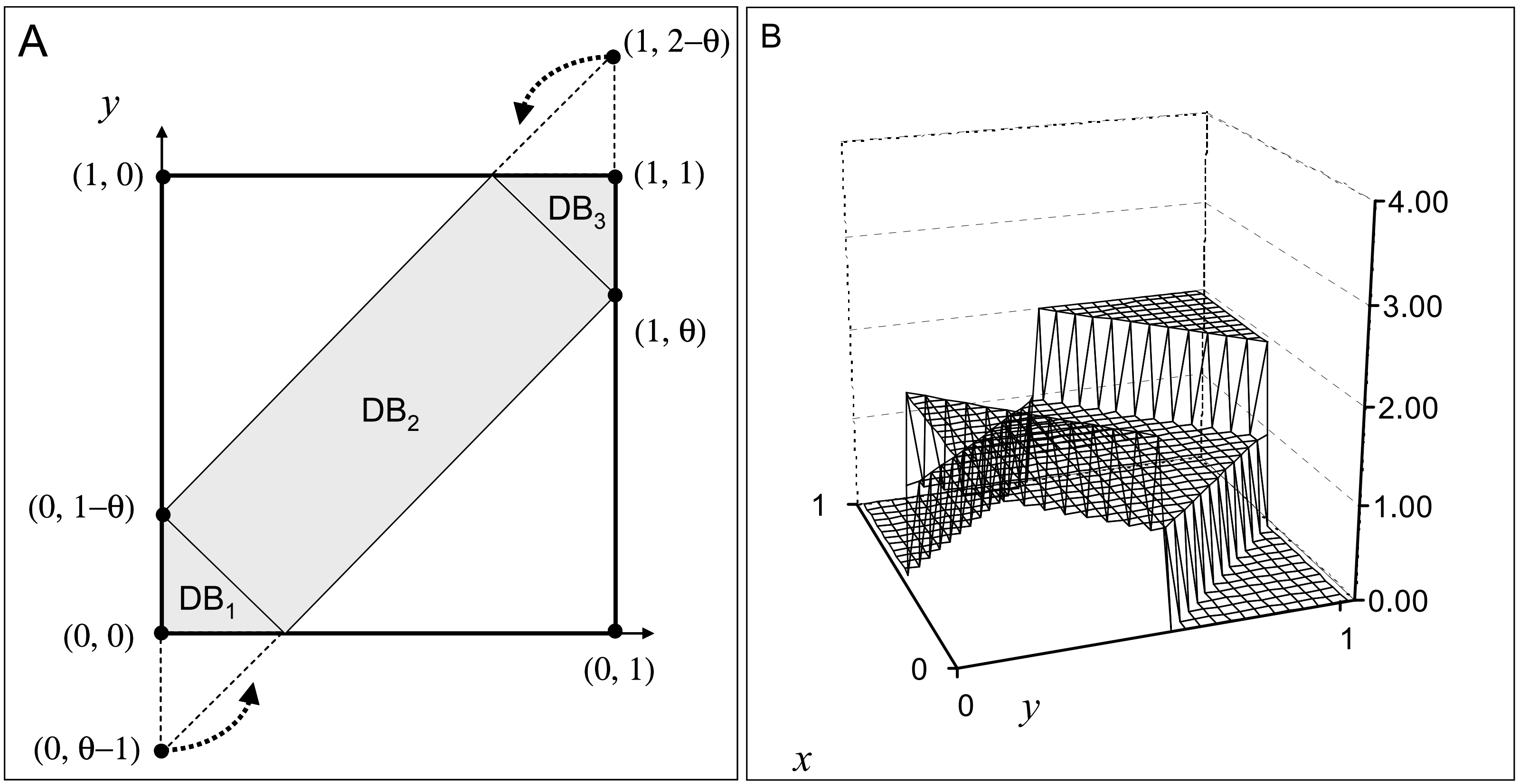
Genealized Diagonal Band

- Cooke and Waij (1986) used **a geometric method** for copula construction



A: Gray area support of a $DB(\theta)$ copula comprised of sub-areas $DB_i, i = 1, 2, 3$; B: Example of a $DB(0.5)$ copula.

Diagonal Band Copula

- Diagonal Band (DB) copula possess pdf:

$$C\{x,y|\theta\} = \begin{cases} 1/(1-\theta) & (x,y) \in DB_1 \cup DB_3 \\ 1/\{2(1-\theta)\} & (x,y) \in DB_2 \\ 0 & \text{elsewhere} \end{cases} \quad (3)$$

- *Analagous to Archimedean copula*, Bojarski (2001) generalized $DB(\theta)$ copula via **a generator function $f(\cdot\,|\theta)$.**

- Generator function $f(\cdot\,|\theta)$ is a **symmetric pdf** with support $[\theta-1, 1-\theta]$.

- Lewandowski (2005) showed that Bojarski's (2001) GDB Copulae are equivalent to Fergusons (1995) family of copulae with joint pdf:

$$c(x,y) = \frac{1}{2}\{g(|x-y| + g(1-|1-x-y|)\}, g(\cdot\,) \text{ pdf on } [0,1] \quad (4)$$

Generalized DB Copula

- For sampling efficiency **inverse cdf** of generator $f(\cdot|\theta)$ would be desirable.

- Consider Van Dorp and Kotz's (2003) **symmetric Two-Sided (TS) pdf's** :

$$f\{z|p(\cdot|\Psi)\} = \frac{1}{2} \times \begin{cases} p(z+1|\Psi), & \text{for } -1 < z \leq 0, \\ p(1-z|\Psi), & \text{for } \quad 0 < z < 1, \end{cases} \tag{5}$$

that too uses the generating pdf $p(z)$ concept. Pdf $p(z)$ has support $[0,1]$.

- The **inverse cdf (or quantile function)** associated with (4)

$$F^{-1}\{u|p(\cdot|\Psi)\} = \begin{cases} P^{-1}(2u|\Psi) - 1, & \text{for } 0 < u \leq \frac{1}{2}, \\ 1 - P^{-1}(2-2u|\Psi), & \text{for } \frac{1}{2} < u < 1, \end{cases} \tag{6}$$

where $P^{-1}(\cdot|\psi)$ is the quantile function of $p(\cdot|\Psi)$.

GDB Copula with TS Gen. PDF

- Bivariate pdf $g(x, y)$ is constructed, where $X \sim U[0,1]$ and **the conditional pdf $g(y|x)$** has the following form :

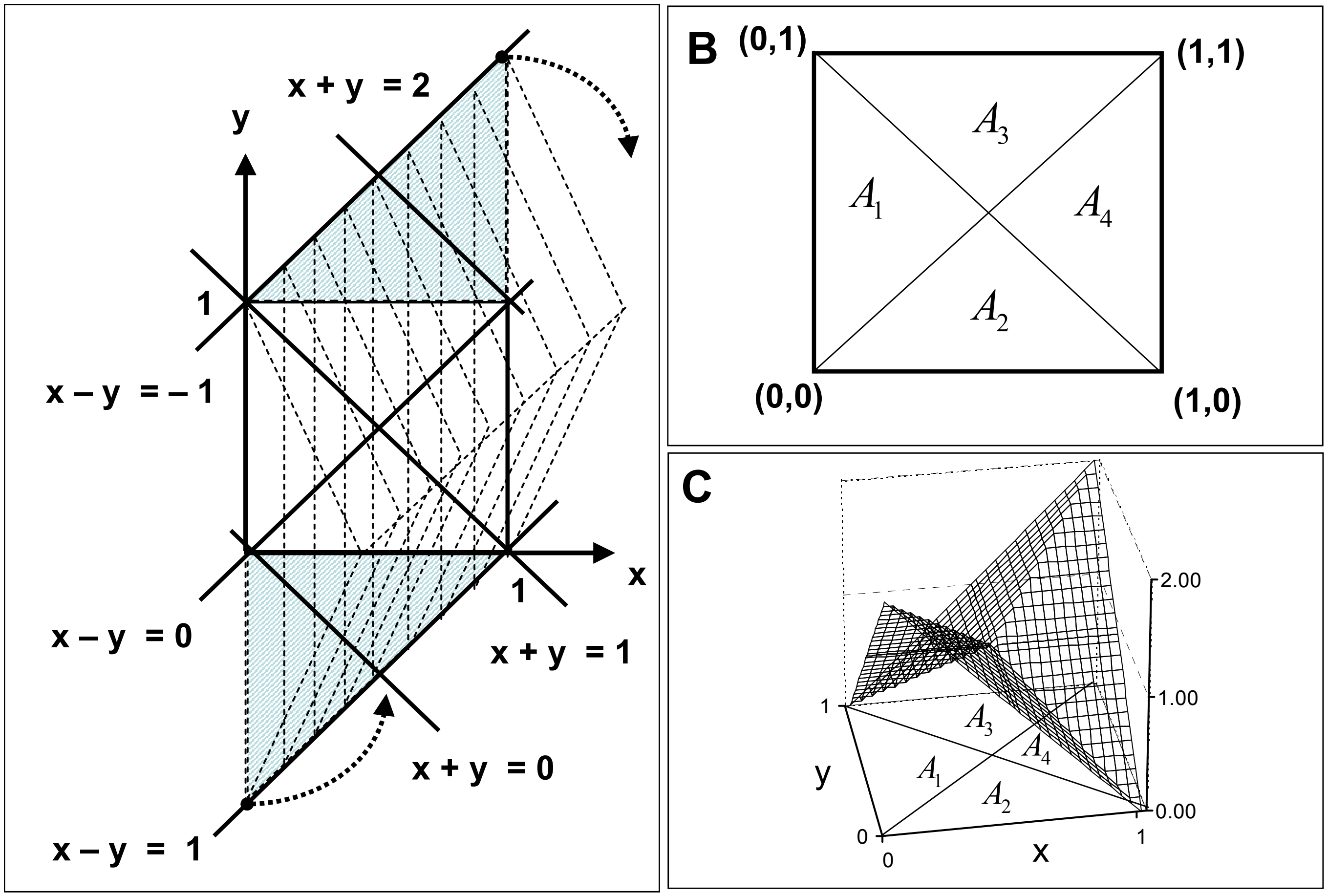
$$g\{y|x, p(\cdot|\Psi)\} = f\{x - y|p(\cdot|\Psi)\}, x - 1 \leq y \leq x + 1, \qquad (7)$$

- From $X \sim U[0,1]$, (7) and **TS framework pdf (4)** it follows that:

$$g\{x, y|p(\cdot|\Psi)\} = \frac{1}{2} \times \begin{cases} p(1 + x - y|\Psi), & -1 < x - y \leq 0, \\ p(1 - x + y|\Psi), & 0 < x - y < 1, \end{cases} \qquad (8)$$

- From (8), a bivariate pdf $c(x, y|p(\cdot|\Psi))$ is constucted on the unit square $[0,1]^2$ **by folding back the probability masses** of $g\{x, y|p(\cdot|\Psi)\}$ outside the unit square $[0,1]^2$ onto it, **using "folding" lines $y = 1$ and $y = 0$.**

GDB Copula with TS Gen. PDF



A: $g(x,y)$ pdf (8); B: Areas $A_i, i = 1,\ldots,4$;
C: $c\{x,y|p(\cdot|\Psi)\}$ pdf (10) with $p(z) = 2z$ on $[0,1]$.

- **Relationship between $c\{x, y|p(\cdot|\Psi)\}$ and $g\{x, y|p(\cdot|\Psi)\}$** in (8) :

$$c\{x, y|p(\cdot|\Psi)\} = \begin{cases} g\{x, y|p(\cdot|\Psi)\} + g\{x, -y|p(\cdot|\Psi)\}, & 0 < x + y \leq 1, \\ g\{x, y|p(\cdot|\Psi)\} + g\{x, 2-y|p(\cdot|\Psi)\}, & 1 < x + y \leq 2. \end{cases} \quad (9)$$

- **Combining (9) with (8)** now yields :

$$c\{x, y|p(\cdot|\Psi)\} = \frac{1}{2} \times \begin{cases} p(1-x-y|\Psi) + p(1+x-y|\Psi), & (x,y) \in A_1, \\ p(1-x-y|\Psi) + p(1-x+y|\Psi), & (x,y) \in A_2, \\ p(x+y-1|\Psi) + p(1+x-y|\Psi), & (x,y) \in A_3, \\ p(x+y-1|\Psi) + p(1-x+y|\Psi), & (x,y) \in A_4. \end{cases} \quad (10)$$

- Note in (10) $\boldsymbol{c(y,x) = c(x,y)}$. Hence, $\boldsymbol{X \sim U[0,1] \Rightarrow Y \sim U[0,1]}$

Joint CDF

- Pdf of GDB copula with **TS pdf with generating pdf $p(z|\Psi)$** :

$$c\{x,y|p(\cdot|\Psi)\} =$$
$$\frac{1}{2}\times\begin{cases} p(1-x-y|\Psi)+p(1+x-y|\Psi), & (x,y)\in A_1, \\ p(1-x-y|\Psi)+p(1-x+y|\Psi), & (x,y)\in A_2, \\ p(x+y-1|\Psi)+p(1+x-y|\Psi), & (x,y)\in A_3, \\ p(x+y-1|\Psi)+p(1-x+y|\Psi), & (x,y)\in A_4. \end{cases}$$

- Cdf of GDB copula with TS gen. pdf $p(z|\Psi)$ and **cdf $P(z|\Psi)$** follows as:

$$C\{x,y|p(\cdot|\Psi)\} = \begin{cases} x-\frac{1}{2}\int_{1-x-y}^{1+x-y}P(z|\Psi)dz, & (x,y)\in A_1, \\ y-\frac{1}{2}\int_{1-x-y}^{1-x+y}P(z|\Psi)\}dz, & (x,y)\in A_2, \\ x-\frac{1}{2}\int_{x+y-1}^{1+x-y}P(z|\Psi)dz, & (x,y)\in A_3, \\ y-\frac{1}{2}\int_{x+y-1}^{1-x+y}P(z|\Psi)dz, & (x,y)\in A_4. \end{cases} \quad (11)$$

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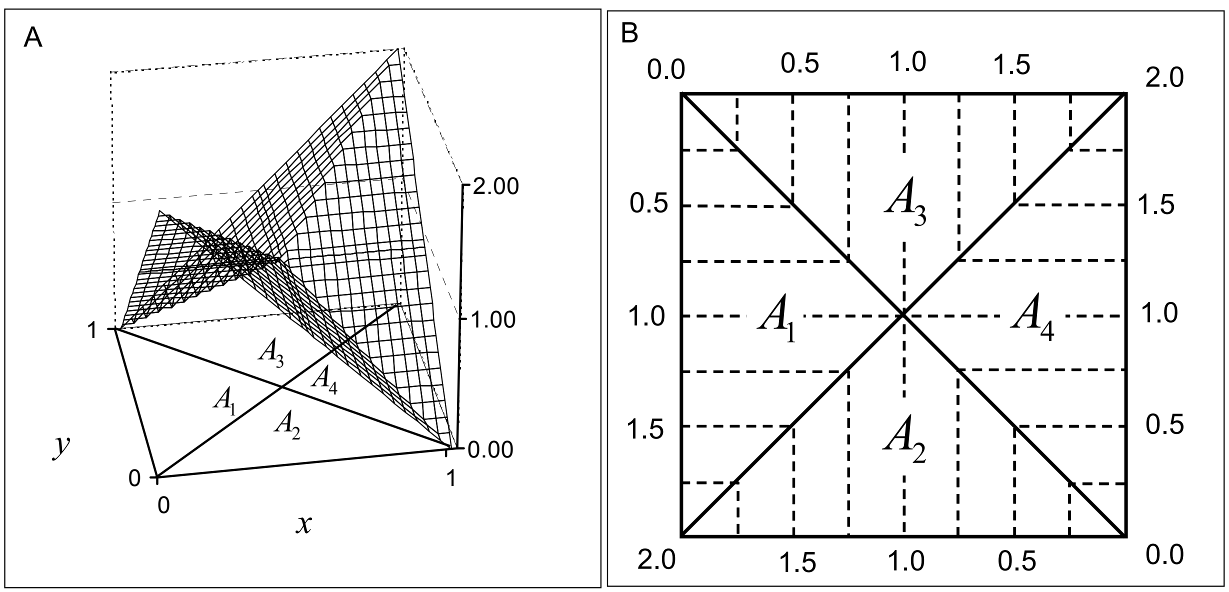
- Substitution of **generating pdf $p(z) = 2z$** with support $[0,1]$ in (10) yields

$$c(x,y) = 2 \times \begin{cases} 1-y, & (x,y) \in A_1, \quad 1-x, \quad (x,y) \in A_2, \\ x, & (x,y) \in A_3, \quad y, \quad\quad\;\; (x,y) \in A_4. \end{cases}$$

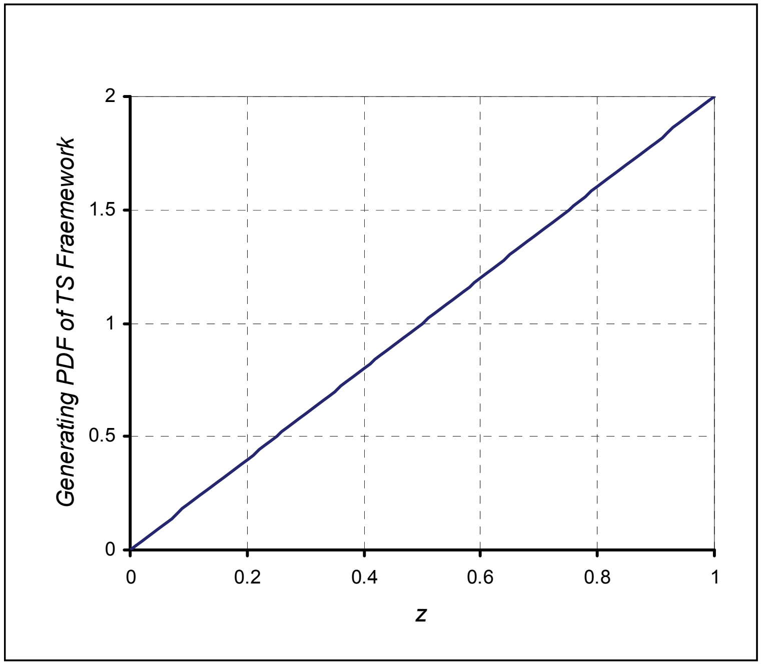


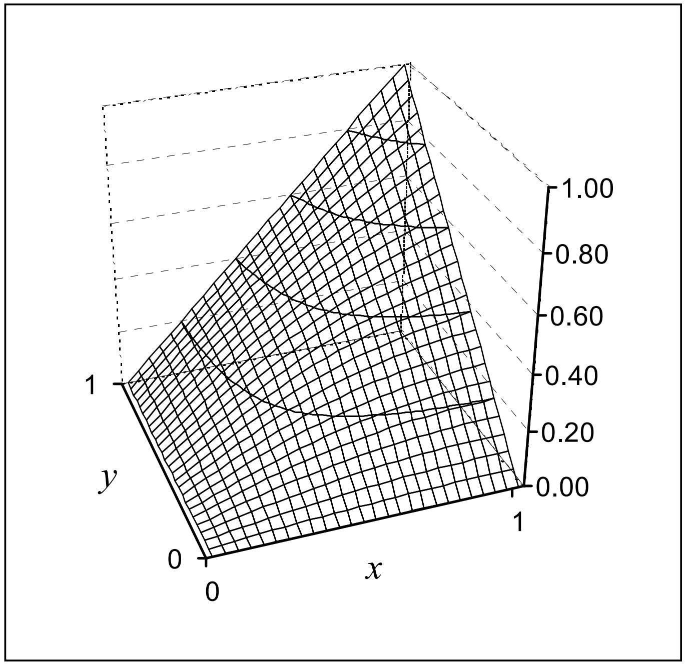
A: Copula density $c\{x,y\}$; B: Density contour plot.

Triangular PDF

- Substitution of pdf $p(z) = 2z$ in (11) and **generating cdf $P(z) = z^2$** yields:

$$C\{x,y\} = \frac{1}{3} \times \begin{cases} -x^3 - 3xy^2 + 6xy\,, & (x,y) \in A_1, \\ -y^3 - 3x^2y + 6xy, & (x,y) \in A_2, \\ y^3 - 3y^2 + 3y(x^2+1) - 3x^2 + 3x - 1, & (x,y) \in A_3, \\ x^3 - 3x^2 + 3x(y^2+1) - 3y^2 + 3y - 1, & (x,y) \in A_4. \end{cases}$$

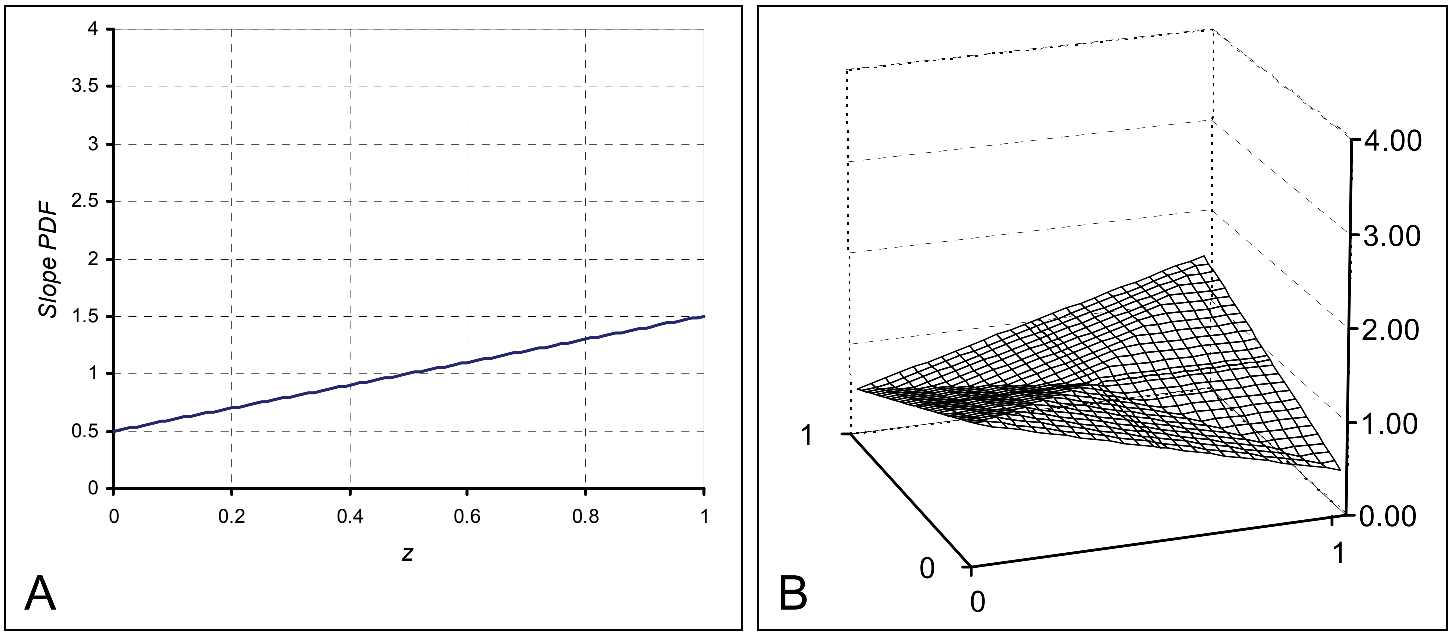




Graph of joint triangular copula cdf $C(x,y)$ given above.

$$p(z|\alpha) = 2 - \alpha + 2(\alpha - 1)z, \; 0 \leq \alpha \leq 2.$$

In figure below: $Pr(Y \leq 0.5 | X \leq 0.5) \approx 0.583$, $\alpha = 1.5$.

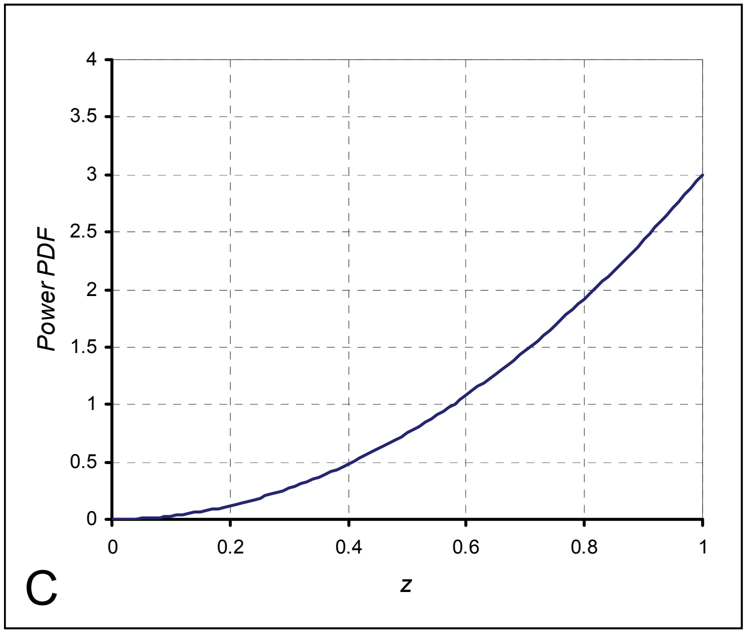


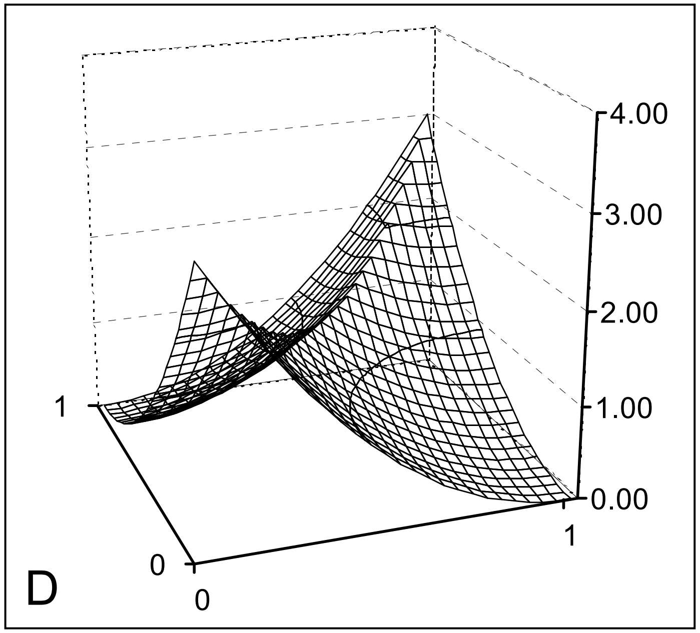
A: Slope generating pdf; B: GDB Copula with TS Gen. PDF in A.

Power PDF

$$p(z|n) = nz^{n-1},\, n > 0.$$

In figure below: $Pr(Y \leq 0.5|X \leq 0.5) = 0.750, n = 3.$

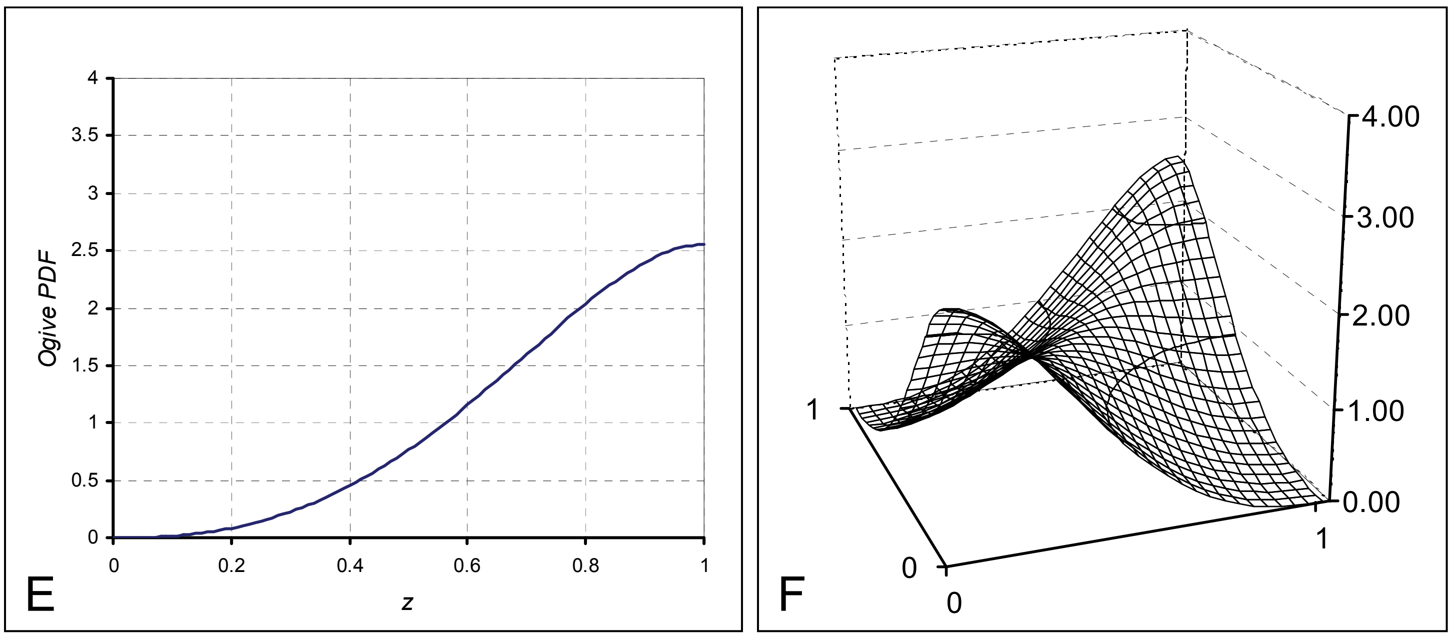




C: Power generating pdf; D: GDB Copula with TS Gen. PDF in A.

$$p(z|m) = \frac{m+2}{3m+4}\{2(m+1)\sqrt{z^m} - mz^{m+1}\}, m > 0.$$

In figure below: $Pr(Y \leq 0.5 | X \leq 0.5) \approx 0.750, m = 4.916.$



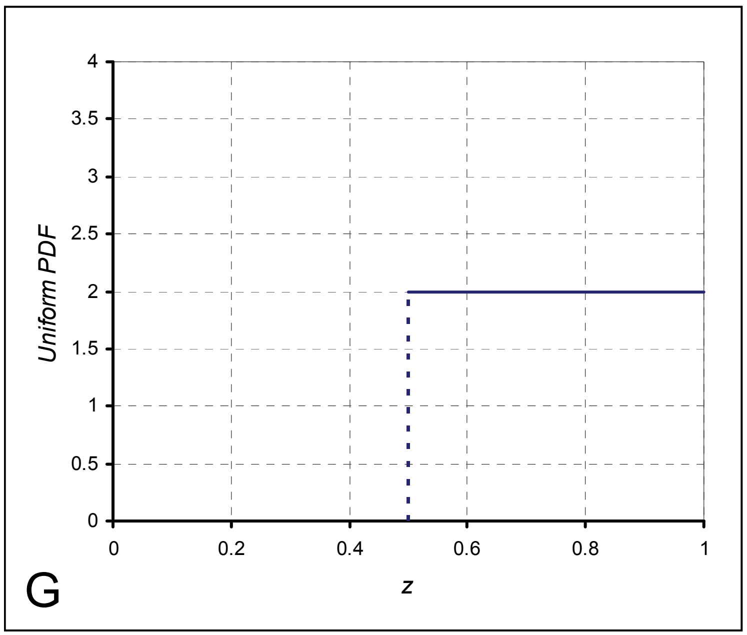
E: Ogive generating pdf; F: GDB Copula with TS Gen. PDF in A.

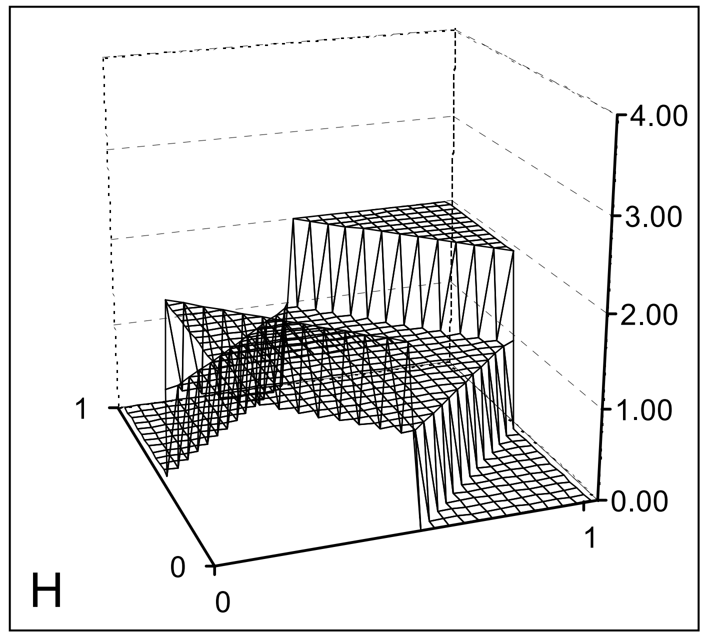
5. GDB EXAMPLES WITH TS GEN. PDF...

Uniform$[\theta, 1]$ PDF

$$p(z|\theta) = \frac{1}{1-\theta}, \theta \leq z \leq 1, 0 \leq \theta \leq 1,$$

In figure below: $Pr(Y \leq 0.5|X \leq 0.5) \approx 0.750, \theta = 0.5.$



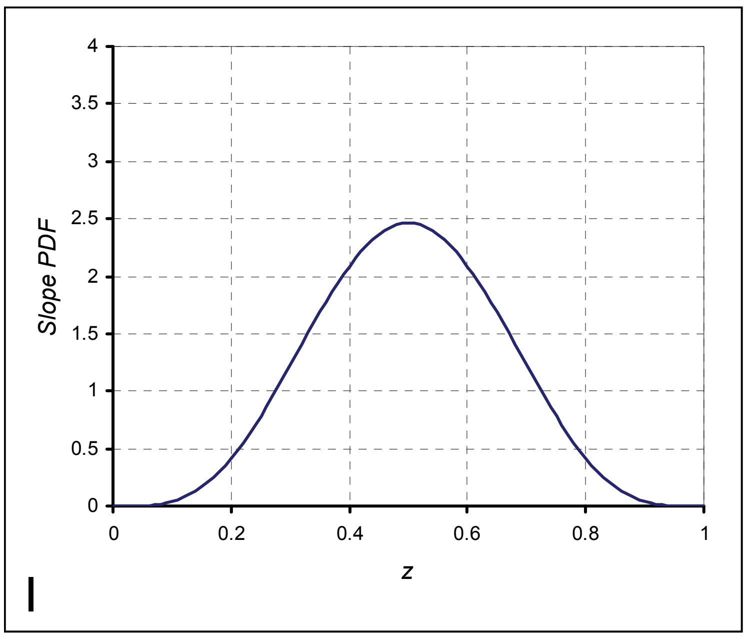


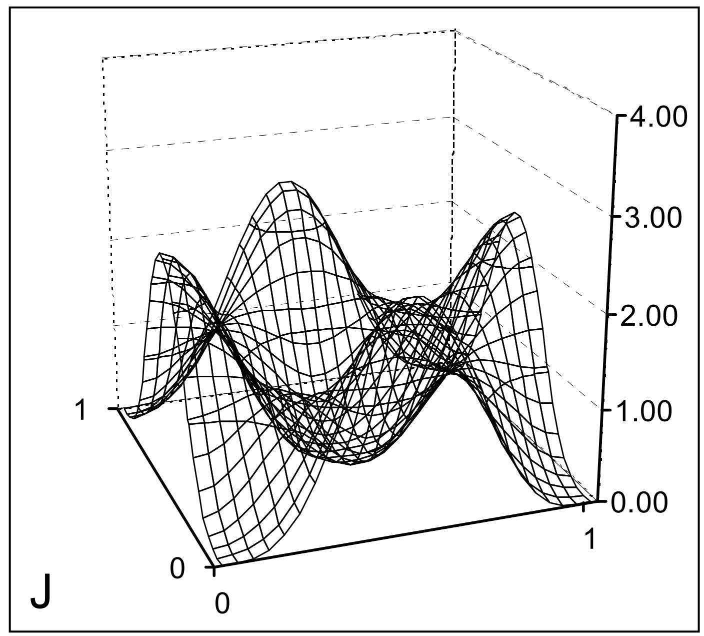
G: Uniform $[\theta, 1]$ gen.pdf; H: GDB Copula with TS Gen. PDF in A.

Beta PDF

$$p(z|a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\, x^{a-1}(1-x)^{b-1}, a > 0,\, b > 0,$$

In figure below: $Pr(Y \leq 0.5|X \leq 0.5) \approx 0.50$, $a = b = 5$.





G: Beta generating pdf; H: GDB Copula with TS Gen. PDF in A.

OUTLINE

1. INTRODUCTION
2. COPULA CONSTRUCTION - ARCHIMEDEAN
3. ARCHIMEDEAN EXAMPLES
4. COPULA CONSTRUCTION - GENERALIZED DIAGONAL BAND
5. GENERALIZED DIAGONAL BAND EXAMPLES
6. **SAMPLING PROCEDURE - ARCHIMEDEAN COPULA**
7. SAMPLING PROCEDURE - GDB COPULA
8. SELECTED REFERENCES

- $\varphi : (0,1] \rightarrow [0,\infty)$ - **The archimedean copula generator function.**

- Let $G(\cdot)$ be the cdf of a random variable Z such that

$$\int_0^\infty G(t)e^{-st}dt = \varphi^{-1}(s).$$

  In other words, $\varphi^{-1}(s)$ is the Laplace transform of the cdf $G(\cdot)$

- **Clayton Copula:** $\varphi^{-1}(s) = (1+s)^{-1/\alpha}$, $\alpha \geq 0$, $Z \sim Gamma(\frac{1}{\alpha}, 1)$

- **Gumbel Copula:** $\varphi^{-1}(s) = exp(-s^{1/\alpha})$, $\alpha \geq 1$, $Z \sim Stable(\frac{1}{\alpha}, 1, \gamma, 0)$, $\gamma = \left[\cos(\frac{\pi}{2\alpha})\right]^{\alpha}$.

- **Frank Copula:** $\varphi^{-1}(s) = \alpha^{-1}\ln[1 + e^s(e^\alpha - 1)]$, $\alpha \in \mathbb{R}\backslash\{0\}$, $Pr(Z = k) = -\frac{\theta^k}{k}[\ln(1-\theta)]^{-1}$, $\theta = 1 - e^{-\alpha}$.

Algorithm (Marshall and Olkin, 1988):

Step 1: Sample u from a uniform random variable U on $[0, 1]$,

Step 2: Sample v from a uniform random variable V on $[0, 1]$,

Step 3: Sample z from $G(\cdot|\alpha)$,

Step 4: Evaluate $u^{\bullet} = -\frac{\ln u}{z}$,

Step 5: Evaluate $v^{\bullet} = -\frac{\ln v}{z}$,

Step 6: $x = \varphi^{-1}(u^{\bullet})$,

Step 7: $y = \varphi^{-1}(v^{\bullet})$.

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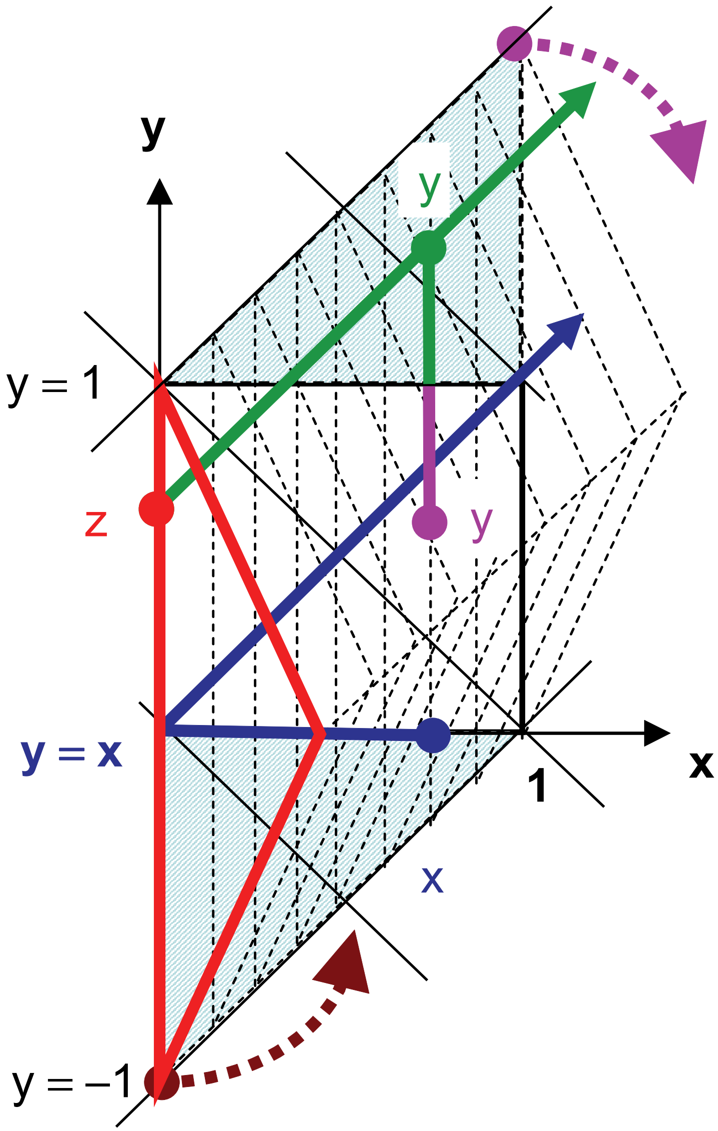
7. SAMPLING PROCEDURE...

GDB Copula



ALGORITHM:

1. **Sample x in [0,1]**
2. **Sample z in [-1,1]**
3. **y = z + x**
4. **If y < 0 then y = –y**
5. **If y > 1 Then y =1–(y–1)**

GDB Copula

- Z random variable with **symmetric Two-Sided (TS) pdf** :

$$f\{z|p(\cdot|\Psi)\} = \frac{1}{2} \times \begin{cases} p(z+1|\Psi), & \text{for } -1 < z \leq 0, \\ p(1-z|\Psi), & \text{for } \quad 0 < z < 1, \end{cases} \tag{5}$$

where $p(z)$ is a generating pdf with support $[0,1]$.

Step 1: Sample x from a uniform random variable X on $[0,1]$.

Step 2: Sample u from a uniform random variable U on $[0,1]$.

Step 3: If $u \leq \frac{1}{2}$ then $z = P^{-1}(2u) - 1$ else $z = 1 - P^{-1}(2 - 2u)$

Step 4: $y = z + x$

Step 5: If $y < 0$ then $y = -y$

Step 6: If $y > 1$ then $y = 1 - (y - 1)$

7. SAMPLING PROCEDURE...

GDB Copula

- For the generating densities herein we have for **arbitrary quantile level $q \in (0,1)$:**

$$P^{-1}(q|\psi) = \begin{cases} \frac{-(2-\alpha)+\sqrt{(2-\alpha)^2+4(\alpha-1)q}}{2(\alpha-1)}, & p(z|\alpha), \alpha \neq 1, \\ q^{1/n}, & p(z|n), \\ \left[\frac{2(m+1)}{m} - \sqrt{\left\{\frac{2(m+1)}{m}\right\}^2 - q\frac{3m+4}{m}}\right]^{2/(m+2)}, & p(z|m), \\ (1-\theta)q + \theta, & p(z|\theta), \end{cases}$$

- One could favor the power pdf and uniform pdf's due to **least number of operations.**

Selected References

Cooke, R.M. and Waij, R. (1986). Monte carlo sampling for generalized knowledge dependence, with application to human reliability. *Risk Analysis*, 6 (3), pp. 335-343.

Genest, C. and Mackay, J. (1986). The joy of copulas, bivariate distributions with uniform marginals. *The American Statistician*, 40 (4), pp. 280-283.

Ferguson, T.F. (1995). A class of symmetric bivariate uniform distributions. *Statistical Papers*, 36 (1), pp. 31-40.

S. Kotz and J.R. van Dorp (2009), Generalized Diagonal Band Copulas with Two-Sided Generating Densities, *Decision Analysis*, published online before print November 25, DOI:10.1287/deca.1090.0162.

Marshall, A. W. and I. Olkin (1988) Families of multivariate distributions. *Journal of the American Statistical Association*, 83, 834–841.

McNeil, A., R. Frey, and P. Embrechts (2005). *Quantitative Risk Management: Concepts,Techniques and Tools*. Princeton University Press.

Nelsen, R.B. (1999). *An Introduction to Copulas*. Springer, New York.

Sklar, A. (1959). Fonctions de répartition à n dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris,* 8, pp. 229-231.

Van Dorp, J.R and Kotz, S. (2003). Generalizations of two sided power distributions and their convolution. *Communications in Statistics: Theory and Methods*, 32 (9), pp. 1703 - 1723.