Session 11: One-Way Analysis of Variance (ANOVA)

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ONE-WAY ANOVA

Summary of Tests

- Univariate $T$-test: $H_0 : \mu = \mu_0$, $H_1 : \mu \neq \mu_0$. (Scalar $\mu_0$ is specified)

  Independence within Sample

  Sample $x$

  $x_{11} \quad x_{12} \quad \cdots \quad x_{1n}$

  Independence between Sample $x$ and Sample $y$

  Sample $y$

  $y_{11} \quad y_{12} \quad \cdots \quad y_{1n}$

- Two Sample Univariate $T$-test: $H_0 : \mu_1 = \mu_2$, $H_1 : \mu_1 \neq \mu_2$. 

  Compare
ONE-WAY ANOVA

Summary of Tests

- Hotelling $T^2$-test: $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$, $H_1 : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$. (Vector $\boldsymbol{\mu}_0$ is specified)

![Diagram of one-way ANOVA with samples and vector observations]
### ONE-WAY ANOVA

**Summary of Tests**

- **Two-Sample Hotelling $T^2$-test:** $H_0 : \mu_1 = \mu_2$, $H_1 : \mu_1 \neq \mu_2$.  

![Diagram showing independence within samples and comparison between samples](image)

<table>
<thead>
<tr>
<th>Sample 1</th>
<th>$x_{11}$</th>
<th>$x_{12}$</th>
<th>...</th>
<th>$x_{1n}$</th>
<th>$\bar{x}_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample 2</td>
<td>$x_{21}$</td>
<td>$x_{22}$</td>
<td>...</td>
<td>$x_{2n}$</td>
<td>$\bar{x}_2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sample p</td>
<td>$x_{p1}$</td>
<td>$x_{p2}$</td>
<td>...</td>
<td>$x_{pn}$</td>
<td>$\bar{x}_p$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sample 1</th>
<th>$y_{11}$</th>
<th>$y_{12}$</th>
<th>...</th>
<th>$y_{1n}$</th>
<th>$\bar{y}_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample 2</td>
<td>$y_{21}$</td>
<td>$y_{22}$</td>
<td>...</td>
<td>$y_{2n}$</td>
<td>$\bar{y}_2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sample p</td>
<td>$y_{p1}$</td>
<td>$y_{p2}$</td>
<td>...</td>
<td>$y_{pn}$</td>
<td>$\bar{y}_p$</td>
</tr>
</tbody>
</table>
Objective of Analysis of Variance (ANOVA):

\[ \begin{align*}
\text{Sample 1:} & \quad x_{11}, x_{12}, \ldots, x_{1n}, \bar{x}_1, \mu_{10} \\
\text{Sample 2:} & \quad x_{21}, x_{22}, \ldots, x_{2n}, \bar{x}_2, \mu_{20} \\
\vdots & \quad \vdots, \quad \vdots, \quad \vdots, \quad \vdots, \quad \vdots \\
\text{Sample p:} & \quad x_{p1}, x_{p2}, \ldots, x_{pn}, \bar{x}_p, \mu_{p0}
\end{align*} \]

**Independence within Sample i**

**Independence between Samples**

**Paired Comparisons**

**Tensile Strength Example:** The tensile strength of synthetic fiber used to make cloth for men's shirts is of interest to a manufacturer. It is suspected that the strength is affected by the percentage of cotton in the fiber. Five levels of cotton percentages are of interest, 15%, 20%, 25%, 30%, and 35%. Five observations are to be taken at each level of cotton percentage, and the 25 total observations are to be run in random order.

Total number of paired comparisons: \( \binom{5}{2} = 10 \)
ONE-WAY ANOVA

Introduction

• It seems that this problem can be solved by performing a two-sample $t$ test on all possible pairs. However, this solution would be incorrect, since it would lead to a considerable distortion in the type I error.

Tensile Strength Example:
There are 10 possible pairs. If the probability of accepting the null hypothesis (there is no difference between a pair) for all 10 tests is $1 - \alpha = 0.95$, then the probability of correctly accepting the null hypothesis for all 10 tests (i.e. there is no difference between the 10 samples) equals:

$$(0.95)^{10} = 0.60 \Leftrightarrow \text{Type I error} = 1 - 0.60 = 0.40.$$ 

if the tests are independent. Thus a substantial increase of Type I error has occurred.

The appropriate procedure for testing equality of several means in the setting above is the ANALYSIS OF VARIANCE.

• It is interesting to note that we are testing equality of means here by analyzing variances. To see how does works we need to study some of the mechanics.
In ANOVA samples are referred to as "treatments". Hence, we have:

\[
X_{ij} = \mu + \tau_i + \epsilon_{ij}, \quad \begin{cases} 
  i = 1, \ldots, p \\
  j = 1, \ldots, n
\end{cases}
\]

- \( \mu \): a parameter common to all treatments called the overall mean.
- \( \tau_i \): a parameter unique to the \( i \)-th treatment called the treatment effect.
- \( \epsilon_{ij} \): a random error component, \( \epsilon_{ij} \sim N(0, \sigma) \) for all \( i, j \) and i.i.d.
ONE-WAY ANOVA

• The mean of treatment \( i \) equals the sum of the overall mean plus the \( i \)-th treatment effect:

\[
E[X_{ij}] = \mu + \tau_i, \quad i = 1, \ldots, p, \quad \text{Convention: } \sum_{i=1}^{p} \tau_i = 0.
\]

• We are interested in testing the equality of the \( p \) treatment means:

\[
H_0 : \mu_1 = \mu_2 = \ldots = \mu_p, \quad H_1 : \mu_i \neq \mu_j, \text{ for a least one } i, j
\]

• If \( H_0 \) is true, all treatments have common mean \( \mu \). An equivalent way to write the above hypotheses is in terms of the treatment effects \( \tau_i \):

\[
H_0 : \tau_1 = \tau_2 = \ldots = \tau_p = 0, \quad H_1 : \tau_i \neq 0, \text{ for a least one } i
\]

Notation:

\[
x_{i.} = \sum_{j=1}^{n} x_{ij}, \quad \bar{x}_{i.} = \frac{1}{n} x_{i.}
\]

\[
x_{..} = \sum_{j=1}^{p} \sum_{j=1}^{n} x_{ij}, \quad \bar{x}_{..} = \frac{1}{N} x_{..}, \quad N = np \text{ (total number of observations)}
\]
ONE-WAY ANOVA

- **ANALYSIS OF VARIANCE (ANOVA) TABLE:**

<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
<th>$F_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between treatments</td>
<td>$SS_{Treatments}$</td>
<td>$p - 1$</td>
<td>$MS_{Treatment}$</td>
<td>$MS_{Treatments} / MS_E$</td>
</tr>
<tr>
<td>Error (within treatments)</td>
<td>$SS_E$</td>
<td>$N - p$</td>
<td>$MS_E$</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>$SS_T$</td>
<td>$N - 1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- **Convenient calculation formulas (also for the unbalanced case):**

\[
SS_T = \sum_{i=1}^{p} \left( \sum_{j=1}^{n_i} x_{ij}^2 \right) - \frac{x^2}{N}, \quad N = \sum_{i=1}^{p} n_i
\]

\[
SS_{Treatments} = \sum_{i=1}^{p} \frac{x_{i}^2}{n_i} - \frac{x^2}{N}, \quad SS_E = SS_T - SS_{Treatments}
\]

unbalanced = different number of observations in each treatment
Tensile Strength Example: The tensile strength of synthetic fiber used to make cloth for men's shirts is of interest to a manufacturer. It is suspected that the strength is affected by the percentage of cotton in the fiber. Five levels of cotton percentage are of interest, 15%, 20%, 25%, 30%, and 35%. Five observations are to be taken at each level of cotton percentage, and the 25 total observations are to be run in random order.

Table: Tensile Strength of Synthetic Fiber (lb/in.$^2$)

<table>
<thead>
<tr>
<th>Percentage of Cotton</th>
<th>Observations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>15%</td>
<td>7</td>
</tr>
<tr>
<td>20%</td>
<td>12</td>
</tr>
<tr>
<td>25%</td>
<td>14</td>
</tr>
<tr>
<td>30%</td>
<td>19</td>
</tr>
<tr>
<td>35%</td>
<td>7</td>
</tr>
</tbody>
</table>

$\bar{x} = 376$
**ONE-WAY ANOVA**

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
<th>( F_0 )</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_{\text{Treatments}} )</td>
<td>475.76</td>
<td>4</td>
<td>118.94</td>
<td>14.76</td>
<td>9.13E-06</td>
</tr>
<tr>
<td>( S_{E} )</td>
<td>161.2</td>
<td>20</td>
<td>8.06</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( S_{T} )</td>
<td>636.96</td>
<td>24</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- \( p \)-value < \( \alpha \) for \( \alpha \in \{1\%, 5\%, 10\%\} \) \( \Rightarrow \) Reject \( H_0 \) for all these \( \alpha \)'s

**Conclusion:** At least one of the treatment means differs!

- **Estimation of parameters** can be done using the least squares approach (similar to linear regression analysis). Recall: \( X_{ij} = \mu + \tau_i + \epsilon_{ij} \)

\[
\hat{\mu} = \overline{X}_{..}, \quad \hat{\tau}_i = \overline{X}_{i..} - \overline{X}_{..}, \quad i = 1, \ldots, p,
\]

\[
\hat{\mu}_i = \hat{\mu} + \hat{\tau}_i = \overline{X}_{i..}, \quad \hat{\sigma}^2 = M S_E = S S_E / (N - p)
\]
ONE-WAY ANOVA

- 100(1 − α)% confidence intervals treatment means $\mu_i$:

$$ \bar{X}_i \pm t_{\alpha/2,N-p} \sqrt{MSE/n} $$

![Confidence Intervals Diagram]

- Lower Bound 5%
- Mean
- Upper Bound 95%
ONE-WAY ANOVA

One-way ANOVA: 15%, 20%, 25%, 30%, 35%

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor</td>
<td>4</td>
<td>475.76</td>
<td>118.94</td>
<td>14.76</td>
<td>0.000</td>
</tr>
<tr>
<td>Error</td>
<td>20</td>
<td>161.20</td>
<td>8.06</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>24</td>
<td>636.96</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

S = 2.839  R-Sq = 74.69%  R-Sq(adj) = 69.63%

Individual 90% CIs For Mean Based on Pooled StDev

<table>
<thead>
<tr>
<th>Level</th>
<th>N</th>
<th>Mean</th>
<th>StDev</th>
</tr>
</thead>
<tbody>
<tr>
<td>15%</td>
<td>5</td>
<td>9.800</td>
<td>3.347</td>
</tr>
<tr>
<td>20%</td>
<td>5</td>
<td>15.400</td>
<td>3.130</td>
</tr>
<tr>
<td>25%</td>
<td>5</td>
<td>17.600</td>
<td>2.074</td>
</tr>
<tr>
<td>30%</td>
<td>5</td>
<td>21.600</td>
<td>2.608</td>
</tr>
<tr>
<td>35%</td>
<td>5</td>
<td>10.800</td>
<td>2.864</td>
</tr>
</tbody>
</table>

Pooled StDev = 2.839
ONE-WAY ANOVA

Confidence Intervals

Minitab Box Plot of Treatment Means

Data

Boxplot of 15%, 20%, 25%, 30%, 35%
ONE-WAY ANOVA

Confidence Intervals

- $100(1 - \alpha)\%$ confidence intervals difference treatment means

$\mu_i - \mu_k: \quad \bar{X}_i - \bar{X}_k \pm t_{\alpha/2, N-p} \sqrt{2MS_E/n}$

Comparison of $\mu_3$ to other treatment means.

Conclusion: We only fail to reject the null-Hypothesis that $\mu_3 = \mu_2$ !!!
ONE-WAY ANOVA

Model Adequacy Checking

- It was assumed in the model that the error terms $\epsilon_{ij}$ are normal distributed with a mean 0 and a variance $\sigma^2$.

- The normality assumptions of the residuals $\epsilon_{ij}$ can be checked via a normal probability plot.

- It is important to recognize that we are testing the equality of treatment means by testing for the equality of variances.

- The required assumption that allows us to do this is that the variance of the error terms $\epsilon_{ij}$ is constant across treatments $i = 1, \ldots, p$.

- The assumption of equality of variance may be visually verified by plotted the residuals of each treatment against one another.

- Alternatively, we may also use Bartlett's test, to test for equality of variance across treatment.
ONE-WAY ANOVA

Model Adequacy Checking

Residual Plots for C1

- Normal Probability Plot of the Residuals
- Residuals Versus the Fitted Values
- Histogram of the Residuals
- Residuals Versus the Order of the Data
ONE-WAY ANOVA

Model Adequacy Checking

Probability Plot of RESI1
Normal - 95% CI

Mean = 0
StDev = 2.839
N = 25
AD = 0.465
P-Value >0.250
ONE-WAY ANOVA

Model Adequacy Checking

Bartlett's Test for Equality of Variance across treatments

\[ H_0 : \sigma_1^2 = \sigma_2^2 = \ldots = \sigma_p^2, \quad H_1 : \text{Not true for at least one } \sigma_i^2 \]

Test Statistic:

\[ \chi_0^2 = \frac{q}{c} \sim \chi_{p-1} \]

\[ q = (N - p) \times \ln S_{\text{pooled}}^2 - \sum_{i=1}^{p} (n_i - 1) \times \ln S_i^2, \quad N = \sum n_i \]

\[ S_{\text{pooled}}^2 = \frac{\sum_{i=1}^{p} (n_i - 1) S_i^2}{(N - p)}, \quad c = 1 + \frac{1}{3(p - 1)} \left[ \sum_{i=1}^{p} (n_i - 1)^{-1} - (N - p)^{-1} \right] \]

Tensile Strength Example:

\[ N = 25, \quad p = 5, \quad S_{\text{pooled}}^2 \approx 8.06, \quad q \approx 1.03, \quad c \approx 1.10, \quad \chi_0^2 \approx 0.93, \]

\[ p - value \approx 0.92. \text{ Conclusion: Fail to Reject the null-Hypothesis} \]
ONE-WAY ANOVA Model Adequacy Checking

Test for Equal Variances for Tensile Strength

Bartlett's Test
Test Statistic 0.93
P-Value 0.920

Levene's Test
Test Statistic 0.32
P-Value 0.863
ONE-WAY ANOVA

Contrasts

• Using the 100(1 − α)% confidence intervals for differences of treatment means of $μ_3 − μ_k$, $k = 1, 2, 4$ and $5$ we tested the hypotheses:

$$H_0 : μ_3 = μ_k, H_1 : μ_3 ≠ μ_k$$

• These hypotheses could be tested by investigating an appropriate linear combination of treatment totals, say:

$$X_{3.} − X_{k.}$$

If we had suggested that the average of cotton percentages 1 and 3 did not differ from the average of cotton percentages 4 and 5, then the hypothesis would have been

$$H_0 : μ_1 + μ_3 = μ_4 + μ_5; H_1 : μ_1 + μ_3 ≠ μ_4 + μ_5$$

which implies the linear combination:

$$X_{1.} + X_{3.} − X_{4.} − X_{5.} = 0$$

• A linear combination of treatments totals $C = \sum_{i=1}^{p} c_i X_{i.}$ such that $\sum_{i=1}^{p} c_i = 0$ is called a contrast.
ONE-WAY ANOVA

Contrasts

- The sum of squares for any contrast equals:

\[ SS_C = \left( \sum_{i=1}^{p} c_i X_i \right)^2 / \left( \sum_{i=1}^{p} n_i c_i^2 \right), \]

where \( n_i \) is the number of observations in treatment \( i \) and it has a single degree of freedom and hence \( MS_C = SS_C \)

- Hence,

\[ \frac{MS_C}{MS_E} = \frac{SS_C}{SS_E/N - p} \sim F_{1,N-p} \]

- Conclusion: Many important comparisons regarding treatment means may be made using contrasts.

- Two contrasts \( \{c_i\} \) and \( \{d_i\} \) are orthogonal when \( \sum_{i=1}^{p} c_i d_i = 0 \)
ONE-WAY ANOVA

Contrasts

• For $p$ treatments, a set of $(p - 1)$ orthogonal contrast partition the sum of squares due to treatments into $(p - 1)$ independent single-degree-of-freedom components.

• Thus, the test performed on orthogonal contrast are independent. Hence, if the type 1 error of each individual test is $(1 - \alpha)$, the type 1 error of the $(p - 1)$ orthogonal contrast tests equals $(1 - \alpha)^{p-1}$.

• Without orthogonality of these contrasts, we cannot say anything about the combined type 1 error probability.

• There are many ways to choose orthogonal contrast coefficient for a given set of treatments. For example, if there are $p = 3$ treatments, with treatment 1 a control and treatments 2 and 3 actual levels of the factor of interest, then appropriate orthogonal contrast might be as follows

<table>
<thead>
<tr>
<th>$c_i$</th>
<th>Treatment 1(Control)</th>
<th>Treatment 2(Level 1)</th>
<th>Treatment 3( Level2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>
Contrasts

• Contrast coefficient must be chosen prior to running the experiment and prior to examining the data. Otherwise bias in Type I error may occur.

**Tensile Strength Example:**

\[ H_0 : \mu_4 = \mu_5 \quad C_1 = -X_4. + X_5. \]
(Compares the average of Treatment 4 and with that of Treatment 5)

\[ H_0 : \mu_1 + \mu_3 = \mu_4 + \mu_5 \quad C_2 = X_1. + X_3. - X_4. - X_5. \]
(Compares the average of Treatments 1 and 3 with that of Treatments 4 and 5)

\[ H_0 : \mu_1 = \mu_3 \quad C_3 = X_1. - X_3. \]
(Compares the average of Treatment 1 and with that of Treatment 3)

\[ H_0 : 4\mu_2 = \mu_1 + \mu_3 + \mu_4 + \mu_5 \quad C_4 = -X_1. + 4X_2. - X_3. - X_4. - X_5. \]
(Compares the average of Treatments 2 with that of Treatments 1, 3, 4 and 5)

Notice that the contrast coefficients are orthogonal!
ONE-WAY ANOVA

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
<th>F₀</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>291.6</td>
<td>1</td>
<td>291.6</td>
<td>36.18</td>
<td>7.01E-06</td>
</tr>
<tr>
<td>C2</td>
<td>31.25</td>
<td>1</td>
<td>31.25</td>
<td>3.88</td>
<td>6.30%</td>
</tr>
<tr>
<td>C3</td>
<td>152.1</td>
<td>1</td>
<td>152.1</td>
<td>18.87</td>
<td>3.15E-04</td>
</tr>
<tr>
<td>C4</td>
<td>0.81</td>
<td>1</td>
<td>0.81</td>
<td>0.10</td>
<td>75.5%</td>
</tr>
<tr>
<td>SSTreatments</td>
<td>475.76</td>
<td>4</td>
<td>118.94</td>
<td>14.76</td>
<td>9.13E-06</td>
</tr>
<tr>
<td>SSE</td>
<td>161.2</td>
<td>20</td>
<td>8.06</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SST</td>
<td>636.96</td>
<td>24</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Conclusion:
- There are differences between the treatment means.
- Furthermore, differences are observed between Treatment 4 and Treatment 5 (C1), and differences of Treatment 1 and Treatment 3 (C3).
- No difference is observed between the average of 1 and 3 and the average of 4 and 5 (C2).
- No difference is observed between treatment 2 and the average of treatments 1, 3, 4 and 5 (C4).
ONE-WAY ANOVA

Mechanics : Optional

- **Total sum of squares:**

\[
SS_T = \sum_{i=1}^{p} \sum_{j=1}^{n} (x_{ij} - \bar{x}_{..})^2 = \sum_{i=1}^{p} \sum_{j=1}^{n} [(x_{ij} - \bar{x}_{i.}) + (\bar{x}_{i.} - \bar{x}_{..})]^2
\]

\[
= \sum_{i=1}^{p} \sum_{j=1}^{n} [(x_{ij} - \bar{x}_{i.})^2 + 2(x_{ij} - \bar{x}_{i.})(\bar{x}_{i.} - \bar{x}_{..}) + (\bar{x}_{i.} - \bar{x}_{..})^2]
\]

\[
= \sum_{i=1}^{p} \sum_{j=1}^{n} (x_{ij} - \bar{x}_{i.})^2 + 2\sum_{i=1}^{p} \sum_{j=1}^{n} (x_{ij} - \bar{x}_{i.})(\bar{x}_{i.} - \bar{x}_{..}) + \sum_{i=1}^{p} \sum_{j=1}^{n} (\bar{x}_{i.} - \bar{x}_{..})^2
\]

\[
= \sum_{i=1}^{p} \sum_{j=1}^{n} (x_{ij} - \bar{x}_{i.})^2 + 2\sum_{i=1}^{p} (\bar{x}_{i.} - \bar{x}_{..}) \sum_{j=1}^{n} (x_{ij} - \bar{x}_{i.}) + n \sum_{i=1}^{p} (\bar{x}_{i.} - \bar{x}_{..})^2
\]

- **Cross product term equals zero, because:**

\[
\sum_{j=1}^{n} (x_{ij} - \bar{x}_{i.}) = \sum_{j=1}^{n} x_{ij} - n \bar{x}_{i.} = n \bar{x}_{i.} - n \bar{x}_{i.} = 0
\]
ONE-WAY ANOVA  

Mechanics: Optional

- Total sum of squares:

\[
SS_T = \sum_{i=1}^{p} \sum_{j=1}^{n} (x_{ij} - \bar{x}_{..})^2 = \sum_{i=1}^{p} \sum_{j=1}^{n} (x_{ij} - \bar{x}_{i.})^2 + n \sum_{i=1}^{p} (\bar{x}_{i.} - \bar{x}_{..})^2
\]

or

\[
SS_T = SS_E + SS_{Treatments}
\]

where:

\[
SS_E = \sum_{i=1}^{p} \sum_{j=1}^{n} (x_{ij} - \bar{x}_{i.})^2 = \sum_{i=1}^{p} \left[ \sum_{j=1}^{n} (x_{ij} - \bar{x}_{i.})^2 \right]
\]

The sum of squares within an treatment \(i\), summed over all treatments

\[
SS_{Treatments} = n \sum_{i=1}^{p} (\bar{x}_{i.} - \bar{x}_{..})^2
\]

The sum of squares of treatment means \(\bar{x}_{i.}\) against the overall mean \(\bar{x}_{..}\).
ONE-WAY ANOVA

Mechanics : Optional

• The sample variance in the $i$-th treatment equals:

$$S_i^2 = \frac{1}{n-1} \sum_{j=1}^{n} (x_{ij} - \bar{x}_{i \bullet})^2 \iff (n-1)S_i^2 = \sum_{j=1}^{n} (x_{ij} - \bar{x}_{i \bullet})^2$$

• These $S_i^2$'s can be combined to get an estimate of overall variance as follows

$$\frac{1}{p} \sum_{i=1}^{p} S_i^2 = \frac{(n-1) \sum_{i=1}^{p} S_i^2}{(n-1)p} = \frac{(n-1)S_1^2 + \ldots + (n-1)S_p^2}{np - p} = \frac{\sum_{i=1}^{p} \left[ \sum_{j=1}^{n} (x_{ij} - \bar{x}_{i \bullet})^2 \right]}{N - p} = SS_E$$

• Recalling $\epsilon_{ij} \sim N(0, \sigma)$ and denoting:

$$MS_E = \frac{SS_E}{N - p} \Rightarrow E[MS_E] = E \left[ \frac{1}{p} \sum_{j=1}^{p} S_i^2 \right] = \frac{1}{p} \sum_{j=1}^{p} \sigma^2 = \sigma^2$$
ONE-WAY ANOVA

Mechanics : Optional

• Recalling $\epsilon_{ij} \sim N(0, \sigma)$, observe that the estimators of the $i$-th treatment means

$$\bar{X}_{i *} = \frac{1}{n} \sum_{j=1}^{n} X_{ij}$$

are all random variables $i = 1, \ldots, p$ with common variance $\sigma^2/n$.

• **If the treatments means are all equal with value $\mu$**, we also have that

$$\bar{X}_{..} = \frac{1}{np} \sum_{i=1}^{p} \sum_{j=1}^{n} X_{ij} = \frac{1}{p} \sum_{i=1}^{p} \left[ \frac{1}{n} \sum_{j=1}^{n} X_{ij} \right] = \frac{1}{p} \sum_{i=1}^{p} \bar{X}_{i *},$$

is an unbiased estimate of the **common** treatment mean $\mu$.

• Hence, **if the treatments means are all equal**,

$$E \left[ \frac{1}{p-1} \sum_{i=1}^{p} (\bar{X}_{i *} - \bar{X}_{..})^2 \right] = \frac{\sigma^2}{n} \Leftrightarrow E \left[ \frac{n}{p-1} \sum_{i=1}^{p} (\bar{X}_{i *} - \bar{X}_{..})^2 \right] = \sigma^2.$$
ONE-WAY ANOVA

Mechanics: Optional

• Denoting:

\[ MS_{Treatments} = \frac{n}{p-1} \sum_{i=1}^{p} (\bar{X}_i - \bar{X}_{..})^2 = \frac{n}{p-1} \sum_{i=1}^{p} (\bar{X}_i - \bar{X}_{..})^2 = \frac{SS_{Treatment}}{p-1} \]

we have, when all the treatments means are equal,

\[ E[MS_{Treatments}] = \sigma^2. \]

• It can be shown that if the treatment means are not necessarily equal,

\[ E[MS_{Treatments}] = \sigma^2 + \frac{n}{p-1} \sum_{i=1}^{p} \tau_i^2. \]

• We have shown that (regardless of the value of the treatment means):

\[ E[MS_E] = E\left[ \frac{SS_E}{N - p} \right] = \sigma^2. \]
ONE-WAY ANOVA

Mechanics : Optional

• Conclusion: If the value of $MS_{Treatments}$ is close to that of $MS_E$ this can be seen as an indication that the treatment means are equal. Moreover, if the treatment means are different it follows that $MS_{Treatments}$ is larger than $MS_E$.

• But how large does $MS_{Treatments}$ have to be before we decide that the treatment means are different?

$$\frac{MS_{Treatments}}{MS_E} = \frac{SS_{Treatments}/(p - 1)}{SS_E/(N - p)} \sim F_{p-1,N-p}$$

• Hence, when

$$F_0 = \frac{MS_{Treatments}}{MS_E} > F_{p-1,N-p}, 1 - \alpha$$

we reject the null-hypothesis of no differences between the treatment means.

• $p$-value of this hypothesis test equals: $Pr(F_{p-1,N-p} > MS_{Treatments}/MS_E)$