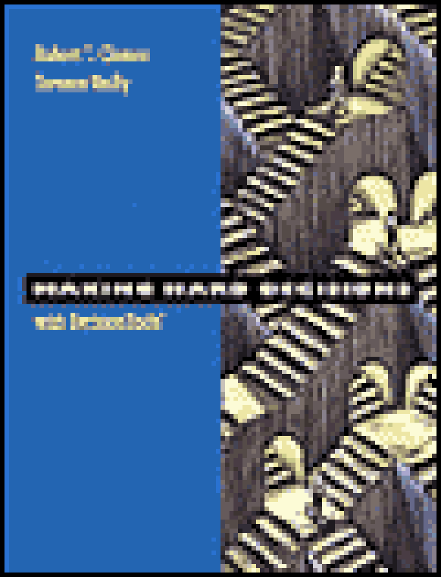

Making Hard Decisions

R. T. Clemen, T. Reilly

Chapter 8

Subjective Probability

Draft: Version 1

A Subjective Interpretation

Frequentist interpretation of probability:

Probability = Relative frequency of occurrence of an event

Frequentist definition requires one to specify **a repeatable experiment**.

Example: Throwing a fair coin, Pr(Heads)=0.5

What about following events?

- Core Meltdown of Nuclear Reactor?
- You being alive at the age of 50?
- Unsure of the final outcome, though event has occurred?
- One basketball team beating the other in next day's match?

A Subjective Interpretation

Without doubt the above events are uncertain and we talk about the probability of these events. These probabilities **HOWEVER** they are **NOT** Relative Frequencies of occurrences.

So what are they?

Subjectivist Interpretation of Probability:

Probability = Degree of belief in the occurrence of an event

By assessing a probability for the events above, one expresses one's **DEGREE OF BELIEF**. High probabilities coincide with a high degree of belief. Low probabilities coincide with a low degree of belief.

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A Subjective Interpretation

Why do we need it?

- Frequentist interpretation is not always applicable.
- It allows to model and structure **individualistic uncertainty** through probability.

ASSESSING SUBJECTIVE DISCRETE PROBABILITIES:

Direct Methods: Directly ask for probability assessments.

- These methods do not work well especially if experts are not familiar with probabilities as a concept.

- These methods do not work well if probabilities in questions are very small (such as for example in risk analyses).

Assessing Discrete Probabilities

Indirect Methods: Formulate questions in expert's domain of expertise and extract probability assessments through probability modeling.

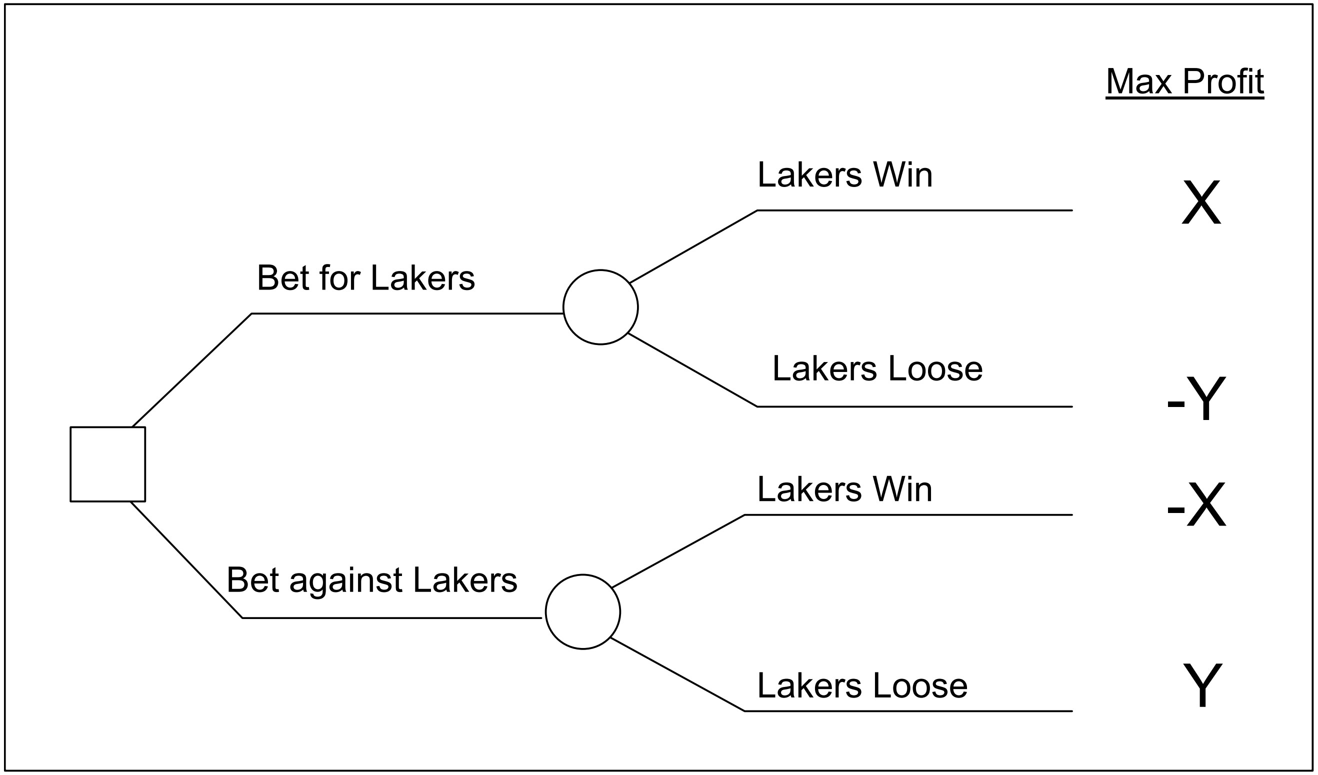
Examples: Betting Strategies, Reference Lotteries, Paired Comparison Method for Relative Probabilities.

1. Betting Strategies:

Event: Lakers winning the NBA title this season

STEP 1: Offer a person to choose between the following bets, where X=100, Y=0.

Assessing Discrete Probabilities



STEP 2: Offer a person to choose between the bets above. where now X=0, Y=100 **(A consistency check).**

The Expert should Switch!

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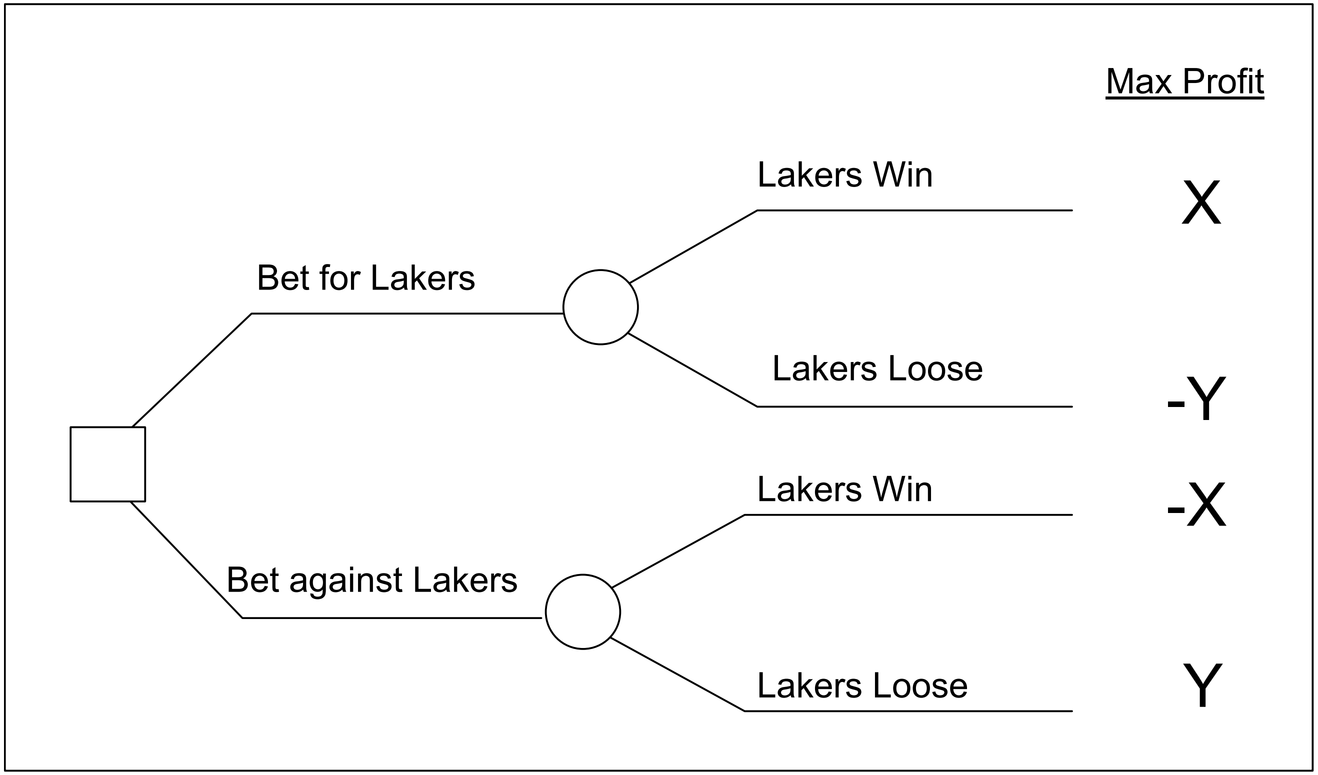
Assessing Discrete Probabilities



STEP 3: Offer a person to choose between the bets above. where now X=100, Y=100.

The comparison is not obvious anymore!

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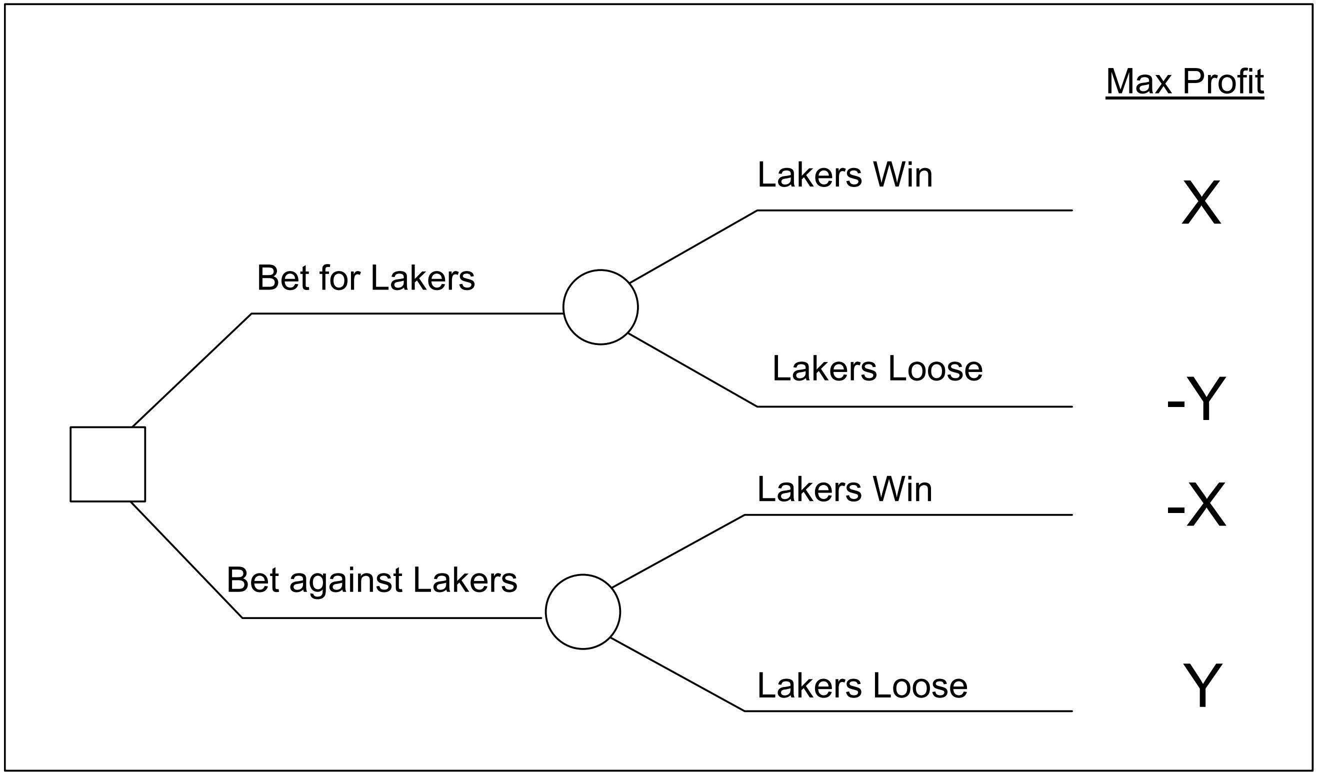
Assessing Discrete Probabilities



STEP 4: Offer a person to choose between the bets above. where now X=100, Y=50.

Continue until the expert is indifferent between these bets!

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Assessing Discrete Probabilities

Assumption: When an expert is **indifferent** between bets the **expected payoffs** from the bets must be the same.

$$\boldsymbol{X} \times \Pr(\boldsymbol{LW}) - \boldsymbol{Y} \times \Pr(\boldsymbol{LL}) = -\boldsymbol{X} \times \Pr(\boldsymbol{LW}) + \boldsymbol{Y} \times \Pr(\boldsymbol{LL}) \Leftrightarrow$$

$$-2\boldsymbol{X} \times \Pr(\boldsymbol{LW}) - 2\boldsymbol{Y} \times \{1 - \Pr(\boldsymbol{LW})\} = 0 \Leftrightarrow$$

$$\Pr(\boldsymbol{LW}) = \frac{\boldsymbol{Y}}{\boldsymbol{X} + \boldsymbol{Y}}$$

Example: Point of indifference at X=50,Y=100

$$\Pr(\boldsymbol{LW})\} = \frac{100}{150} = \frac{2}{3} \approx 66.7\%$$

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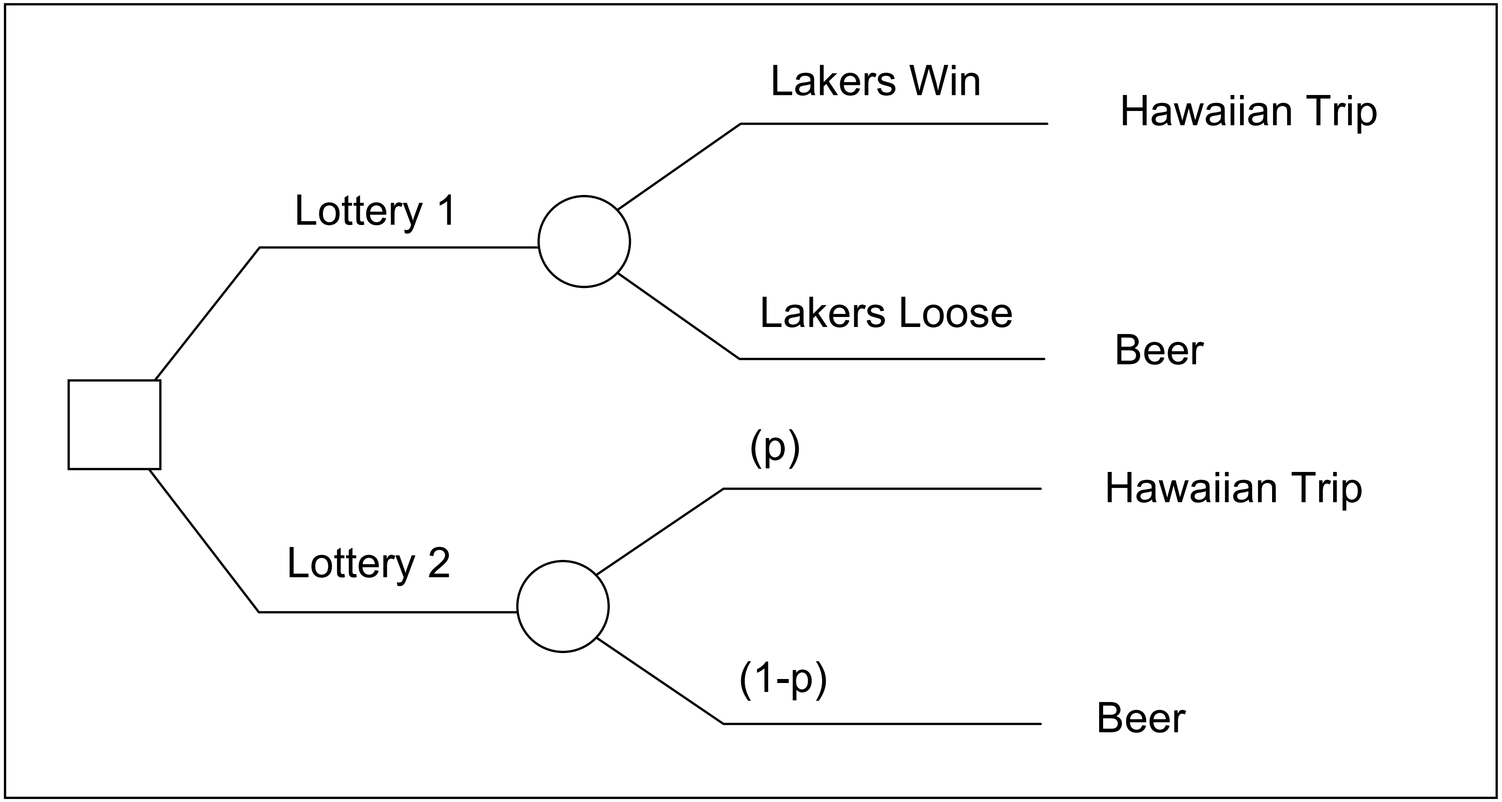
2. Reference Lotteries:

Event: Lakers winning the NBA title this season

Choose two prices A and B, such that A>>B.



Assessing Discrete Probabilities

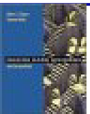
Lottery 2 is the **REFERENCE LOTTERY** and a probability mechanism is specified for lottery 2.

Examples of Probability Mechanisms:

- Throwing a fair coin
- Ball in an urn
- Throwing a die,
- Wheel of fortune

Strategy:

1. Specify p_1. Ask which one do you prefer?
2. If Lottery 1 is preferred offer change p_i to $p_{i+1} > p_i$.
3. If Lottery 2 is preferred offer change pi to $p_{i+1} < p_i$.
4. When indifference point is reached STOP, else Goto 2.



Assessing Discrete Probabilities

Assumption:

When Indifference Point has been reached $\Rightarrow$ $Pr(LW) = p$

Consistency Checking:

Subjective Probabilities must follow the laws of probability

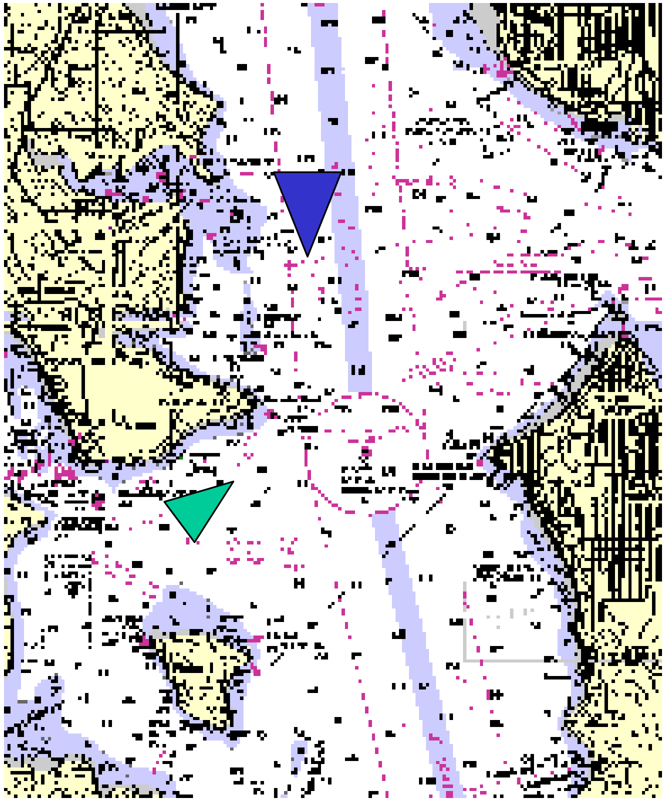
Example:

If an expert specifies $Pr(A)$, $Pr(B|A)$ and $Pr(A \cap B)$ then

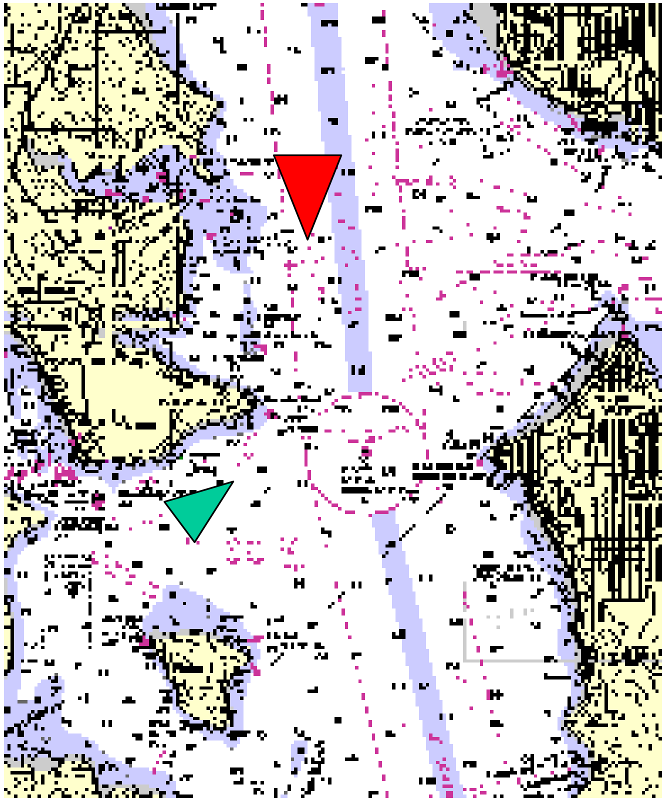
$$Pr(B|A) * Pr(A) = Pr(A \cap B)$$

3. Paired Comparisons of Situations (Optional)

Issaquah class ferry on the Bremerton to Seattle route in a crossing situation within 15 minutes, no other vessels around, good visibility, negligible wind.



Other vessel is a navy vessel



Other vessel is a product tanker

Example of Question Format

Question: 1 89

Situation 1	**Attribute**	**Situation 2**
Issaquah	**Ferry Class**	-
SEA-BRE(A)	**Ferry Route**	-
Navy	**1st Interacting Vessel**	**Product Tanker**
Crossing	**Traffic Scenario 1st Vessel**	-
0.5 – 5 miles	**Traffic Proximity 1st Vessel**	-
No Vessel	**2nd Interacting Vessel**	-
No Vessel	**Traffic Scenario 2nd Vessel**	-
No Vessel	**Traffic Proximity 2nd Vessel**	-
> 0.5 Miles	**Visibility**	-
Along Ferry	**Wind Direction**	-
0	**Wind Speed**	-
	Likelihood of Collision	
	9 8 7 6 5 4 3 2 1 2 3 4 5 6 7 8 9	
Situation 1 is worse	**<==================X==================>>**	**Situation 2 is worse**

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Probability Model that uses the Expert's Responses

1 $\text{Traffic Scenario } 1 = \underline{X}^1 \xleftrightarrow{\text{Paired Comparison}} \text{Traffic Scenario } 2 =$

$$\underline{Y}(\underline{X}) = \text{Vector including 2 - way interactions}$$

2 $$\Pr(\text{Accident} \mid \text{Propulsion Failure}, \underline{X}^1) = P_0 e^{\underline{B}^T \underline{Y}(\underline{X}^1)}$$

3 $$\frac{\Pr(\text{Accident} \mid \text{Prop. Failure}, \underline{X}^1)}{\Pr(\text{Accident} \mid \text{Prop. Failure}, \underline{X}^2)} = \frac{P_0 e^{\underline{\beta}^T \underline{Y}(\underline{X}^1)}}{P_0 e^{\underline{\beta}^T \underline{Y}(\underline{X}2)}} = e^{\underline{\beta}^T \left(\underline{Y}(\underline{X}^1) - \underline{Y}(\underline{X}\right.}$$

4 $$LN\left\{\frac{\Pr(\text{Accident} \mid \text{Prop. Failure}, \underline{X}^1)}{\Pr(\text{Accident} \mid \text{Prop. Failure}, \underline{X}^2)}\right\} = \underline{\beta}^T \left(\underline{Y}(\underline{X}^1) - \underline{Y}(\underline{X}^2\right.$$

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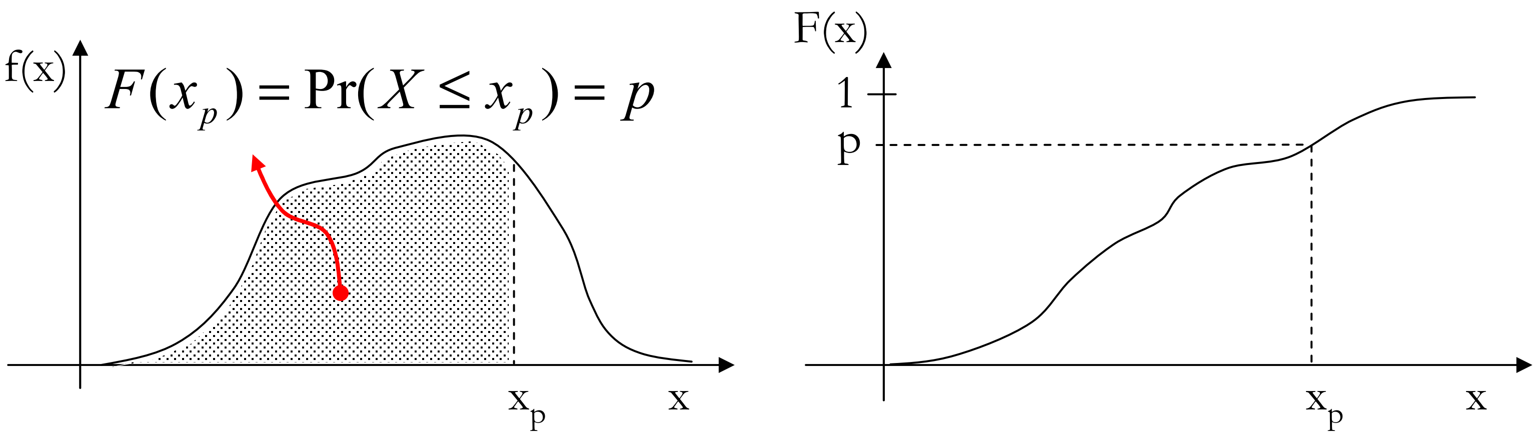
Assessing Continuous Probabilities

Method 1: Ask for distribution parameters e.g. Normal(μ,σ)
Method 2: Ask for distribution quantities and solve for the parameters (more in Chapter 10)
Method 3: Ask for the shape of a CDF e.g. by assessing a number of quantiles.

The p-th quantile x_p:



Terminology: quantile, fractile, percentile, quartile

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Assessing Continuous Probabilities

Assessing CDF is often conducted by assessing a number of quantiles.

Method 3A: Use quantile estimates to solve a distribution's parameters.

Method 3B: Connect multiple quantile estimates by straight lines to approximate the CDF

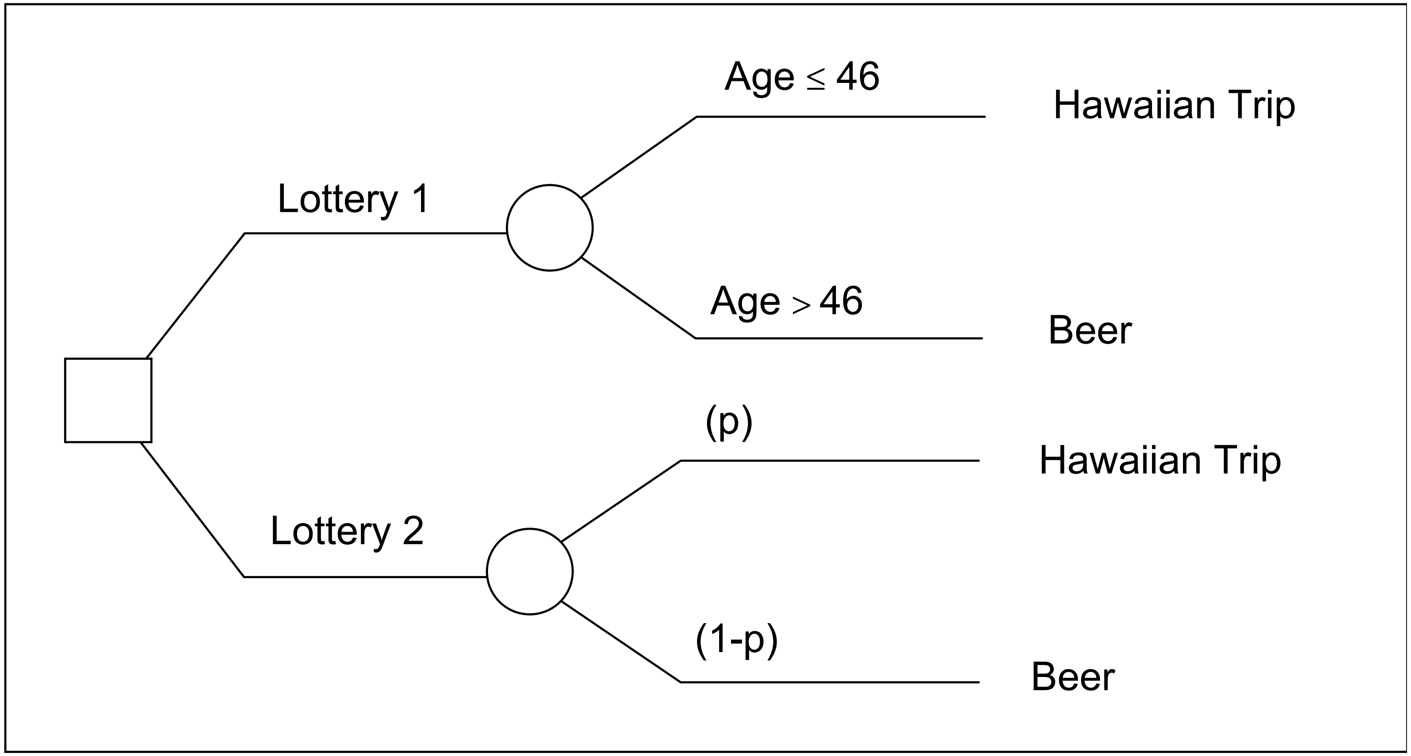
Example:

Uncertain Event: Current Age of a Movie Actress (e.g. ….)

Assessing Continuous Probabilities

STEP 1: You know the age of this actress is between 30 and 65.

STEP 2: Consider **a Reference Lottery**



Suppose you are indifferent for p=0.5 $\Rightarrow$ **Pr(Age $\leq$ 46) = 0.5**

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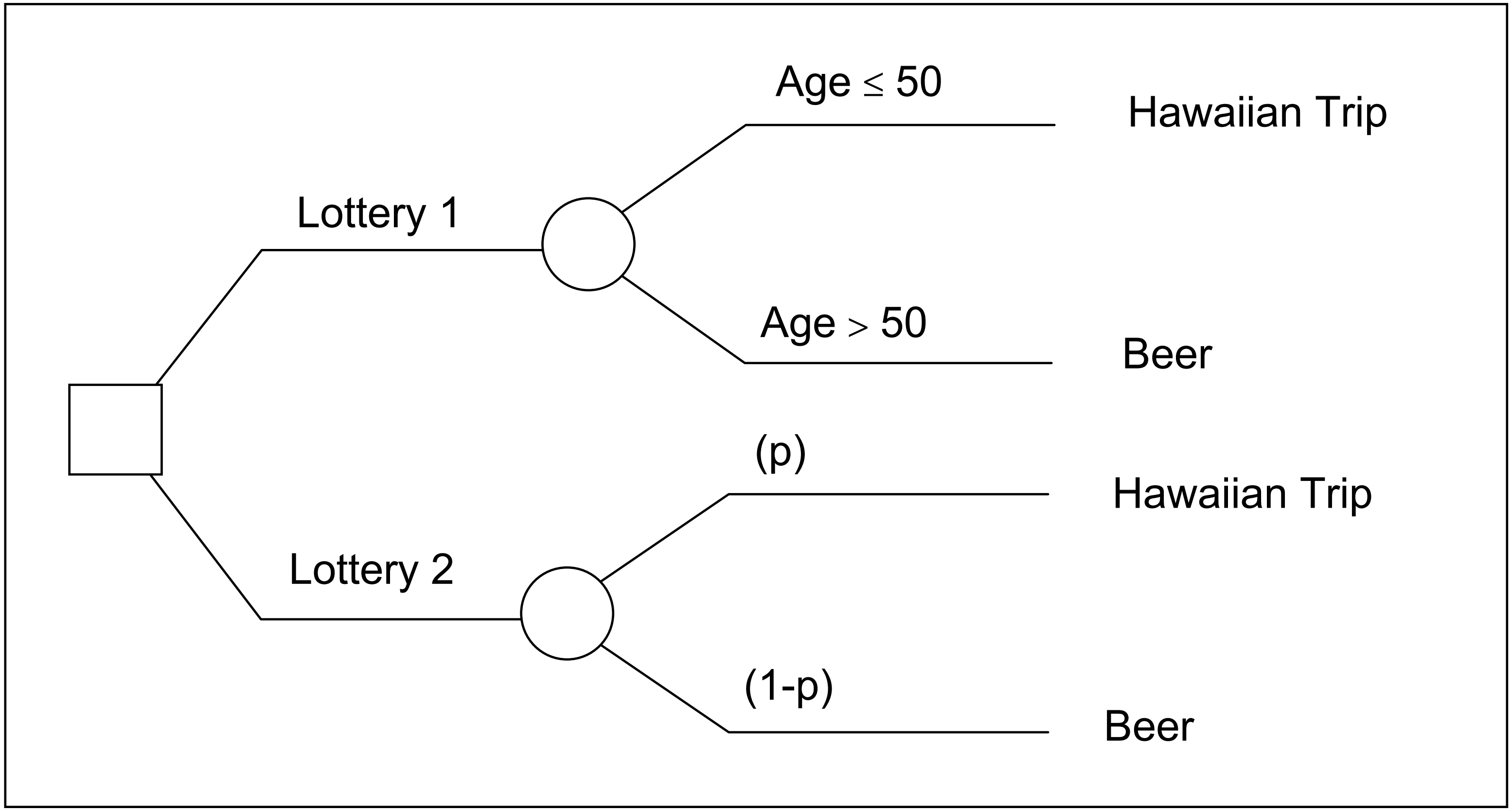
STEP 3: Consider a Reference Lottery



Suppose you are indifferent for p=0.8 ⇒ **Pr(Age ≤ 50) = 0.8**

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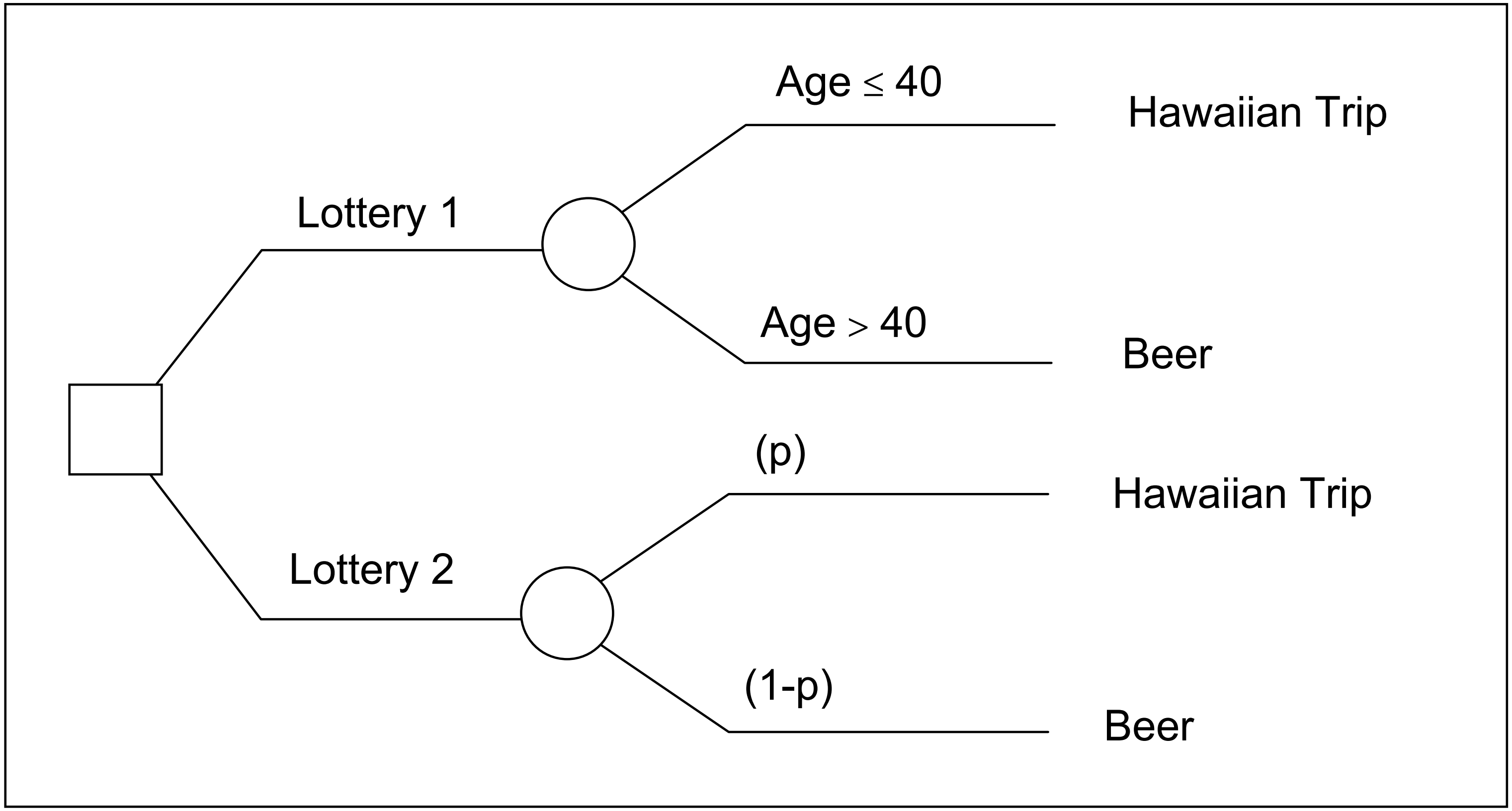
Assessing Continuous Probabilities

STEP 4: Consider **a Reference Lottery**



Suppose you are indifferent for p=0.05 ⇒ **Pr(Age ≤ 40) = 0.05**

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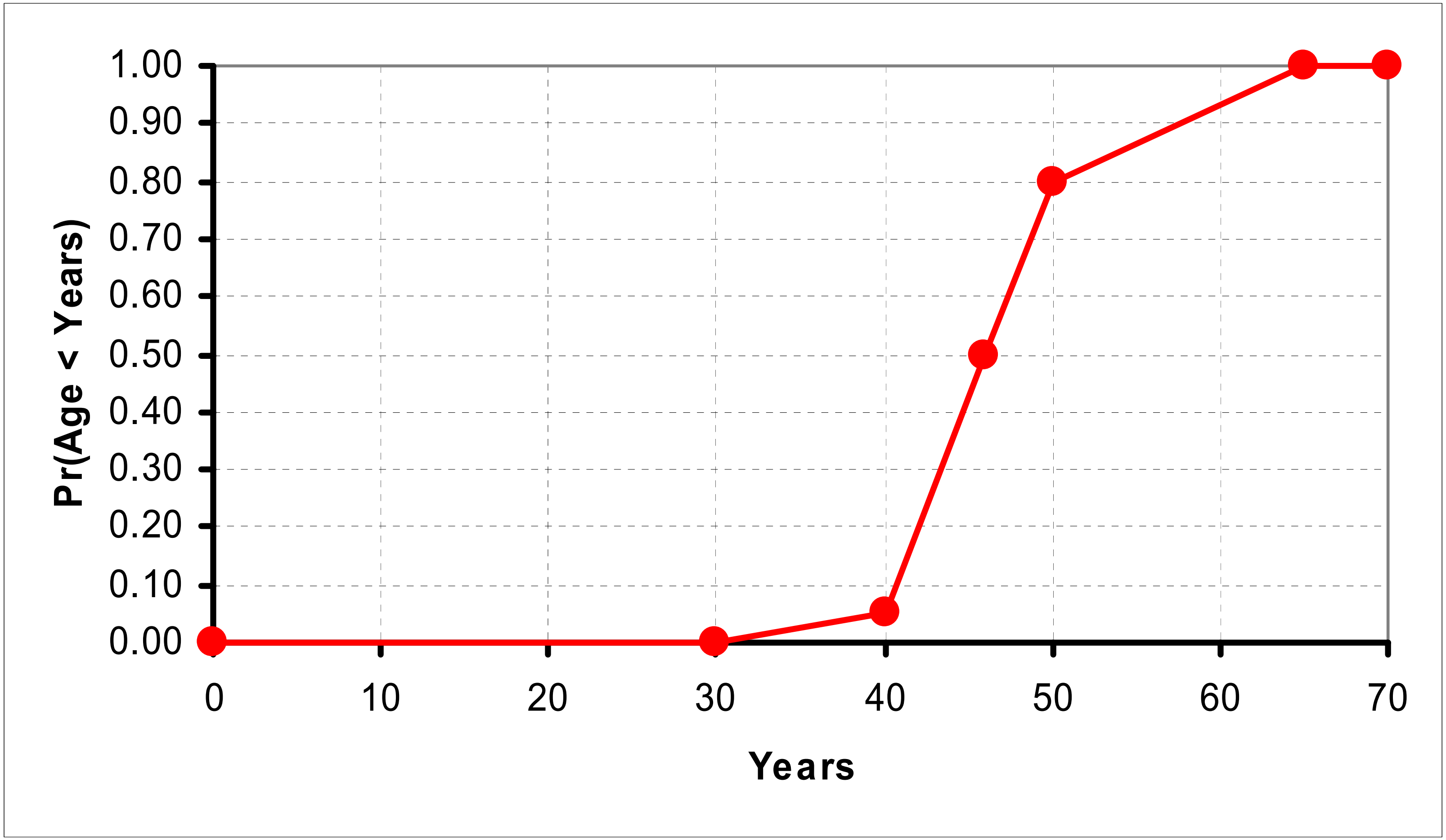
Assessing Continuous Probabilities

STEP 5: Approximate cumulative distribution function (CDF)



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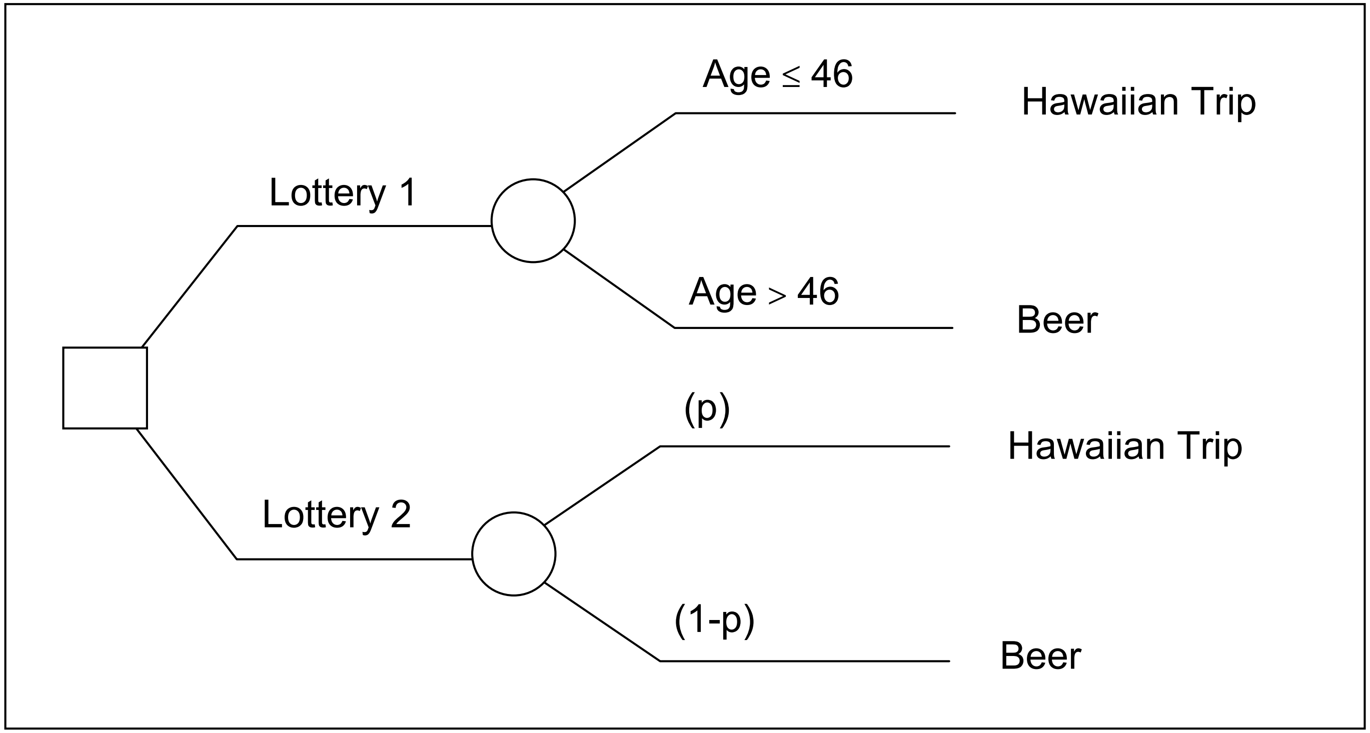
Two ways for using of **REFERENCE LOTTERIES**

1. Fix Horizontal Axis:



Strategy: Adjust **probability p** until indifference point has been reached by using a **probability mechanism**.

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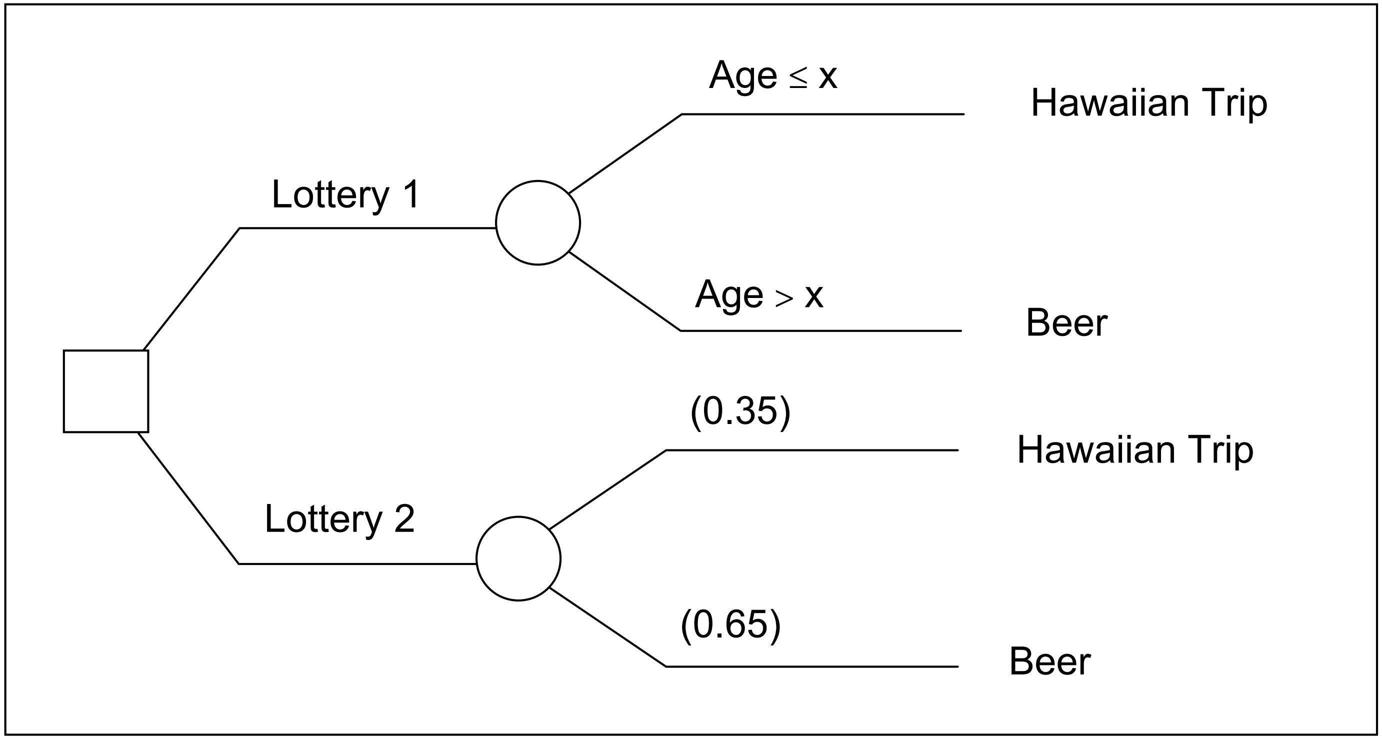
Two ways for using of **REFERENCE LOTTERIES**

2. Fix Vertical Axis:



Strategy: Adjust **Age x** until indifference point has been reached by using a **probability mechanism**.

Assessing Continuous Probabilities

Overall Strategy To Assess Continuous Cumulative Distribution Function:

STEP 1: Ask for the Median (50% Quantile).

STEP 2: Ask for Extreme Values (0% quantile, 100% quantile).

STEP 3: Ask for High/Low Values (5% Quantile, 95% Quantile).

STEP 4: Ask for 1st and 3rd Quartile (25% Quantile, 75% Quantile).

STEP 5 A: Approximate CDF through straight line technique

STEP 5 B: Model CDF between assessed points

STEP 5 C: Calculate "Best Fit" in a family of CDF's.

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Using Continuous CDF's in Decision Trees

Advanced Approach: Monte Carlo Simulation
(More in Chapter 11)

Simple Approach: Use a discrete approximation that well approximates the expected value of the underlying continuous distribution.

1.Extended Pearson Tukey-Method:

A continuous fan node is replaced by a **three branch uncertainy node**. Extended Pearson Tukey - method specifies what three outcomes to choose and which three probabilities to assign to these outcomes. It works well for **symmetric continuous distribution functions.**

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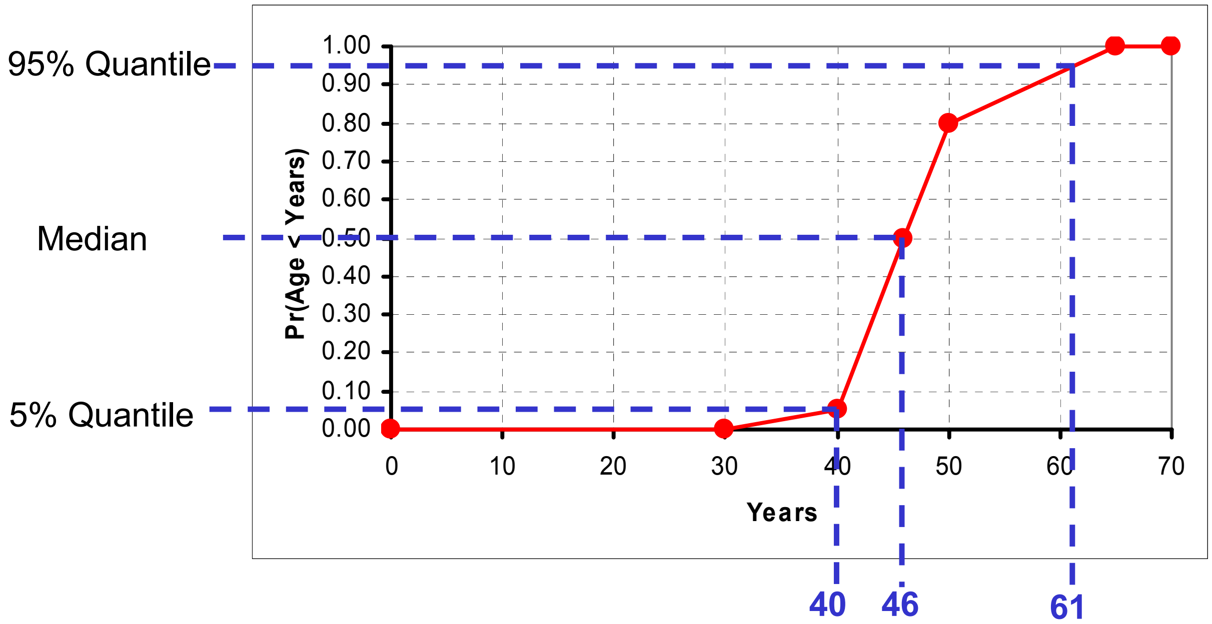
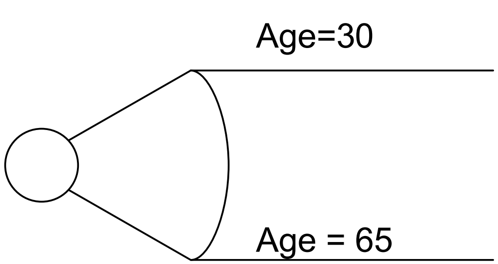
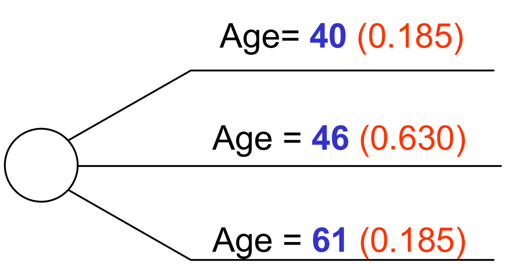
Extended Pearson Tukey Method





Continuous Fan



Discrete Approximation

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Extended Pearson Tukey Method

Next, calculate **the expected value** of **the discrete approximation**:

Age	Pr(Age)	Age*Pr(Age)
40	0.185	7.4
46	0.63	29.0
61	0.185	11.3
	E[Age]	47.7

2. Four Point Bracket Median Method:

A continuous fan node is replaced by a **four branch uncertainy node**:

STEP 1: Divide total range in four equally likely intervals
STEP 2: Determine bracket median in each interval
STEP 3: Assign equal probabilities to all bracket medians (0.25 in this case)

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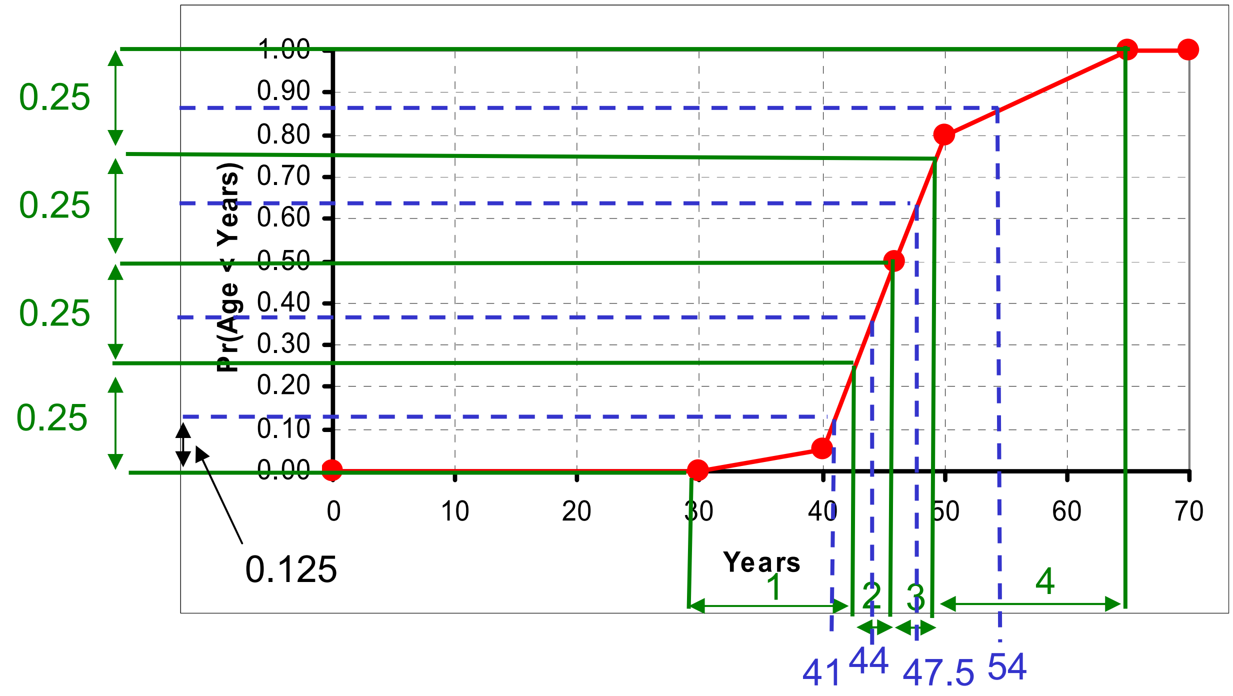
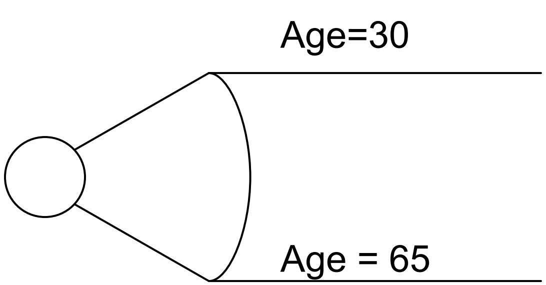
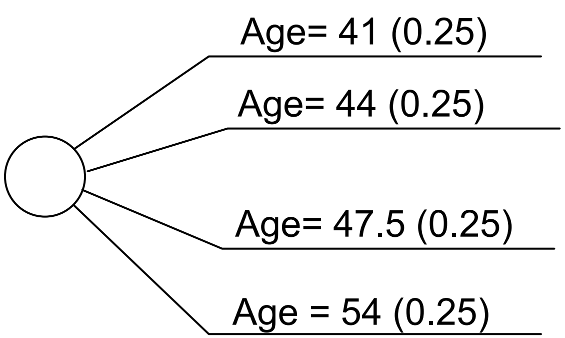
Four Point Bracket Median Method





Continuous Fan



Discrete Approximation

Draft: Version 1

Extended Pearson Tukey Method

Next, calculate **the expected value** of **the discrete approximation**:

Age	Pr(Age)	Age*Pr(Age)
41	0.25	10.3
44	0.25	11.0
47.5	0.25	11.9
54	0.25	13.5
	E[Age]	46.6

Accuracy of bracket median method can be improved by using **a five point approximation, a six point approximation**, etc. … until the approximated expected value does not change any more (beyond a specified accuracy level).

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Pit falls: Heuristics and Biases

Thinking probabilistically is not easy!!!!

When eliciting expert judgment, expert use primitive cognitive techniques to make their assessments. These techniques are in general simple and intuitively appealing, however they may result in a number of biases.

Representative Bias:
Probability estimate is made on the basis of **similarities within a group**. One tend to ignore relevant information such as incidence/base rate.

Example: X is the event that **"a person is sloppy dressed".** In your judgement Managers (M) are well dressed, i.e. **Pr(X|M)=0.1**, and Computer Scientist are badly dressed, i.e. **Pr(X|C)=0.8**.

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Pit falls: Heuristics and Biases

At a conference with **90% attendence of managers** and **10% attendance of computer scientist** you Observe a Person and notices that he dresses (particularly) sloppy. What do think is more likely?:

"The person is a computer scientist"

or

"The persion is a manager"

WHAT THE ANSWER SHOULD BE?

Pit falls: Heuristics and Biases

$$\frac{\Pr(C \mid X)}{\Pr(M \mid X)} = \frac{\dfrac{\Pr(X \mid C)\Pr(C)}{\Pr(X)}}{\dfrac{\Pr(X \mid M)\Pr(M)}{\Pr(X)}} = \frac{\Pr(X \mid C)\Pr(C)}{\Pr(X \mid M)\Pr(M)} = \frac{0.8 * 0.1}{0.1 * 0.9} < 1$$

IN OTHER WORDS: It is **more likely** that this person is **a manager** than **a computer scientist**.

Availability Bias:
Probability estimate is made according to the ease with which one can retrieve similar events.

Anchoring Bias:
One makes first assessment (anchor) and make subsequent assessments relative to this anchor.

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Motivational Bias:
Incentives are always present, such that people do not really say what they believe.

DECOMPOSITION AND PROBABILITY ASSESSMENTS

Break down problem into finer detail using probability laws until you have reached a point at which experts are **comfortable** in making the assessment in **a meaning full manner.** Next, aggregate the detail assessment using probability laws to obtain probability estimates at a lower level of detail.

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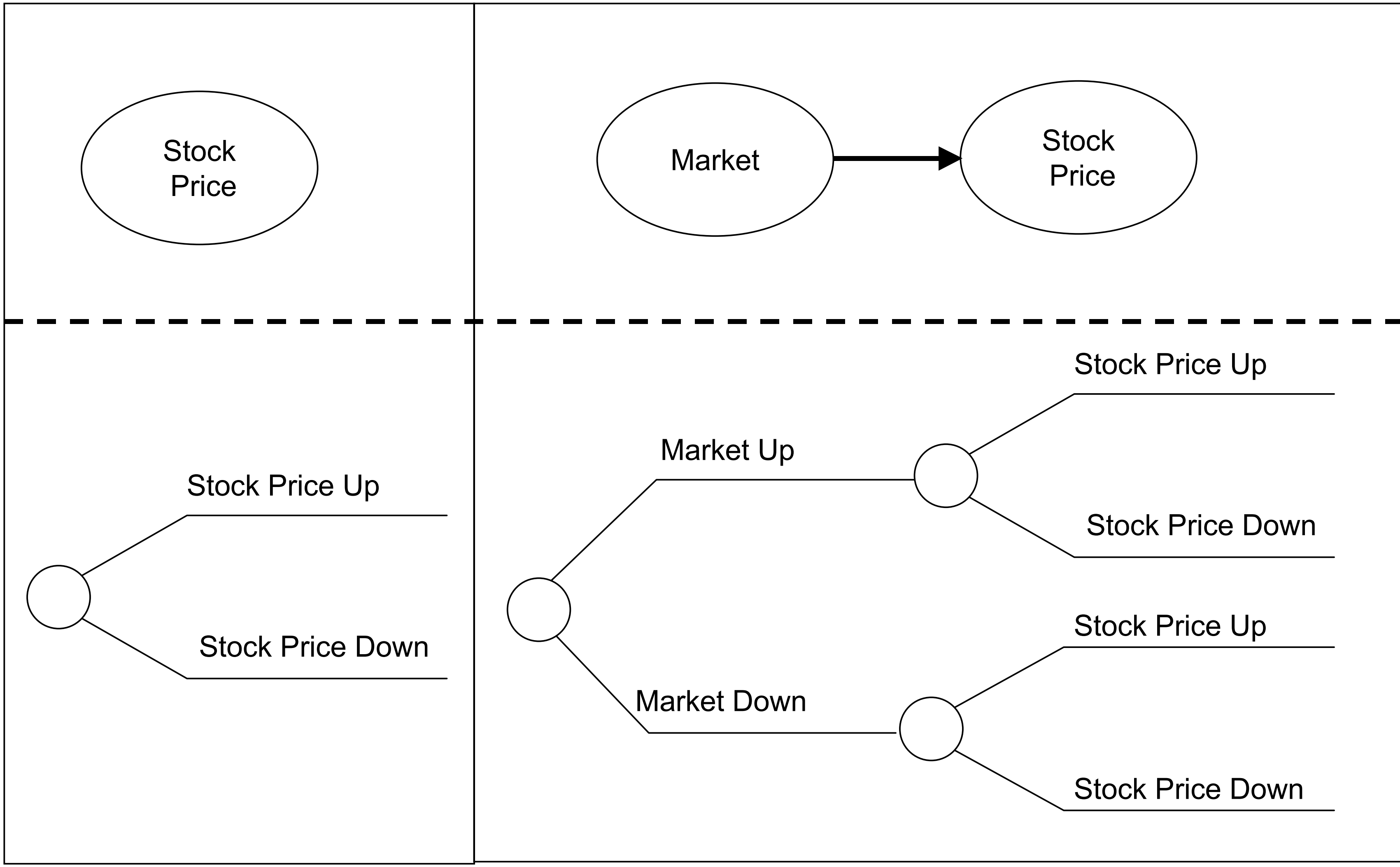
Motivational Bias:
Incentives are always present, such that people do not really say what they believe.

DECOMPOSITION AND PROBABILITY ASSESSMENTS

Break down problem into finer detail using probability laws until you have reached a point at which experts are **comfortable** in making the assessment in **a meaning full manner.** Next, aggregate the detail assessment using probability laws to obtain probability estimates at a lower level of detail.

Decomposition And Probability Assessments

Stock Market Example:



Decomposition And Probability Assessments

EXPERT JUDGEMENT ELICITATION PROCEDURE

STRUCTURED APPROACH TO CAPTURING AN EXPERTS **KNOWLEDGE BASE** AND CONVERT HIS\HER KNOWLEDGE BASE INTO **QUANTITATIVE ASSESSMENTS**.

MODELERS SKILLED IN DECOMPOSITION AND AGGREGATION OF ASSESSMENTS

NORMATIVE

EXPERTS

SUBSTANTIVE

KNOWLEDGABLE ABOUT THE SUBJECT MATTER AND EXTENSIVE EXPERIENCE

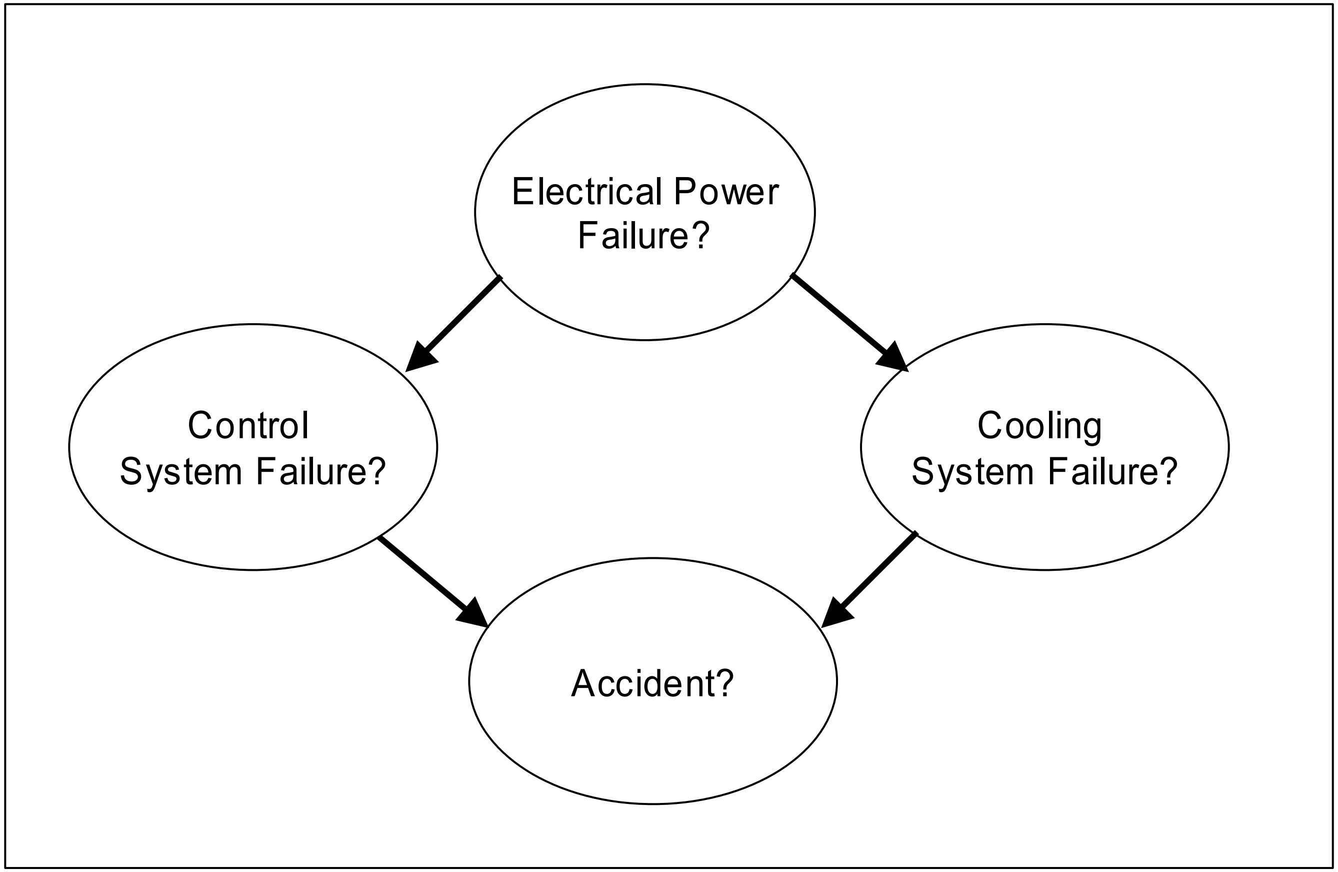
ELICITATION PROCESS = MULTIPLE CYCLES (AT LEAST 2)

1. **DECOMPOSITION OF EVENT OF INTEREST TO A MEANINGFULL LEVEL FOR SUBSTANTIVE EXPERT**

2. **ELICITATION OF JUDGMENT OF SUBSTANTIVE EXPERT FACILI-TATED BY NORMATIVE EXPERT**

3. **AGGREGATION OF JUDGEMENTS BY NORMATIVE EXPERT**

Nuclear Regulatory Example:



Which probability estimates do we need to calculate:

The Probability of An Accident?

Decomposition And Probability Assessments

A = Accident, L = Cooling System Failure,
N = Control System Failure, E = Electrical System Failure.

STAGE 1: WORK BACKWORDS

STEP 1: Assess the four conditional probabilities:

$$\Pr(A \mid L, N), \Pr(A \mid \overline{L}, N), \Pr(A \mid L, \overline{N}), \Pr(A \mid \overline{L}, \overline{N})$$

STEP 2: Assess the two conditional probabilities:

$$\Pr(L \mid E), \Pr(L \mid \overline{E})$$

STEP 3: Assess the two conditional probabilities

$$\Pr(N \mid E), \Pr(N \mid \overline{E})$$

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STEP 4: Assess the probability: $\Pr(E)$

STAGE 2: AGGREGATE DETAILED PROBABILITY ESTIMATES TO ASSESS THE PROBABILITY OF AN ACCIDENT

STEP 5: Apply Law of Total Probability

$$\Pr(A) = \Pr(A \mid L, N) \times \Pr(L, N) + \Pr(A \mid \overline{L}, N) \times \Pr(\overline{L}, N)$$

$$+ \Pr(A \mid L, \overline{N}) \times \Pr(L, \overline{N}) + \Pr(A \mid \overline{L}, \overline{N}) \times \Pr(L, \overline{N})$$

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Decomposition And Probability Assessments

STEP 6: Apply Law of Total Probability

$$\Pr(L,N) = \Pr(L,N \mid E) \times \Pr(E) + \Pr(L,N \mid \overline{E}) \times \Pr(\overline{E})$$

Do the same with: $\Pr(\overline{L},N), \Pr(L,\overline{N}), \Pr(\overline{L},\overline{N})$

STEP 7: Apply Conditional Independence assumption

$$\Pr(L,N \mid E) = \Pr(L \mid E) \times \Pr(N \mid E)$$

STEP 8: Go Back to STEP 5 and substitute the appropriate value to calculate the probability of an accident.

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Expert Judgment Elicitation Principles

(Source: **"Experts in Uncertainty"** ISBN: 0-019-506465-8
by Roger M. Cooke)

Reproducibility:
It must be possible for Scientific peers to review and if necessary **reproduce** all calculations. This entails that the calculation model must be fully specified and the ingredient data must be made available.

Accountability:
The source of Expert Judgment must be identified (who do they work for and what is their level of expertise).

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Expert Judgment Elicitation Principles

(Source: **"Experts in Uncertainty"** ISBN: 0-019-506465-8 by Roger M. Cooke)

Empirical Control:
Expert probability assessment must in principle be susceptible to empirical control.

Neutrality:
The method for combining/evaluating expert judgments should encourage experts to state true opinions.

Fairness:
All Experts are treated equally, prior to processing the results of observation

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Practical Expert Judgment Elicitation Guidelines

1. **The questions must be clear.** Prepare an attractive format for the questions and graphic format for the answers.

2. Perform a **dry run**. Be prepared to change questionnaire format.

3. **An analyst must be present** during the elicitation.

4. Prepare **a brief explanation** of the elicitation format and of **the model** for processing the responses.

5. Avoid **coaching.** (You are not the expert)

6. The elicitation session should **not exceed 1 hour**.

Coherence and The Dutch Book

Subjective probabilities must follow The Laws of Probability. If they do not, the person assessing the probabilities is **incoherent.**

Incoherence $\Rightarrow$ Possibility of a Dutch Book

Dutch Book:

A series of bets in which **the opponent is guaranteed to loose** and you win.

Example: There will be a Basketball Game tonight between Lakers and Celtics and your friend says that **Pr(Lakers Win)=40%** and **Pr(Celtic Win)=50%.** You note that the probabilities do not add up to 1, but your friend **stubbornly** refuses to change his initial estimates.

You think, "GREAT!" let's set up a series of bets!

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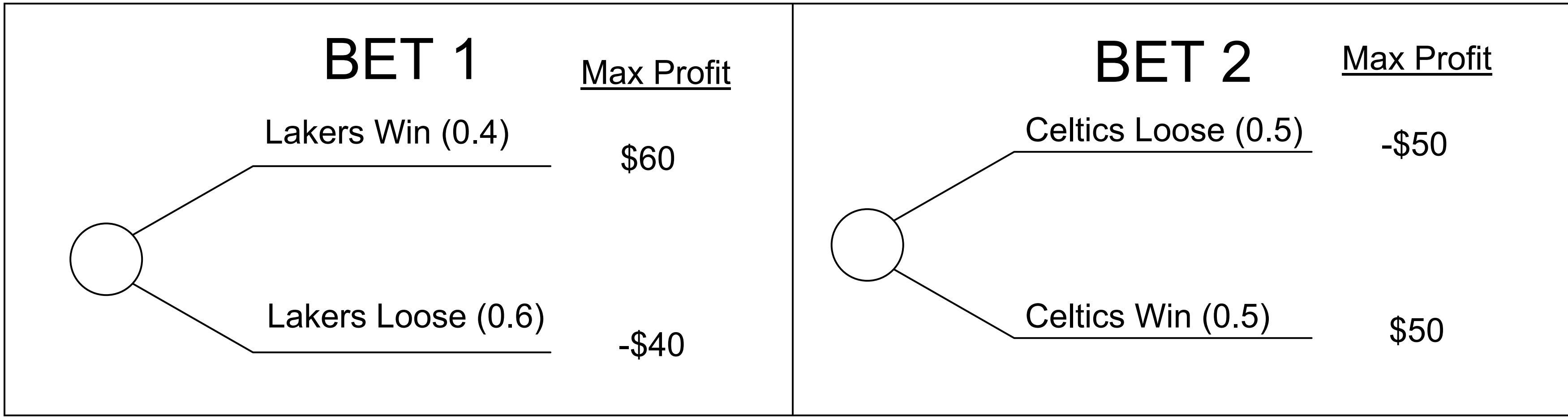
Coherence and The Dutch Book



Note that, **EMV of both bets equal $0** according to his probability assessments and **can thus be considered fair** and he should be willing to engage in both.

Lakers Win: Bet 1 - You win $60, Bet 2: You Loose $50,
Net Profit: $10

Lakers Loose: Bet 1 - You loose $40, Bet 2: You win $50,
Net Profit: $10