Generalized Diagonal Band Copulae with Two-Sided Generating Densities

"Presentation Short Course: Beyond Beta and Applications"

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OUTLINE

1. **INTRODUCTION**
2. COPULA CONSTRUCTION
3. GENERALIZED DIAGONAL BAND EXAMPLES
4. SAMPLING PROCEDURE
5. ORDINAL MEASURES OF ASSOCIATION
6. COPULA PARAMETER ELICITATION
7. A VALUE OF INFORMATION EXAMPLE
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- X', Y': **Continuous random variables** such that $X' \sim G(\cdot)$, $Y' \sim H(\cdot)$
- $G(\cdot)$, H($\cdot$): **Cumulative distribution functions** - cdf's.
- The mapping $X' \rightarrow X = G(X') \Rightarrow$ **$X \sim$ Uniform[0, 1]** is called the *probability integral transformation* e.g. Nelsen (1999).
- Any bivariate joint distribution of (X', Y') can be transformed to a bivariate copula $(X, Y) = \{G(X'), H(Y')\}$ - Sklar (1959).
- Thus, a bivariate copula is **a bivariate distribution with uniform marginals.**
- As such, many authors studied copulae **indirectly.**
- Gaussian and Student-t Copulae (of this construct) were studied **explicitly.**

- Genest and Mackay (1986) used **an algabraic method** for copula construction.

- $\varphi : (0,1] \rightarrow [0,\infty)$, *a convex decreasing function* with $\varphi(1) = 0$ - **The generator function.**

- They possess **joint cdf and probability density function (pdf)**:

$$C\{x,y|\varphi(\cdot)\} = \begin{cases} \varphi^{-1}\{\varphi(x)+\varphi(y)\} & \varphi(x)+\varphi(y) \leq 0 \\ 0 & \text{elsewhere} \end{cases} \tag{1}$$

$$c\{x,y|\varphi(\cdot)\} = -\frac{\varphi''\{C(x,y)\}\varphi'(x)\varphi'(y)}{[\varphi'\{C(x,y)\}]^3} \tag{2}$$

- Cooke and Waij (1986) used **a geometric method** for copula construction

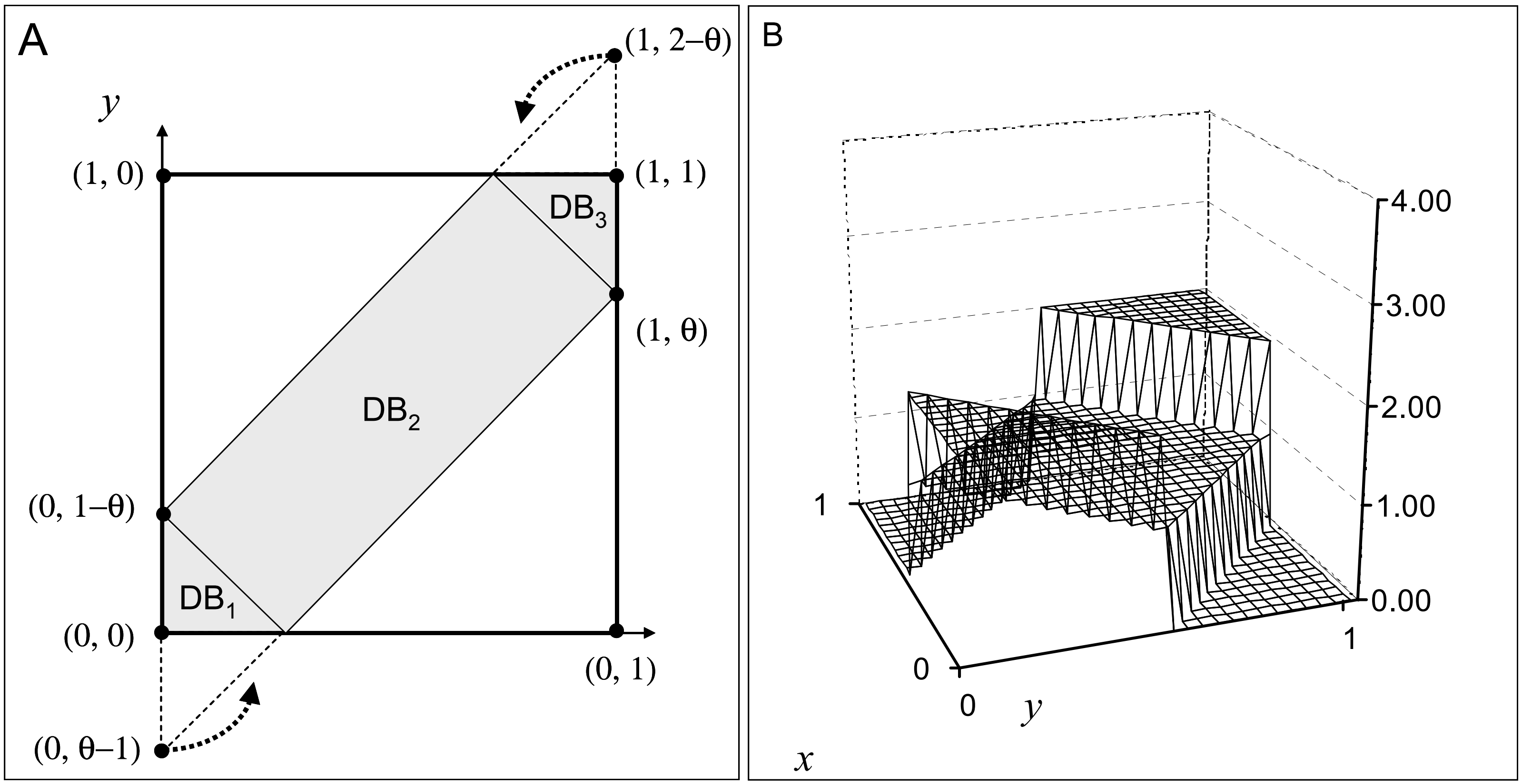

Figure 1: A: Gray area support of a $DB(\theta)$ copula comprised of sub-areas $DB_i, i = 1, 2, 3$; B: Example of a $DB(0.5)$ copula.

- Diagonal Band (DB) copula possess pdf:

$$C\{x,y|\theta\} = \begin{cases} 1/(1-\theta) & (x,y) \in DB_1 \cup DB_3 \\ 1/\{2(1-\theta)\} & (x,y) \in DB_2 \\ 0 & \text{elsewhere} \end{cases} \quad (3)$$

- *Analagous to Archimedean copula*, Bojarski (2001) generalized $DB(\theta)$ copula via **a generator function $f(\cdot\,|\theta)$.**

- Generator function $f(\cdot\,|\theta)$ is a **symmetric pdf** with support $[\theta-1, 1-\theta]$.

- Lewandowski (2005) showed that Bojarski's (2001) GDB Copulae are equivalent to Fergusons (1995) family of copulae with joint pdf:

$$c(x,y) = \frac{1}{2}\{g(|x-y| + g(1-|1-x-y|)\}, g(\cdot\,) \text{ pdf on } [0,1] \quad (4)$$

- For sampling efficiency **inverse cdf** of generator $f(\cdot|\theta)$ would be desirable.

- Consider Van Dorp and Kotz's (2003) **symmetric Two-Sided (TS) pdf's** :

$$f\{z|p(\cdot|\Psi)\} = \frac{1}{2} \times \begin{cases} p(z+1|\Psi), & \text{for } -1 < z \leq 0, \\ p(1-z|\Psi), & \text{for } \quad 0 < z < 1, \end{cases} \tag{5}$$

that too uses the generating pdf $p(z)$ concept. Pdf $p(z)$ has support $[0, 1]$.

- The **inverse cdf (or quantile function)** associated with (4)

$$F^{-1}\{u|p(\cdot|\Psi)\} = \begin{cases} P^{-1}(2u|\Psi) - 1, & \text{for } 0 < u \leq \frac{1}{2}, \\ 1 - P^{-1}(2 - 2u|\Psi), & \text{for } \frac{1}{2} < u < 1, \end{cases} \tag{6}$$

where $P^{-1}(\cdot|\psi)$ is the quantile function of $p(\cdot|\Psi)$.

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- Bivariate pdf $g(x,y)$ is constructed, where $X \sim U[0,1]$ and **the conditional pdf $g(y|x)$** has the following form :

$$g\{y|x, p(\cdot|\Psi)\} = f\{x-y|p(\cdot|\Psi)\}, x-1 \leq y \leq x+1, \tag{7}$$

- From $X \sim U[0,1]$, (7) and **TS framework pdf (4)** it follows that:

$$g\{x,y|p(\cdot|\Psi)\} = \frac{1}{2} \times \begin{cases} p(1+x-y|\Psi), & -1 < x-y \leq 0, \\ p(1-x+y|\Psi), & 0 < x-y < 1, \end{cases} \tag{8}$$

- From (8), a bivariate pdf $c(x,y\,|\,p(\cdot|\Psi))$ is constucted on the unit square $[0,1]^2$ **by folding back the probability masses** of $g\{x,y|p(\cdot|\Psi)\}$ outside the unit square $[0,1]^2$ onto it, **using "folding" lines $y=1$ and $y=0$.**

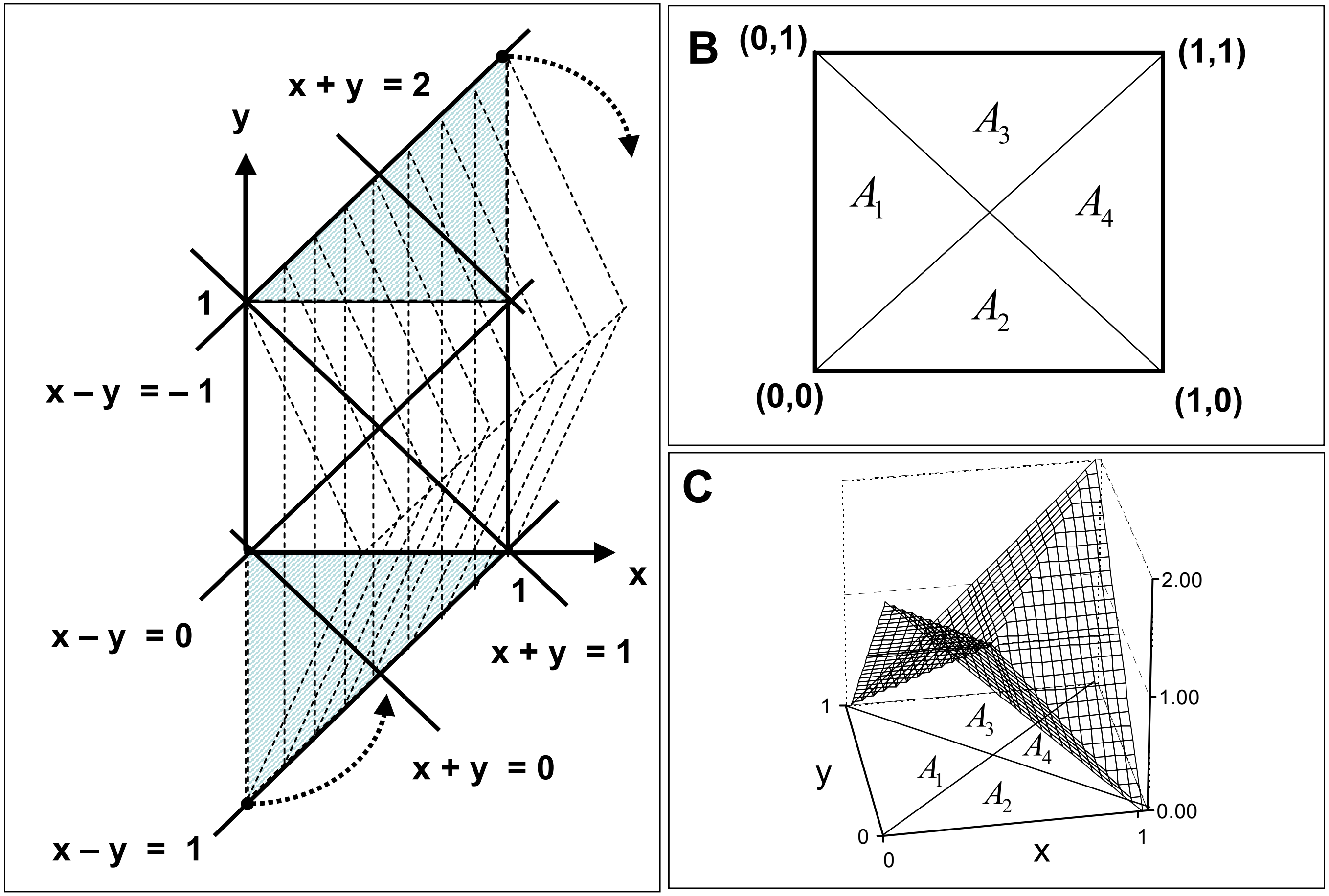

Figure 1. A: $g(x,y)$ pdf (8); B: Areas $A_i, i = 1, \ldots, 4$;
C: $c\{x, y|p(\,\cdot\,|\Psi)\}$ pdf (10) with $p(z) = 2z$ on $[0,1]$.

- **Relationship between $c\{x, y|p(\cdot|\Psi)\}$ and $g\{x, y|p(\cdot|\Psi)\}$** in (8) :

$$c\{x, y|p(\cdot|\Psi)\} = \tag{9}$$
$$\begin{cases} g\{x, y|p(\cdot|\Psi)\} + g\{x, -y|p(\cdot|\Psi)\}, & 0 < x + y \leq 1, \\ g\{x, y|p(\cdot|\Psi)\} + g\{x, 2 - y|p(\cdot|\Psi)\}, & 1 < x + y \leq 2. \end{cases}$$

- **Combining (9) with (8)** now yields :

$$c\{x, y|p(\cdot|\Psi)\} = \tag{10}$$
$$\frac{1}{2} \times \begin{cases} p(1 - x - y|\Psi) + p(1 + x - y|\Psi), & (x, y) \in A_1, \\ p(1 - x - y|\Psi) + p(1 - x + y|\Psi), & (x, y) \in A_2, \\ p(x + y - 1|\Psi) + p(1 + x - y|\Psi), & (x, y) \in A_3, \\ p(x + y - 1|\Psi) + p(1 - x + y|\Psi), & (x, y) \in A_4. \end{cases}$$

- Note in (10) $c(y, x) = c(x, y)$. Hence, $X \sim U[0, 1] \Rightarrow Y \sim U[0, 1]$

- Pdf of GDB copula with **TS pdf with generating pdf $p(z|\Psi)$** :

$$c\{x, y|p(\,\cdot\,|\Psi)\} = \frac{1}{2} \times \begin{cases} p(1-x-y|\Psi) + p(1+x-y|\Psi), & (x,y) \in A_1, \\ p(1-x-y|\Psi) + p(1-x+y|\Psi), & (x,y) \in A_2, \\ p(x+y-1|\Psi) + p(1+x-y|\Psi), & (x,y) \in A_3, \\ p(x+y-1|\Psi) + p(1-x+y|\Psi), & (x,y) \in A_4. \end{cases}$$

- Cdf of GDB copula with TS gen. pdf $p(z|\Psi)$ and **cdf $P(z|\Psi)$** follows as:

$$C\{x, y|p(\,\cdot\,|\Psi)\} = \begin{cases} x - \frac{1}{2}\int_{1-x-y}^{1+x-y} P(z|\Psi)dz, & (x,y) \in A_1, \\ y - \frac{1}{2}\int_{1-x-y}^{1-x+y} P(z|\Psi)\}dz, & (x,y) \in A_2, \\ x - \frac{1}{2}\int_{x+y-1}^{1+x-y} P(z|\Psi)dz, & (x,y) \in A_3, \\ y - \frac{1}{2}\int_{x+y-1}^{1-x+y} P(z|\Psi)dz, & (x,y) \in A_4. \end{cases} \qquad (11)$$

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- Substitution of **generating pdf $p(z) = 2z$** with support $[0, 1]$ in (10) yields

$$c(x,y) = 2 \times \begin{cases} 1-y, & (x,y) \in A_1, \quad 1-x, \quad (x,y) \in A_2, \\ x, & (x,y) \in A_3, \quad y, \quad (x,y) \in A_4. \end{cases}$$

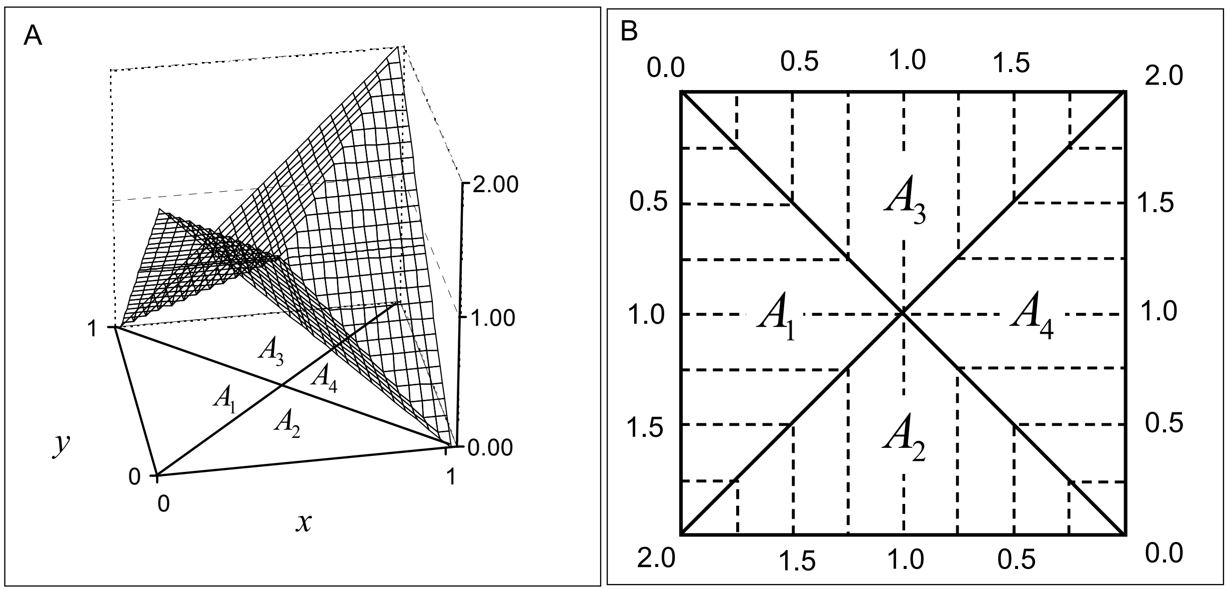


Figure 2. A: Copula density $c\{x, y\}$; B: Density contour plot.

- Substitution of pdf $p(z) = 2z$ in (11) and **generating cdf $P(z) = z^2$** yields:

$$C\{x,y\} = \frac{1}{3} \times \begin{cases} -x^3 - 3xy^2 + 6xy\,, & (x,y) \in A_1, \\ -y^3 - 3x^2y + 6xy, & (x,y) \in A_2, \\ y^3 - 3y^2 + 3y(x^2+1) - 3x^2 + 3x - 1, & (x,y) \in A_3, \\ x^3 - 3x^2 + 3x(y^2+1) - 3y^2 + 3y - 1, & (x,y) \in A_4. \end{cases}$$

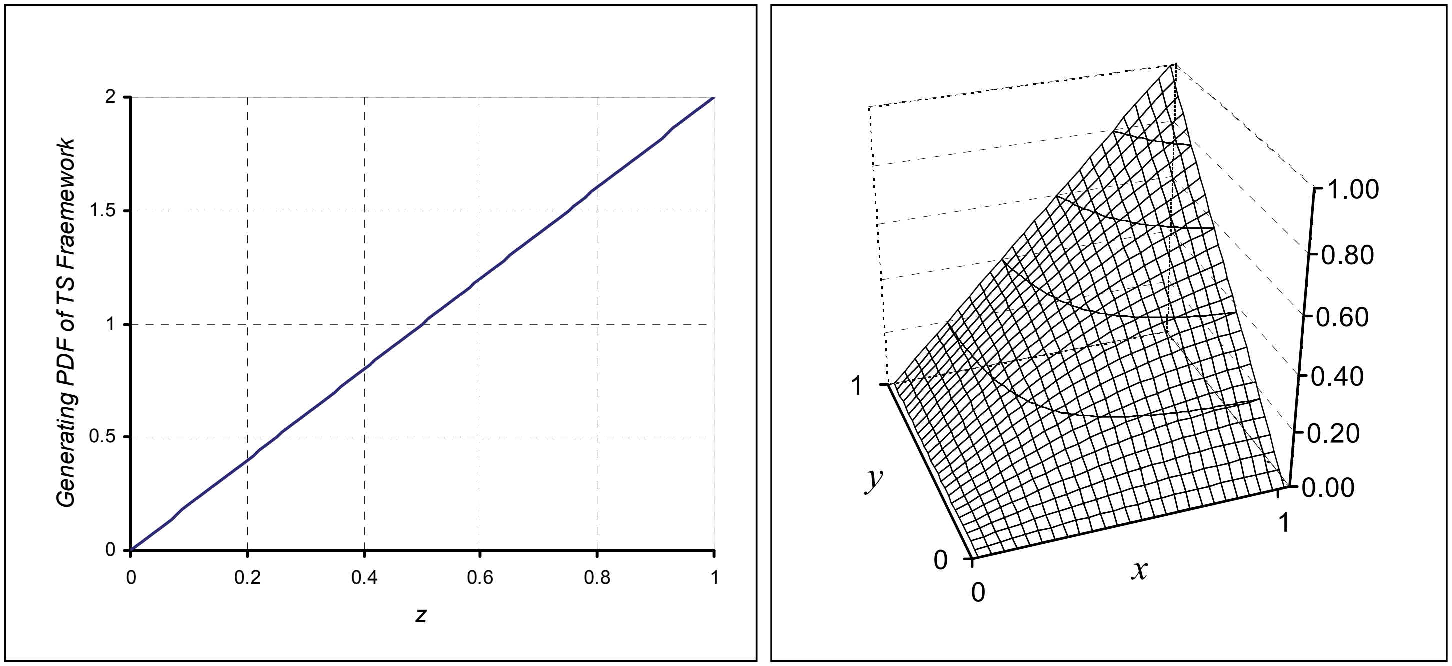


Figure 3. Graph of joint triangular copula cdf $C(x,y)$ given above.

$$p(z|\alpha) = 2 - \alpha + 2(\alpha - 1)z,\ 0 \leq \alpha \leq 2,$$

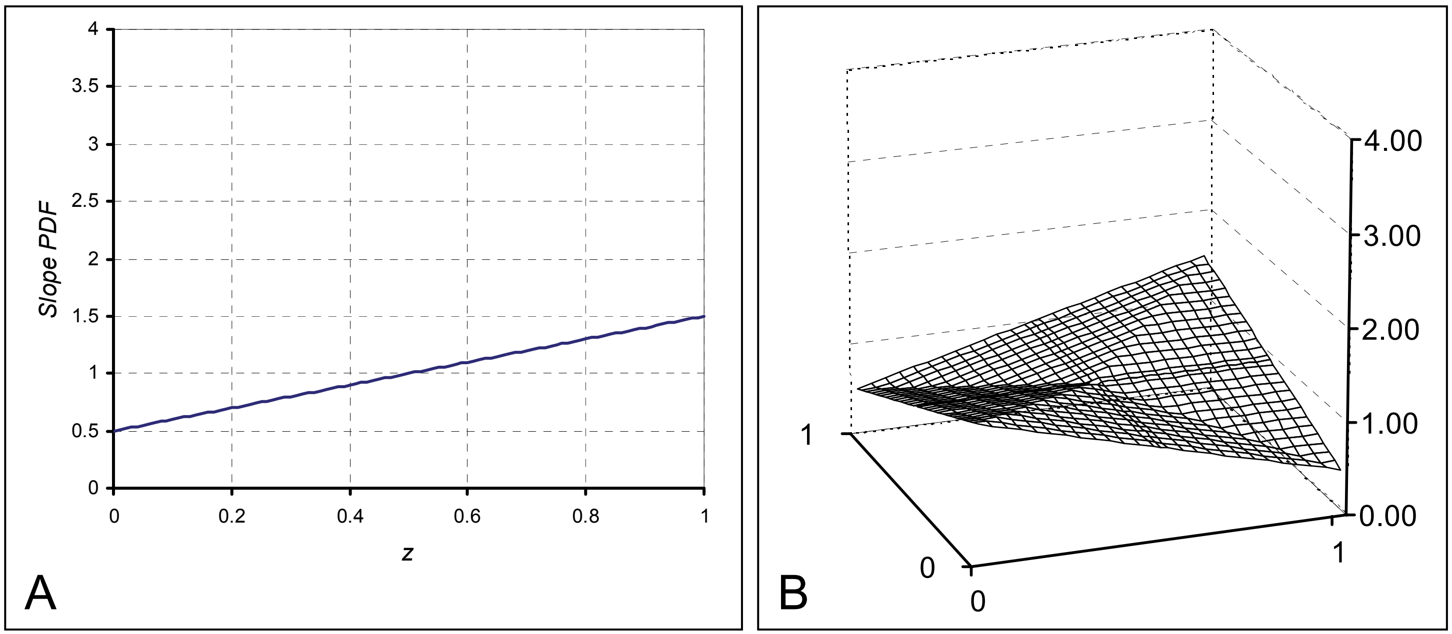


Figure 4. A: Slope generating pdf; B: GDB Copula with TS Gen. PDF in A.

$$p(z|n) = nz^{n-1},\ n > 0,$$

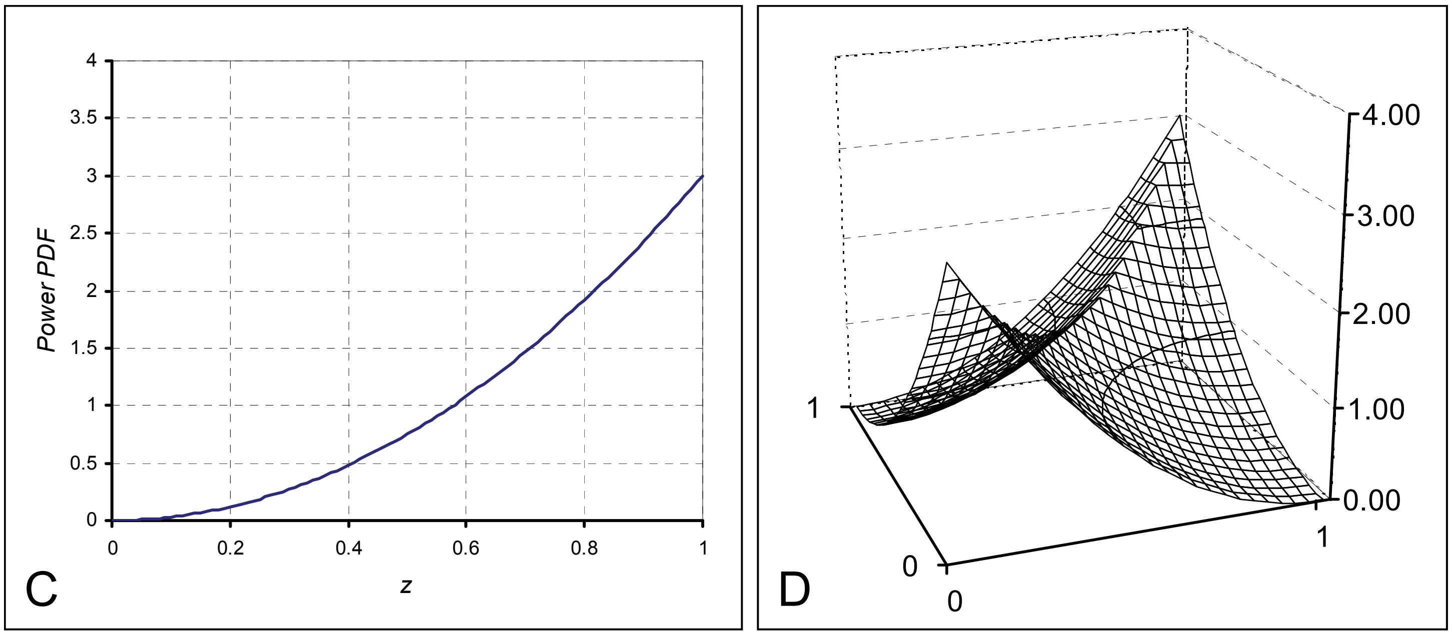


Figure 5. C: Power generating pdf; D: GDB Copula with TS Gen. PDF in A.

$$p(z|m) = \frac{m+2}{3m+4}\{2(m+1)\sqrt{z^m} - mz^{m+1}\}, m > 0.$$

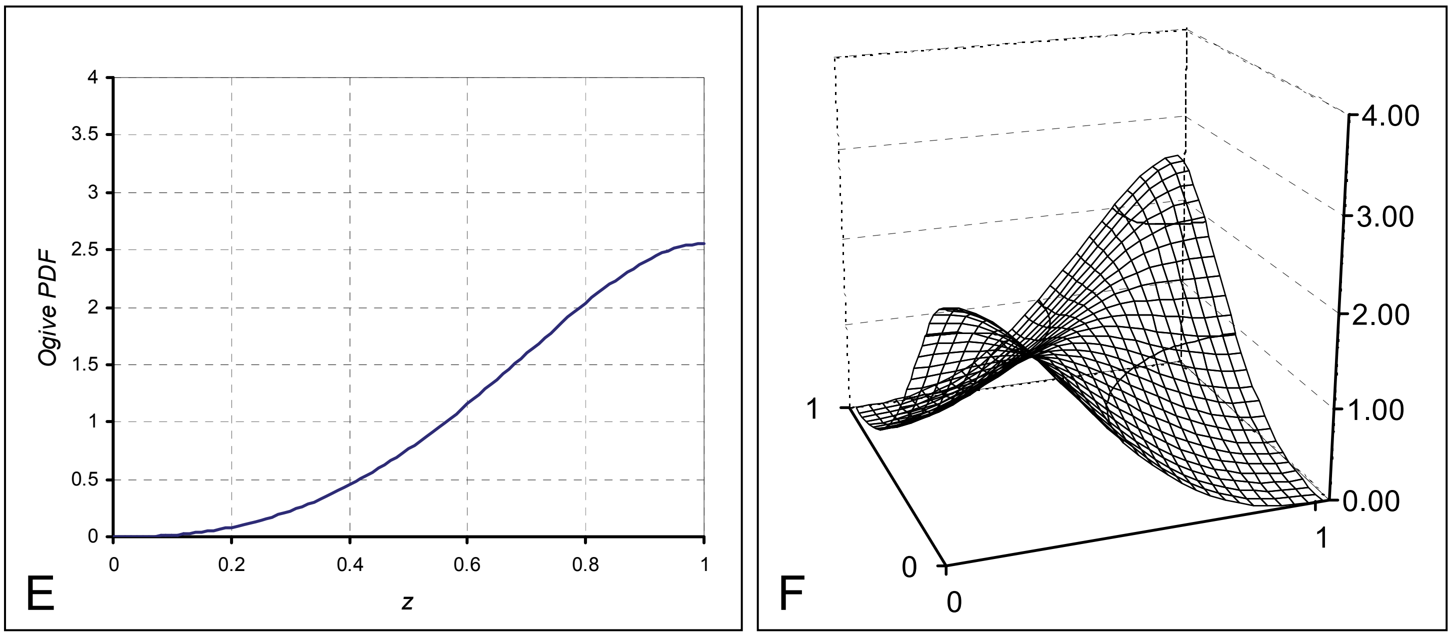
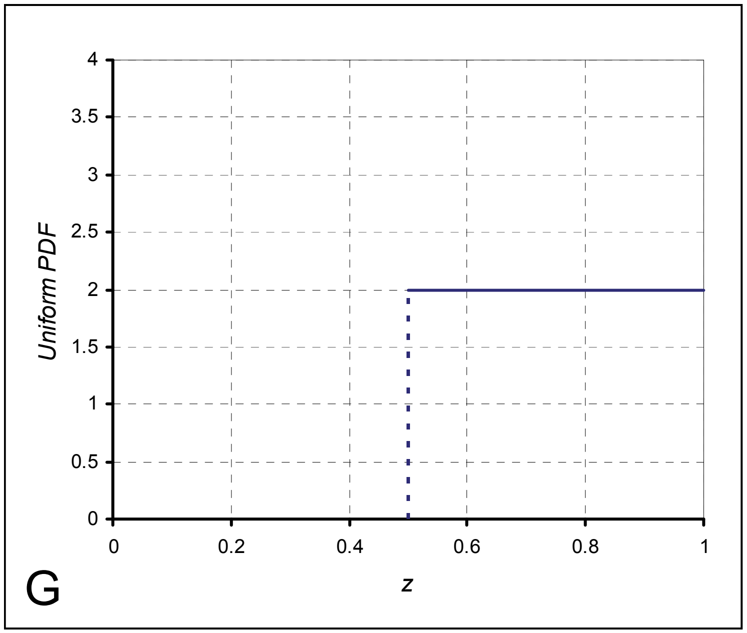


Figure 6. E: Ogive generating pdf; F: GDB Copula with TS Gen. PDF in A.

$$p(z|\theta) = \frac{1}{1-\theta}, \theta \leq z \leq 1, 0 \leq \theta \leq 1,$$



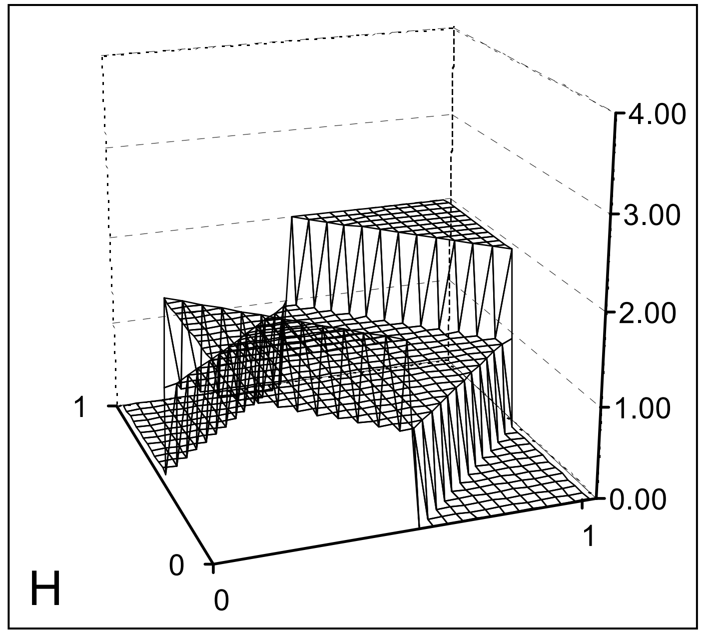


Figure 7. G: Uniform $[\theta, 1]$ gen.pdf; H: GDB Copula with TS Gen. PDF in A.

$$p(z|a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\, x^{a-1}(1-x)^{b-1}, a > 0,\, b > 0,$$

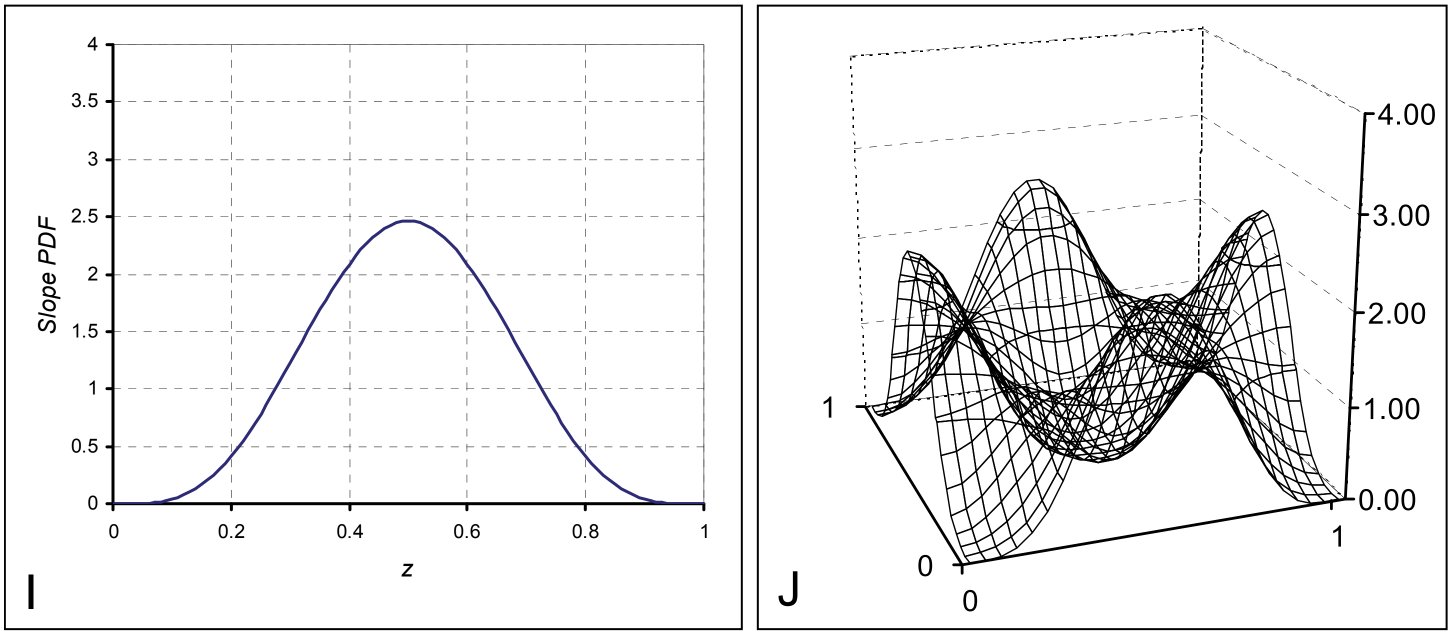


Figure 8. G: Beta generating pdf; H: GDB Copula with TS Gen. PDF in A.

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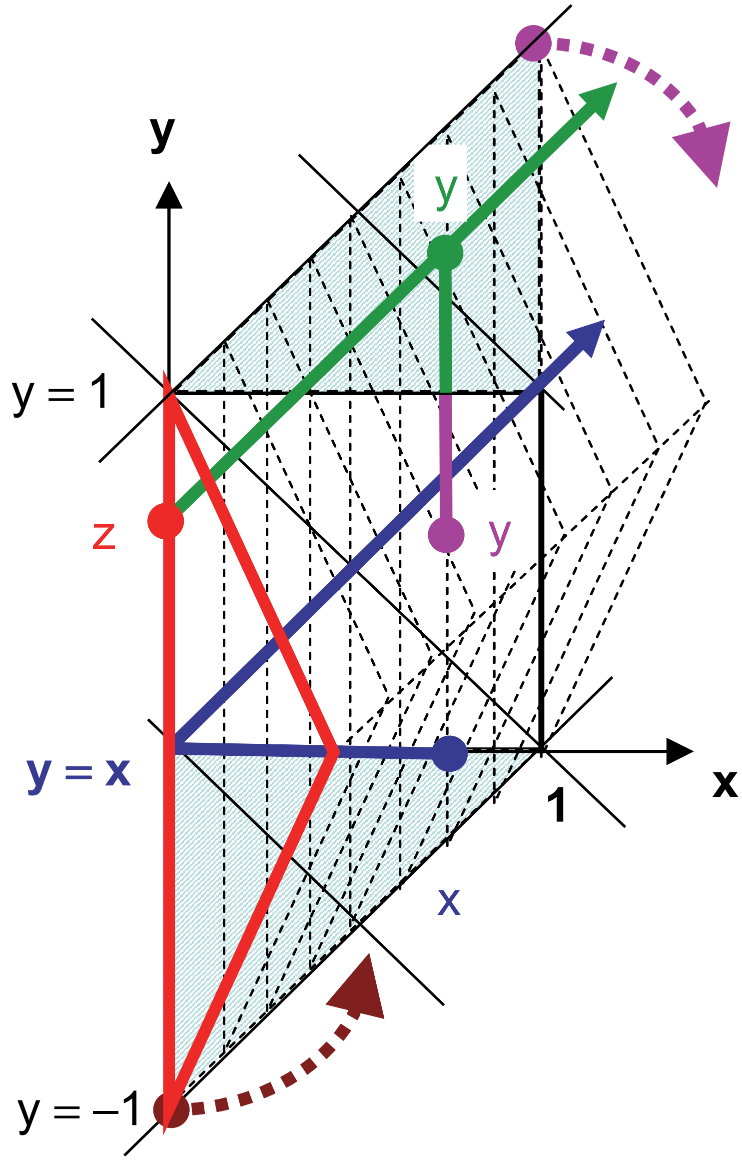

ALGORITHM:

1. Sample x in [0,1]
2. Sample z in [-1,1]
3. $y = z + x$
4. If $y < 0$ then $y = -y$
5. If $y > 1$ Then $y = 1 - (y - 1)$

- Thus, sampling algorithm **mimics construction method** :

Step 1: Sample x from a uniform random variable X on $[0, 1]$.

Step 2: Sample u from a uniform random variable U on $[0, 1]$.

Step 3: If $u \leq \frac{1}{2}$ then $z = P^{-1}(2u) - 1$ else $z = 1 - P^{-1}(2 - 2u)$

Step 4: $y = z + x$

Step 5: If $y < 0$ then $y = -y$

Step 6: If $y > 1$ then $y = 1 - (y - 1)$

- For the generating densities herein we have for **arbitrary quantile level $q \in (0, 1)$:**

$$P^{-1}(q|\psi) =$$
$$\begin{cases} \frac{-(2-\alpha)+\sqrt{(2-\alpha)^2+4(\alpha-1)q}}{2(\alpha-1)}, & p(z|\alpha), \alpha \neq 1, \\ q^{1/n}, & p(z|n), \\ \left[\frac{2(m+1)}{m} - \sqrt{\left\{\frac{2(m+1)}{m}\right\}^2 - q\frac{3m+4}{m}}\right]^{2/(m+2)}, & p(z|m), \\ (1-\theta)q + \theta, & p(z|\theta), \end{cases}$$

- One could favor the power pdf and uniform pdf's due to **least number of operations.**

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- **Positive (negative) dependence** between $X' \sim G(\cdot)$ and $Y' \sim H(\cdot)$ when **large values** of one go with **large (small) values** of the other.

- In case of positive (negative) dependence, X' and Y' are said to be **concordant (disconcordant)**.

- Classical measures for *the degree of positive or negative dependence*:

  Blomquist's (1950) β , Kendall's (1938) τ and Spearman's (1904) ρ_s.

- All three measures attain **values ranging from -1 to 1**.

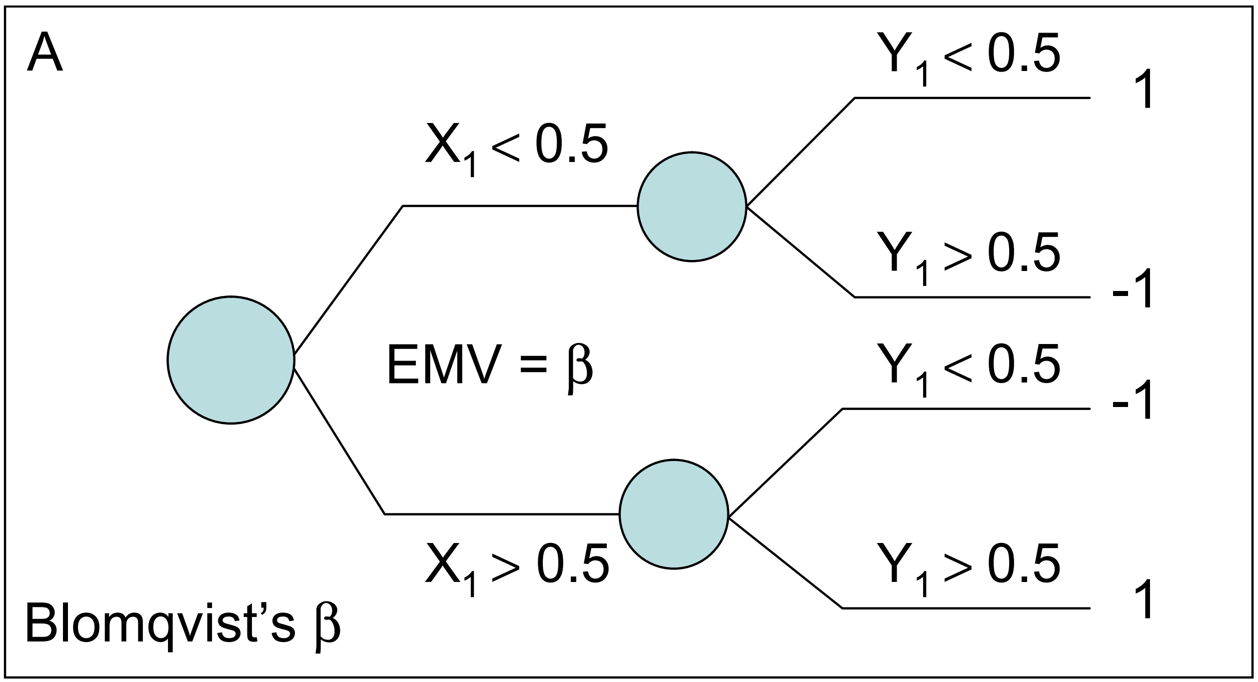
- All three are *ordinally invariant*. Hence,

  $$\rho_s(X', Y') = \rho_s(X, Y), \text{ where } (X, Y) = \{G(X'), H(Y')\}, \text{ etc.}$$

- Recall, X and $Y \sim U[0,1]$ and thus the joint pdf of (X, Y) is a copula.

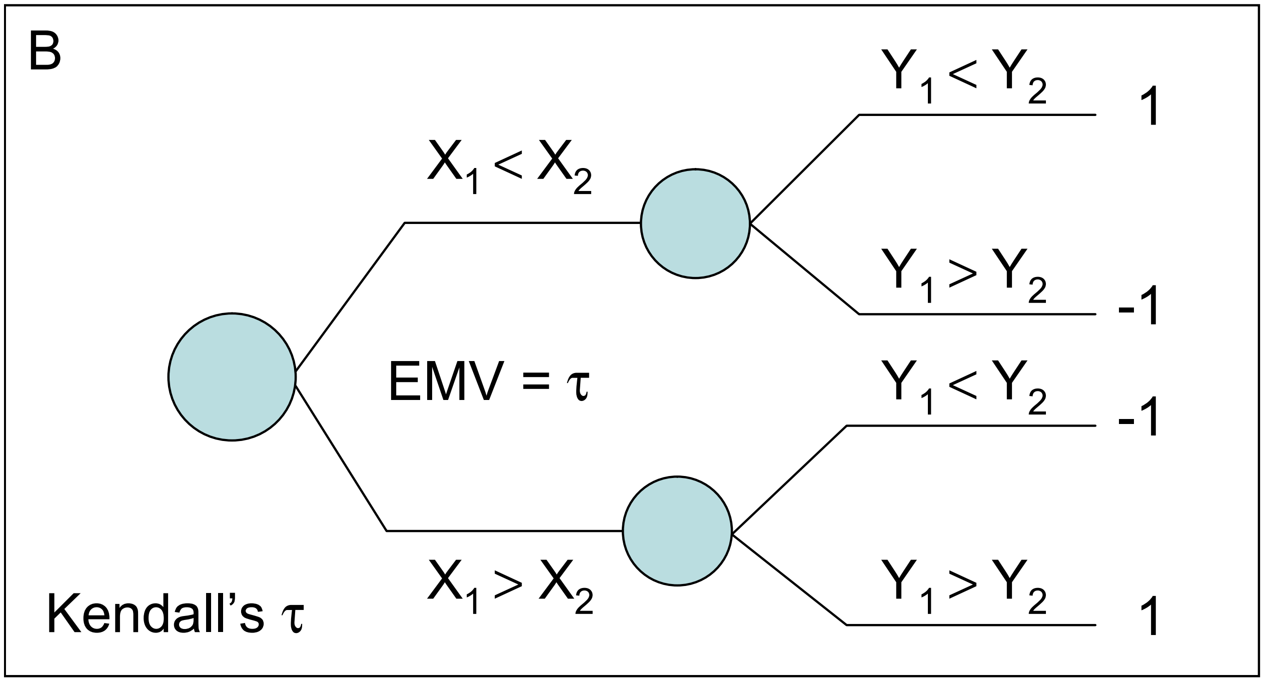
- **Excellent review** of classical measures β, τ and ρ_s is given by Kruskal (1958).

$$\beta(X,Y) = 4C(\frac{1}{2},\frac{1}{2}) - 1, \text{where } C(\cdot\,,\,\cdot) \text{ is copula cdf}$$



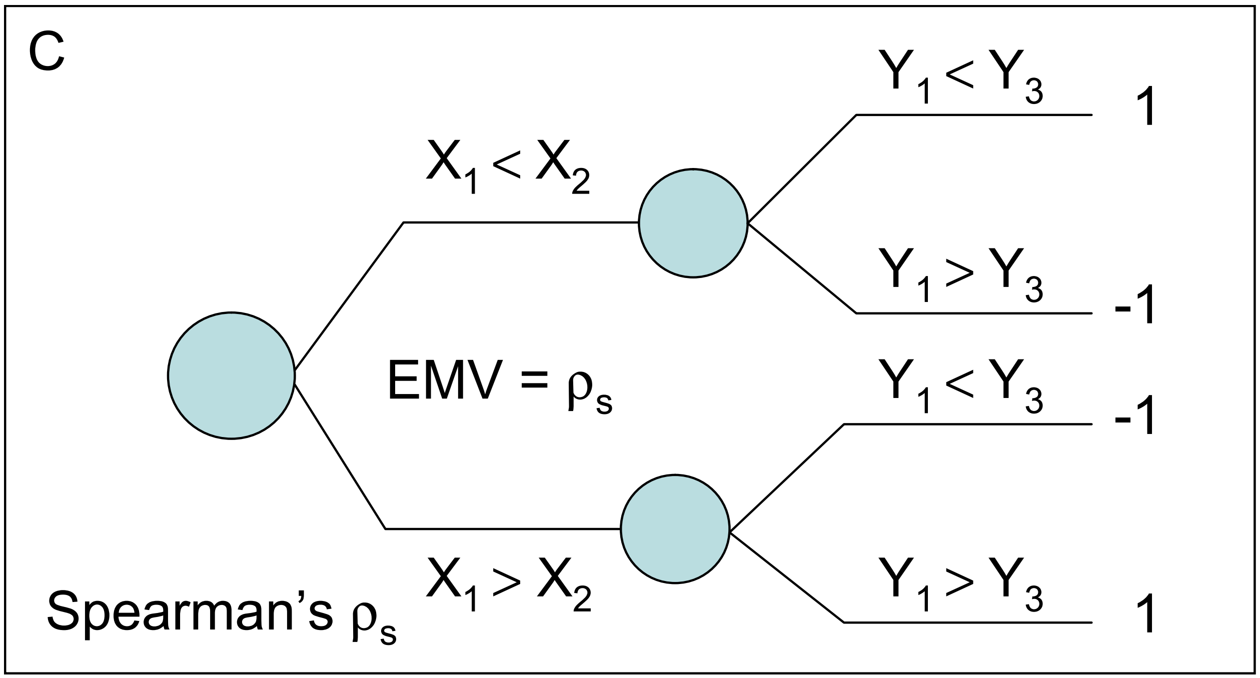
- Let $c(\cdot\,,\,\cdot), C(\cdot\,,\,\cdot)$ be the copula pdf and cdf and let $(X_i, Y_i) \sim C(\cdot\,,\,\cdot)$, $i = 1, 2$ be two independent bivariate samples from the copula.

$$\tau(X,Y) = 4\int_0^1\int_0^1 C(x,y)c(x,y)dxdy - 1.$$



- Let $c(\cdot\,,\,\cdot), C(\cdot\,,\,\cdot)$ be the copula pdf let $(X_i, Y_i) \sim C(\cdot\,,\,\cdot), i = 1, 2, 3$ be three independent bivariate samples from the copula.

$$\rho_s(X,Y) = 12\int_0^1\int_0^1 xyc(x,y)dxdy - 3.$$



- Summarizing, **population expressions for β, τ and ρ_s are**:

$$\begin{cases} \beta(X,Y) = 4C(\frac{1}{2},\frac{1}{2}) - 1, \\ \tau(X,Y) = 4\int_0^1\int_0^1 C(x,y)c(x,y)dxdy - 1, \\ \rho_s(X,Y) = 12\int_0^1\int_0^1 xyc(x,y)dxdy - 3, \end{cases}$$

- We have for GDB copula with **TS pdf with generating pdf $p(\cdot|\Psi)$ and $Z \sim p(\cdot|\Psi)$:**

$$\begin{cases} \beta\{X,Y|p(\cdot|\Psi)\} = 2E[Z|\Psi] - 1, \\ \tau\{X,Y|p(\cdot|\Psi)\} = 2E[Z^2|\Psi] - 2\int_0^1 P^2(s|\Psi)ds + 4\int_0^1 sP^2(s|\Psi)ds - 1, \\ \rho_s\{X,Y|p(\cdot|\Psi)\} = -4E[Z^3|\Psi] + 6E[Z^2|\Psi] - 1. \end{cases}$$

- **Slope pdf :** $p(z|\alpha) = 2 - \alpha + 2(\alpha - 1)z,\ 0 \le \alpha \le 2,$

$$\begin{cases} \beta\{X,Y|p(\cdot|\alpha)\} = -\frac{1}{3} + \frac{1}{3}\alpha, & \in [-\frac{1}{3}, \frac{1}{3}], \\ \tau\{X,Y|p(\cdot|\alpha)\} = -\frac{4}{15} + \frac{4}{15}\alpha, & \in [-\frac{4}{15}, \frac{4}{15}], \\ \rho_s\{X,Y|p(\cdot|\alpha) = -\frac{2}{5} + \frac{2}{5}\alpha, & \in [-\frac{2}{5}, \frac{2}{5}]. \end{cases}$$

- **Power pdf :** $p(z|n) = nz^{n-1},\ n > 0,$

$$\begin{cases} \beta\{X,Y|p(\cdot|n)\} = \frac{n-1}{n+1}, & \in [-1,1], \\ \tau\{X,Y|p(\cdot|n)\} = \frac{n-1}{n+2} + \frac{n-1}{(n+1)(n+2)(2n+1)}, & \in [-1,1], \\ \rho_s\{X,Y|p(\cdot|n)\} = \frac{(n-1)(n+6)}{(n+2)(n+3)}, & \in [-1,1]. \end{cases}$$

- **Ogive pdf:** $p(z|m) = \frac{m+2}{3m+4}\{2(m+1)\sqrt{z^m} - mz^{m+1}\}, m > 0,$

$$\begin{cases} \beta\{X,Y|p(\cdot|m)\} = \frac{m(m+1)(3m+8)}{(m+3)(m+4)(3m+4)}, & \in [0,1], \\ \tau\{X,Y|p(\cdot|m)\} = \frac{m(m+1)(162m^6+2643m^5+18132m^4+66108m^3+140032m+58880)}{(m+3)(m+4)(m+6)(2m+5)(3m+4)^2(3m+8)(3m+10)}, & \in [0,1], \\ \rho_s\{X,Y|p(\cdot|m)\} = \frac{m(m+1)(3m^3+70m^2+424m+736)}{(m+4)(m+5)(m+6)(m+8)(3m+4)}, & \in [0,1]. \end{cases}$$

- **U[θ, 1] pdf :** $p(z|\theta) = \frac{1}{1-\theta}, \theta \leq z \leq 1, 0 \leq \theta \leq 1,$

$$\begin{cases} \beta\{X,Y|p(\cdot|\theta)\} = \theta, & \in [0,1], \\ \tau\{X,Y|p(\cdot|\theta)\} = \theta(\theta+2)/3, & \in [0,1], \\ \rho_s\{X,Y|p(\cdot|\theta)\} = \theta(1+\theta-\theta^2), & \in [0,1]. \end{cases}$$

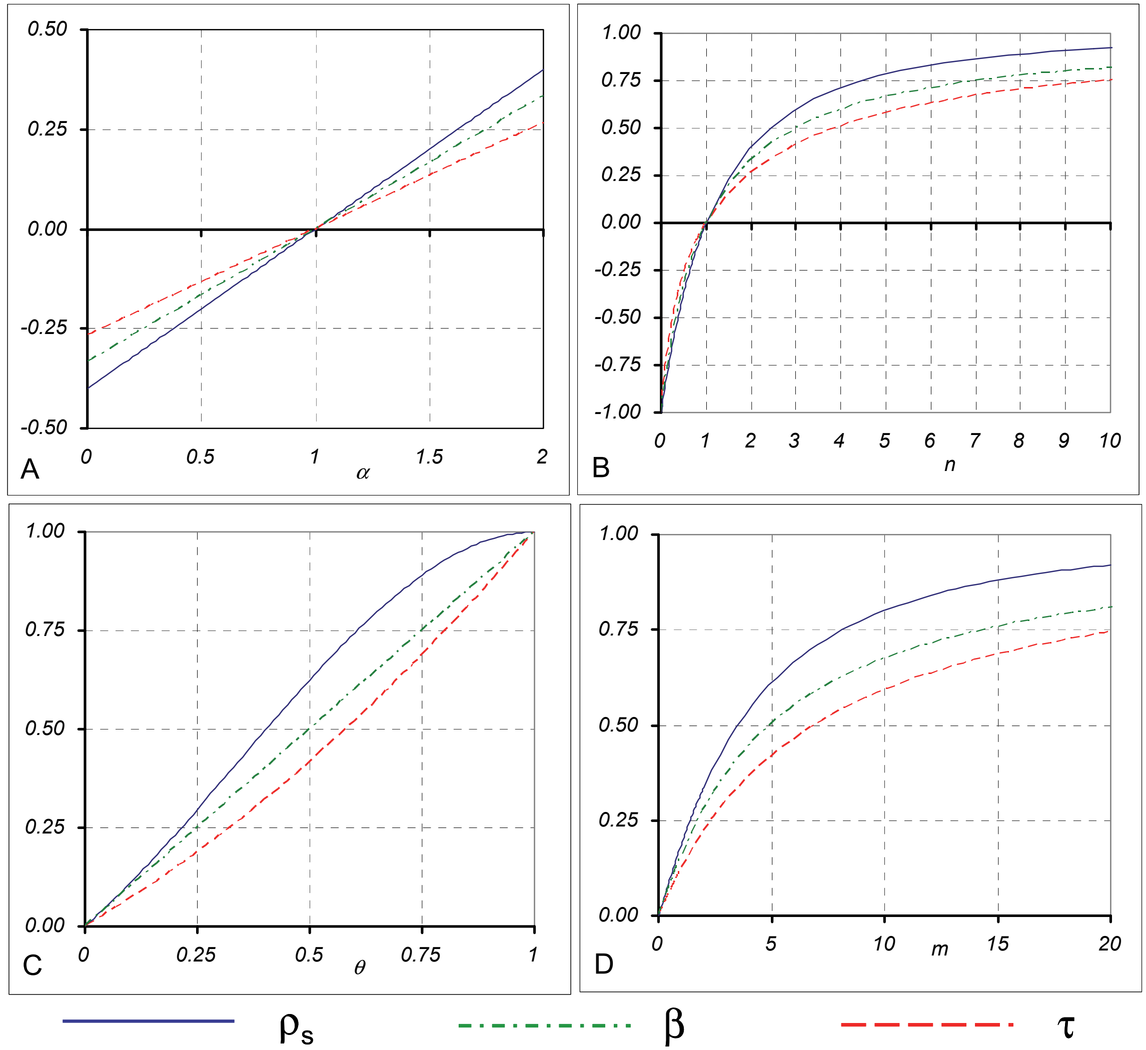


Figure 8. A: Slope(α); B: Power(n); C: Uniform[θ, 1]; D: Ogive(m).

- Let $q(z|\Psi)$ be pdf $Z' = 1 - Z, Z \sim p(z|\Psi) \Rightarrow$ $\boldsymbol{q(z|\Psi) = p(1-z|\Psi)}$.
- $c\{x, y|q(z|\Psi)\} = c\{x, y|p(1-z|\Psi)\}$ obtained via **a right angle rotation.**

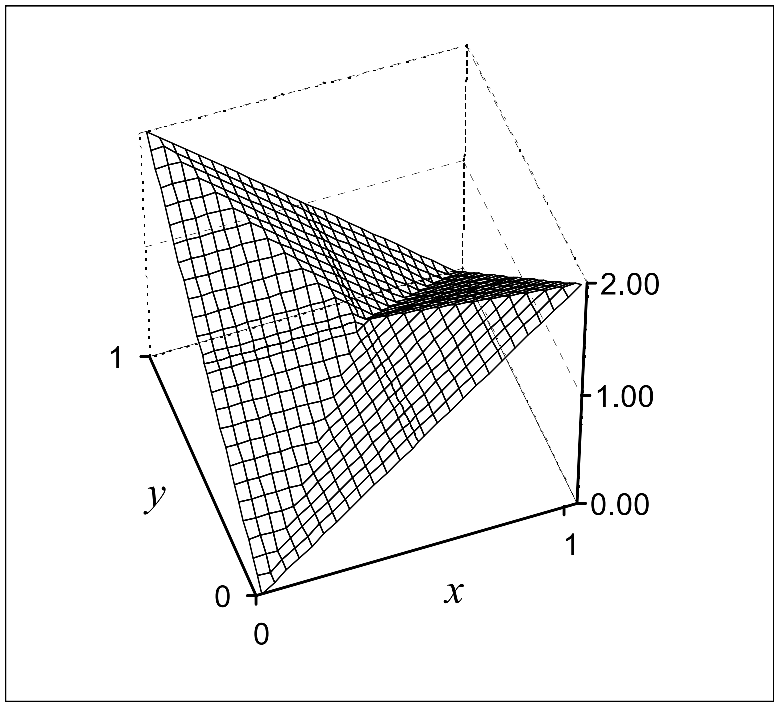


Figure 9. Graph of rotated copula using $\boldsymbol{p(1-z|\Psi) = 2(1-z)}$.

- We have for GDB copula with **TS gen. pdf $p(\cdot|\Psi)$ and $Z \sim p(\cdot|\Psi)$:**

$$\begin{cases} \beta\{X,Y|p(\cdot|\Psi)\} = 2E[Z|\Psi] - 1, \\ \tau\{X,Y|p(\cdot|\Psi)\} = 2E[Z^2] - 2\int_0^1 P^2(s|\Psi)ds + 4\int_0^1 sP^2(s|\Psi)ds - 1, \\ \rho_s\{X,Y|p(\cdot|\Psi)\} = -4E[Z^3|\Psi] + 6E[Z^2|\Psi] - 1. \end{cases}$$

- **Let $q(z|\Psi)$ be pdf $Z' = 1 - Z,\ Z \sim p(z|\Psi)$** $\Rightarrow$

$$\begin{cases} \beta\{X,Y|q(z|\Psi)\} = \beta\{X,Y|p(1-z|\Psi)\} = -\beta\{X,Y|p(z|\Psi)\}, \\ \tau\{X,Y|q(z|\Psi)\} = \tau\{X,Y|p(1-z|\Psi)\} = -\tau\{X,Y|p(z|\Psi)\}, \\ \rho_s\{X,Y|q(z|\Psi)\} = \rho_s\{X,Y|p(1-z|\Psi)\} = -\rho_s\{X,Y|p(z|\Psi)\}. \end{cases}$$

- $p(z|\Psi)$ **symmetric** on $[0,1] \Rightarrow$ **$p(1-z|\Psi) = p(z|\Psi) \Rightarrow \beta,\ \tau,\ \rho_s \equiv 0$**

$$p(z|a) = \frac{\Gamma(2a)}{\Gamma(a)\Gamma(a)} x^{a-1}(1-x)^{a-1}, a > 0 \Rightarrow \beta, \tau, \rho_s \equiv 0, \forall a > 0$$

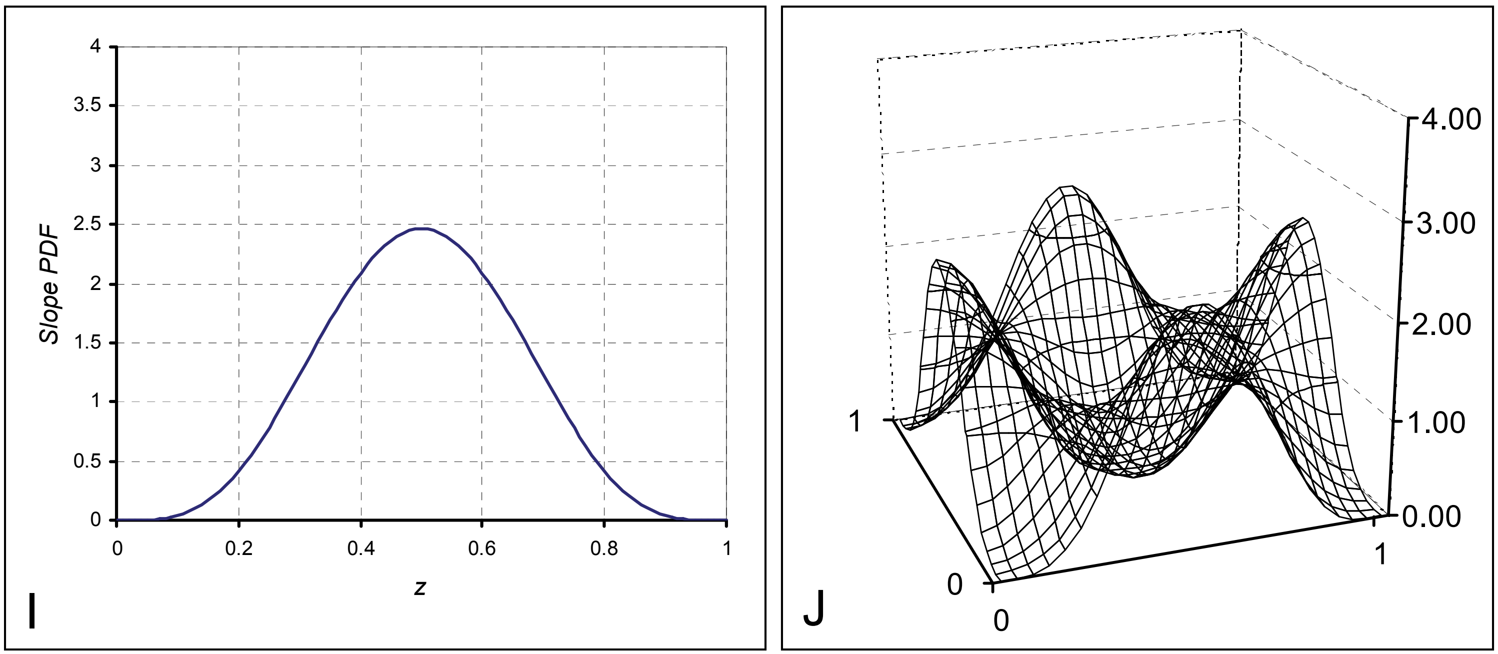


Figure 10. G: Beta generating pdf; H: GDB Copula with TS Gen. PDF in A.

- **Lower and upper tail dependence measures are in vogue**, particularly in problem contexts dealing with modeling **the joint occurrence of extreme events**, such as insurance and modeling of default risk in finance.

- Recent **burst of attention** to the copula approach may be credited to **the Gaussian copula** which has been widely adopted by the "financial quants" .

- Embrechts (2008) even refers to this attention as **"the copula craze"**.

- Unfortunately, some (see, e.g., Salmon, 2009) **blamed** Gaussian copula for the **2008 financial crash**, in part due to **lack of** lower and upper tail dependence.

- These measures too are **ordinal measures of association**, although they focus primarily on modeling positive dependence and not negative dependence.

- $X' \sim G(\cdot), Y' \sim H(\cdot)$, **Lower tail dependence λ_L**:

$$\lambda_L = \lim_{x \downarrow 0} Pr\{Y' \leq H^{-1}(x) | X' \leq G^{-1}(x)\}$$
$$= \lim_{x \downarrow 0} Pr(Y \leq x | X \leq x) = \lim_{x \downarrow 0} \frac{C(x,x)}{x},$$

- $X' \sim G(\cdot), Y' \sim H(\cdot)$, **Upper tail dependence λ_U**:

$$\lambda_U = \lim_{x \uparrow 1} Pr\{Y' > H^{-1}(x) | X' > G^{-1}(x)\}$$
$$= \lim_{x \uparrow 1} Pr(Y > x | X > x) = \lim_{x \uparrow 1} \frac{1 - 2x + C(x,x)}{1 - x}.$$

- Clayton, Frank and Gumbel copulae exhibit lower or upper tail dependence. Clayton, Frank and Gumbel copulae belong to the **Archimedean class**.

- **For GDB copula cdf $C\{x,y|p(\cdot|\Psi)\}$ we have $\lambda_L = \lambda_U = 0$,** similar to the Gaussian copulae (Embrechts et. al, 2002).

- Blomquist's β, Kendall's τ and Spearman's ρ_S are more applicable in contexts dealing with **the modeling of joint events in general, not extremes per se.**

- Blomquist's β, Kendall's τ and Spearman's ρ_S pertain to full copula support and not just to their asymptotic extreme values.

- **Caution** to those who believe that the Clayton, Frank and Gumbel copulae could serve as the panacea instead of Gaussian Copula.

- Heteroscedastic behavior of financial processes **suggests** their dependence cannot be modelled using a copula with **a constant correlation over time**, **regardless** of the copula displaying tail dependence or not.

OUTLINE

- $X' \sim G(\cdot),\ Y' \sim H(\cdot), \{G(X'), H(Y')\} = (X, Y) \sim C\{x, y | p(\cdot | \Psi)\}$

- Elicit: $\boldsymbol{Pr(Y' \leq y'_{0.5} | X' \leq x'_{0.5}) = Pr(Y \leq 0.5 | X \leq 0.5) = \pi.}$

- This *elicitation procedure* falls within **the conditional fractile estimation method for eliciting degree of dependence** - Clemen and Reilly (1999).

- We have for Blomquist's β

$$\beta\{X, Y | p(\cdot | \Psi)\} = 2\pi\{X, Y | p(\cdot | \Psi)\} - 1$$

- Hence, elicitation of $\pi\{X, Y | p(\cdot | \Psi)\}$ is equivalent to **an indirect elicitation of Blomquist's $\boldsymbol{\beta}$** which has a more straightforward interpretation as τ and ρ_s.

$$C\{x,y|p(\cdot|\Psi)\} = \begin{cases} x - \frac{1}{2}\int_{1-x-y}^{1+x-y} P(z|\Psi)dz, & (x,y) \in A_1, \\ y - \frac{1}{2}\int_{1-x-y}^{1-x+y} P(z|\Psi)\}dz, & (x,y) \in A_2, \\ x - \frac{1}{2}\int_{x+y-1}^{1+x-y} P(z|\Psi)dz, & (x,y) \in A_3, \\ y - \frac{1}{2}\int_{x+y-1}^{1-x+y} P(z|\Psi)dz, & (x,y) \in A_4. \end{cases} \Rightarrow$$

$$\frac{1}{2}\pi = Pr(Y \leq 0.5, X \leq 0.5) = C\{\frac{1}{2},\frac{1}{2}|p(\cdot|\Psi)\} = \frac{1}{2}\int_0^1 \{1 - P(z|\Psi)\}dz$$

- With $Z \sim P(z|\Psi)$, $E[Z|\Psi] = \int_0^1 \{1 - P(z|\Psi)\}dz$, thus we have:

$$\pi\{X,Y|p(\cdot|\Psi)\} = E[Z|\Psi]. \qquad (12)$$

- Thus, having elicited π one solves for ψ using **the method of moments**.

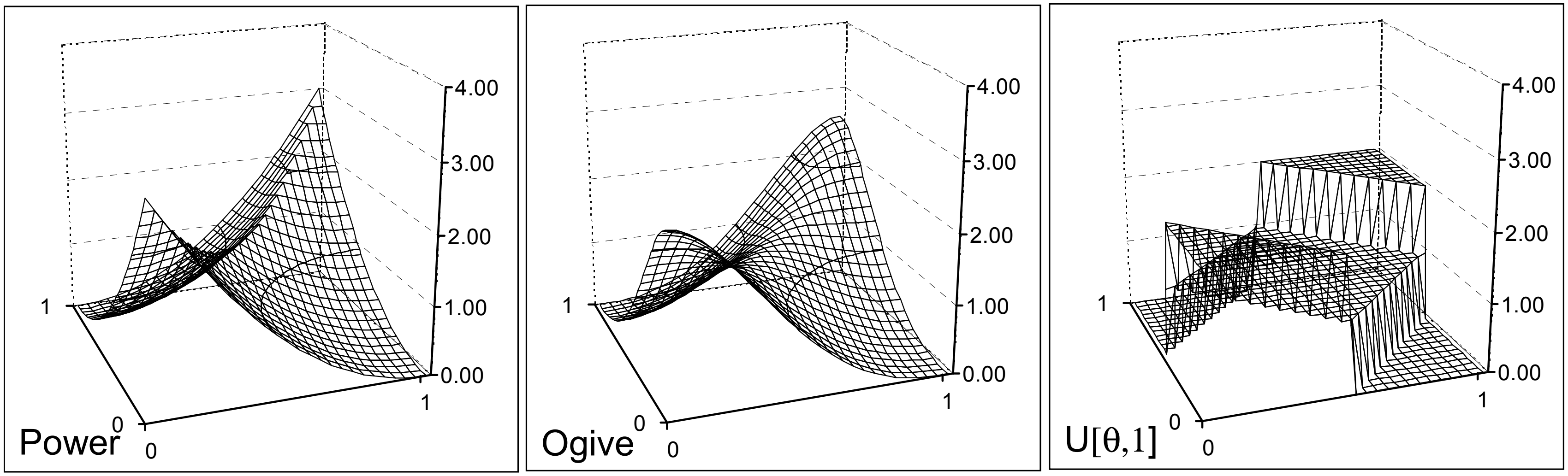
- We have for **the different pdf's herein**

$$\pi\{X,Y|p(\cdot|\Psi)\} = \begin{cases} (2+\alpha)/6 \in [\frac{1}{3},\frac{2}{3}], & p(z|\alpha), \quad \text{Slope pdf,} \\ n/(n+1) \in [0,1], & p(z|n), \quad \text{Power pdf,} \\ \frac{(m+2)^2}{3m+4}\left[\frac{3m+6}{(m+4)(m+3)}\right] \in [0.5,1], & p(z|m), \quad \text{Ogive pdf,} \\ (\theta+1)/2 \in [0.5,1], & p(z|\theta), \quad \text{U}[\theta,1]\text{ pdf.} \end{cases}$$

- **Assume that an expert has assessed a value**

$$\pi\{X,Y|p(\cdot|\Psi)\} = Pr(Y \leq 0.5|X \leq 0.5) = 0.75.$$

- **How does one select a GDB copula with a TS gen. pdf that matches?**



- **All three match the constraint**

$$\pi\{X,Y|p(\cdot|\Psi)\} = Pr(Y \leq 0.5|X \leq 0.5) = 0.75$$

- How do we select one? If **smoothness is required $\Rightarrow$ Ogive pdf.**

- What if smoothness is not required? **Pick one that is uniform as possible?**

- In figures above $n = 3, m = 4.916, \theta = 1/2$ and subsitution in ρ_s formulas:

$$\rho_s(n) = \frac{3}{5},\ \rho_s(m) = 0.6059 \text{ and } \rho_s(\theta) = \frac{5}{8}. \qquad (38)$$

- **Can we select the one with the smallest rank correlation in general?**

- Pdf $p(\cdot|\Psi)$ symmetric on $[0,1] \Rightarrow E[Z|\psi] = \pi\{X, Y|p(\cdot|\Psi)\} = \frac{1}{2}$.

- Pdf $p(\cdot|\Psi)$ symmetric on $[0,1] \Rightarrow \rho_s\{X, Y|p(\cdot|\Psi)\} \equiv 0$.

- When elicited $\pi\{X, Y|p(\cdot|\Psi)\} = Pr(Y \leq 0.5|X \leq 0.5) = \frac{1}{2}$ it seems to intuitive to select copula with independent uniform marginals.

- **Hence, answer is: No.**

- Select GDB copula that **minimizes distance** between it and copula with independent uniform marginals.

- **Kullback-Liebler distance** measures **the relative information** of one candidate pdf $f(x,y)$ with respect to pdf $g(x,y)$ given by

$$I(f|g) = \int\int f(x,y)\ln\{f(x,y)/g(x,y)\}dxdy.$$

- Setting $f(x,y) = c\{x,y\}$ and $g(x,y) = u(x,y) = 1,$ yields:

$$I(c|u) = \int\int_{S_c} c(x,y)\ln\{c(x,y)\}dxdy, \qquad (40)$$

- The quantity $E = -I(c|u) \geq$ is known as **the entropy** of the pdf $c(x,y)$.

- Bedford and Meeuwissen (1997) constructed **maximum entropy copulae given a correlation constraint** that are *minimally informative*.

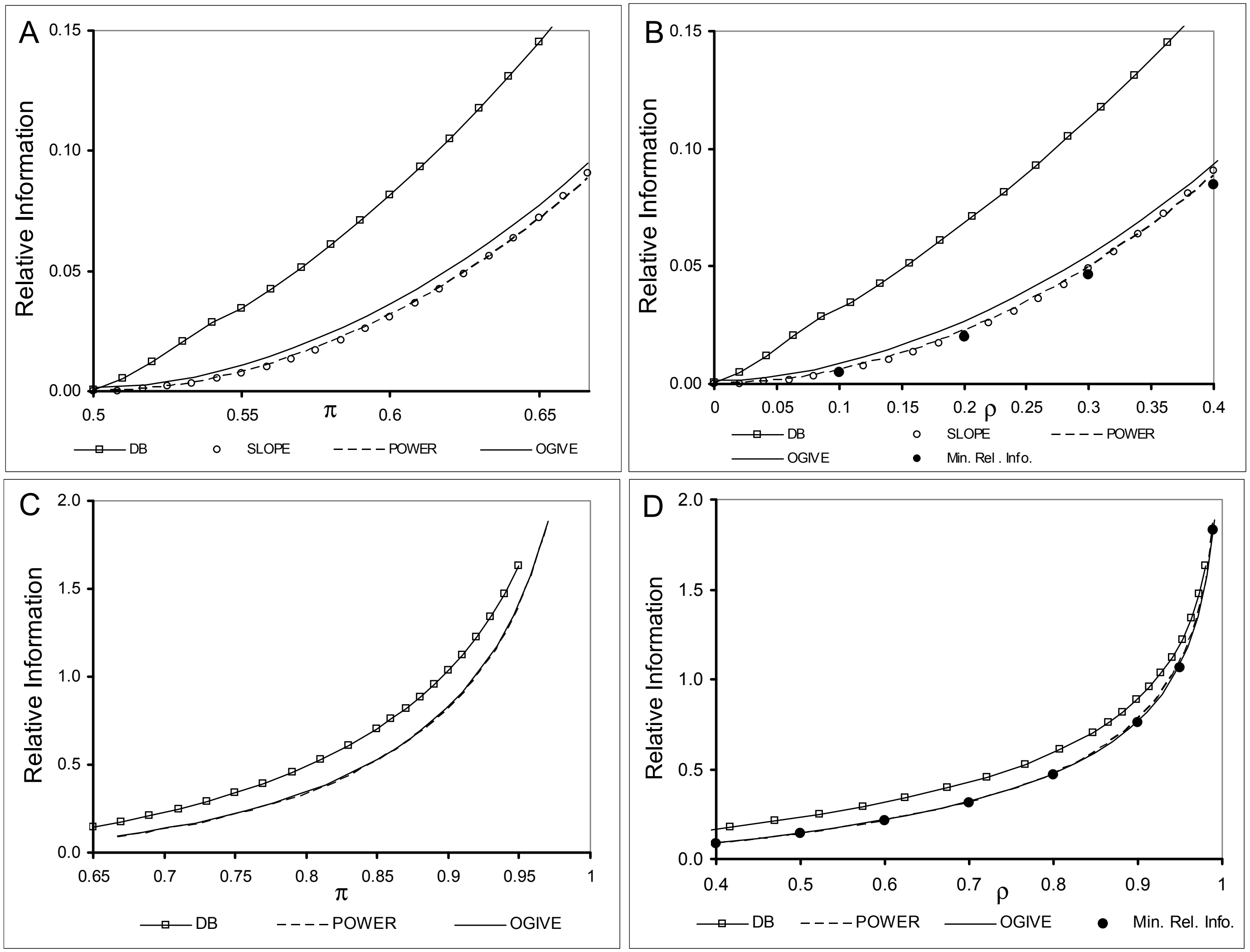
- Bedford and Meeuwissen's (1997) **maximum entropy copulae given a correlation constraint**, do not possess closed form pdf and cdf.

- Select amongst a set of GDB copula that matches a specified constraint that one that is **minimally informative** (or has maximum entropy).

- Utilizing **numerical integration** over a 100 by 100 grid over $[0, 1]^2$, we have

$$I\{c(x,y)|p(\,\cdot\,|\psi)\} = \begin{cases} 0.2136, & p(z|n), n = 3, \quad \text{Power pdf,} \\ 0.2222, & p(z|m), m = 4.916, \quad \text{Ogive pdf} \\ 0.3400, & p(z|\theta), \theta = 0.5. \quad \text{U}[\theta, 1] \text{ pdf} \end{cases}$$

- Summarizing, given *the constraint* set by $\pi = Pr(Y \leq 0.5|X \leq 0.5) = 0.75$, the relative information approach above would suggest to **use the GDB copula with the power(n) generating pdf (26) with $n = 3$**.



$$\textbf{A: } 0.5 \le \pi \le \tfrac{2}{3};\ \textbf{B: } 0 \le \rho \le 0.4;\ C\textbf{: } \tfrac{2}{3} \le \pi \le 0.95;\ \textbf{D: } 0.4 \le \rho \le 0.99$$

OUTLINE

1. INTRODUCTION
2. COPULA CONSTRUCTION
3. GENERALIZED DIAGONAL BAND EXAMPLES
4. SAMPLING PROCEDURE
5. ORDINAL MEASURES OF ASSOCIATION
6. COPULA PARAMETER ELICITATION
7. **A VALUE OF INFORMATION EXAMPLE**
8. SELECTED REFERENCES

- The farmer needs to **protecting his/her crop of oranges** (with a total worth of \$50,000) against freezing weather with the objective of **minimizing losses.**

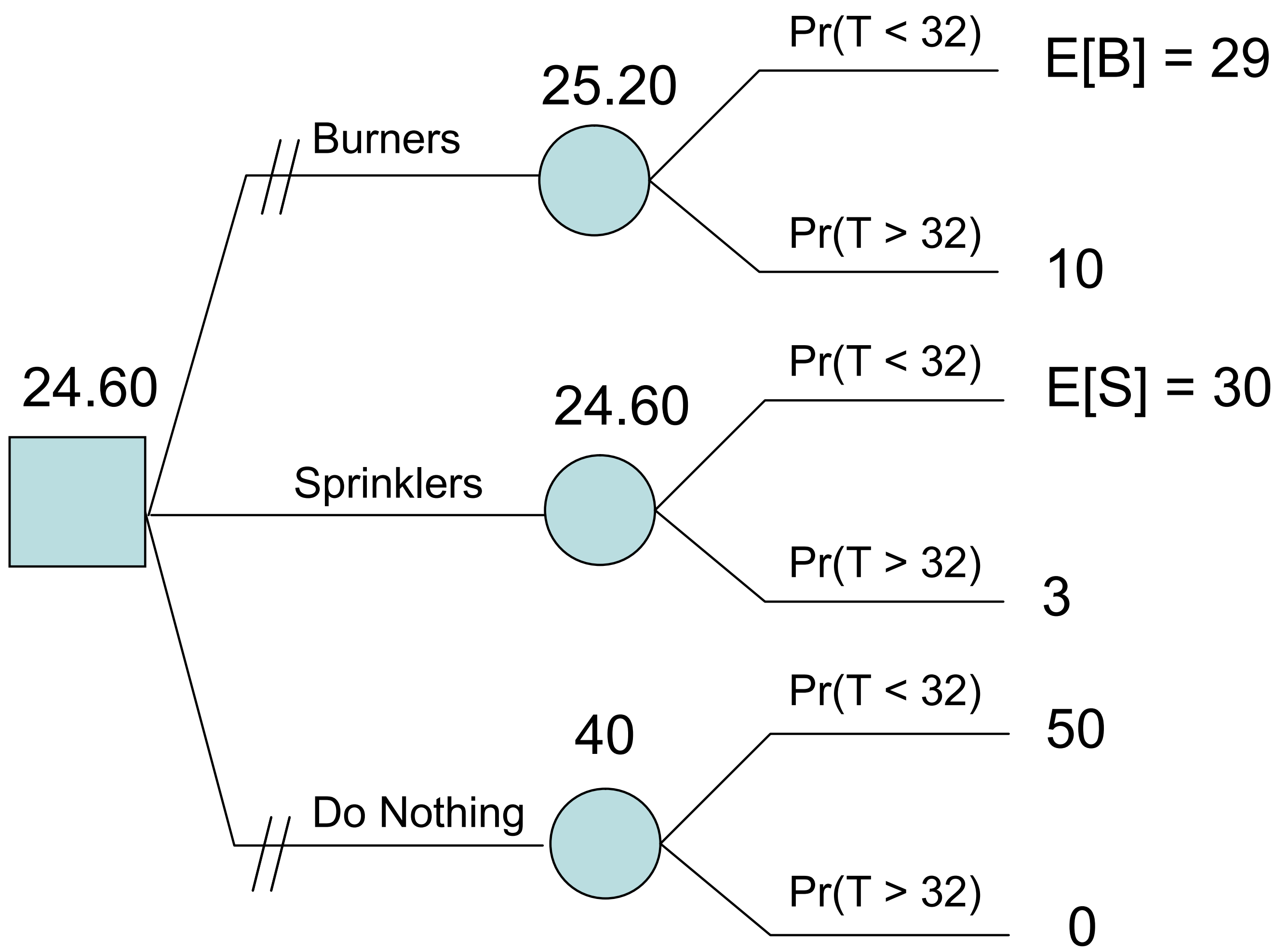
- **Temperature T** (in Fahrenheit) $\sim U[24, 34]$ that night.

- $T < 32$ (below freezing) $\Rightarrow$ Farmer looses entire crop **without protection.**

- Two protection alternatives: **Burners** or **Sprinklers** with $\$10,000$ or $\$3,000$, respectively, in mobilization cost.

- Effectiveness of both is uncertain. Farmer assesses **all-in loss** $B\ (S)$ to vary between $a =$ \$25000 (\$28,000) and $b =$ \$35,000 (\$33,000) with a most likely value of $m =$ \$27,000 (\$29,000) **if it freezes**.

- Assume B and S to be **triangular distributed** $\Rightarrow E[B] =$ \$29,000, $E[S] =$ \$30,000.


24.60
Burners
25.20
Pr(T < 32)
E[B] = 29
Pr(T > 32)
10
Sprinklers
24.60
Pr(T < 32)
E[S] = 30
Pr(T > 32)
3
Do Nothing
40
Pr(T < 32)
50
Pr(T > 32)
0

- **Effectiveness sprinkler** option is based on an **insular layer of freezing water** on the oranges.

- **Effectiveness burner** option is based on **gas usage.**

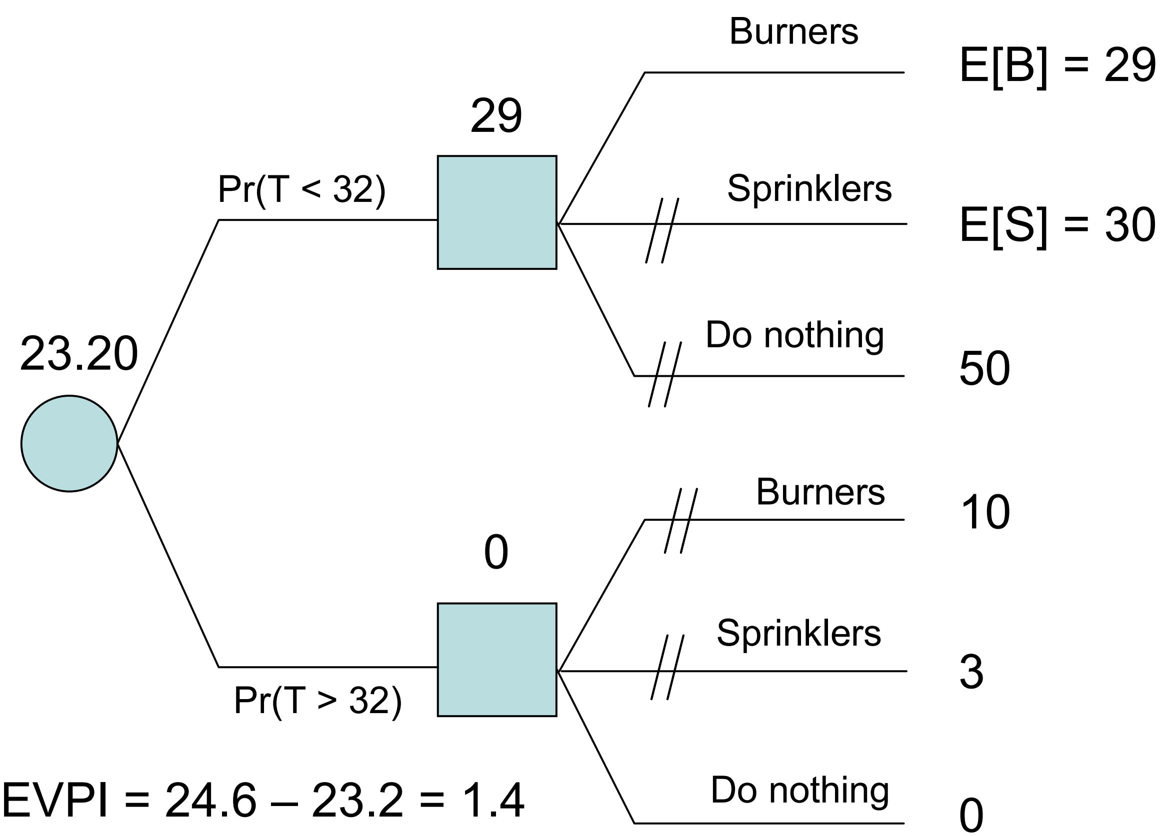
- The farmer assesses a 90% chance (60% chance) that the burning loss B (sprinkler loss S) is **above it median value** $b_{0.5}$ ($s_{0.5}$) when the temperature T is **below its median value** $29F$. Hence, we have:

$$Pr(B < b_{0.5} | T < 29) = 0.1, Pr(S < s_{0.5} | T < 29) = 0.4,$$

  where $b_{0.5} \approx \$28{,}675$ and $s_{0.5} \approx \$29{,}838$.

- **Model dependence** between B (S) and T using a GDB copula with a power (slope) generating density with $n = 1/11$ ($\alpha = 0.4$).

- To reduce losses further, the farmer considers consulting either *a clairvoyant Expert A* on **"Freezing"** or *a clairvoyant Expert B* on **the temperature *T*.**



EVPI = 24.6 – 23.2 = 1.4

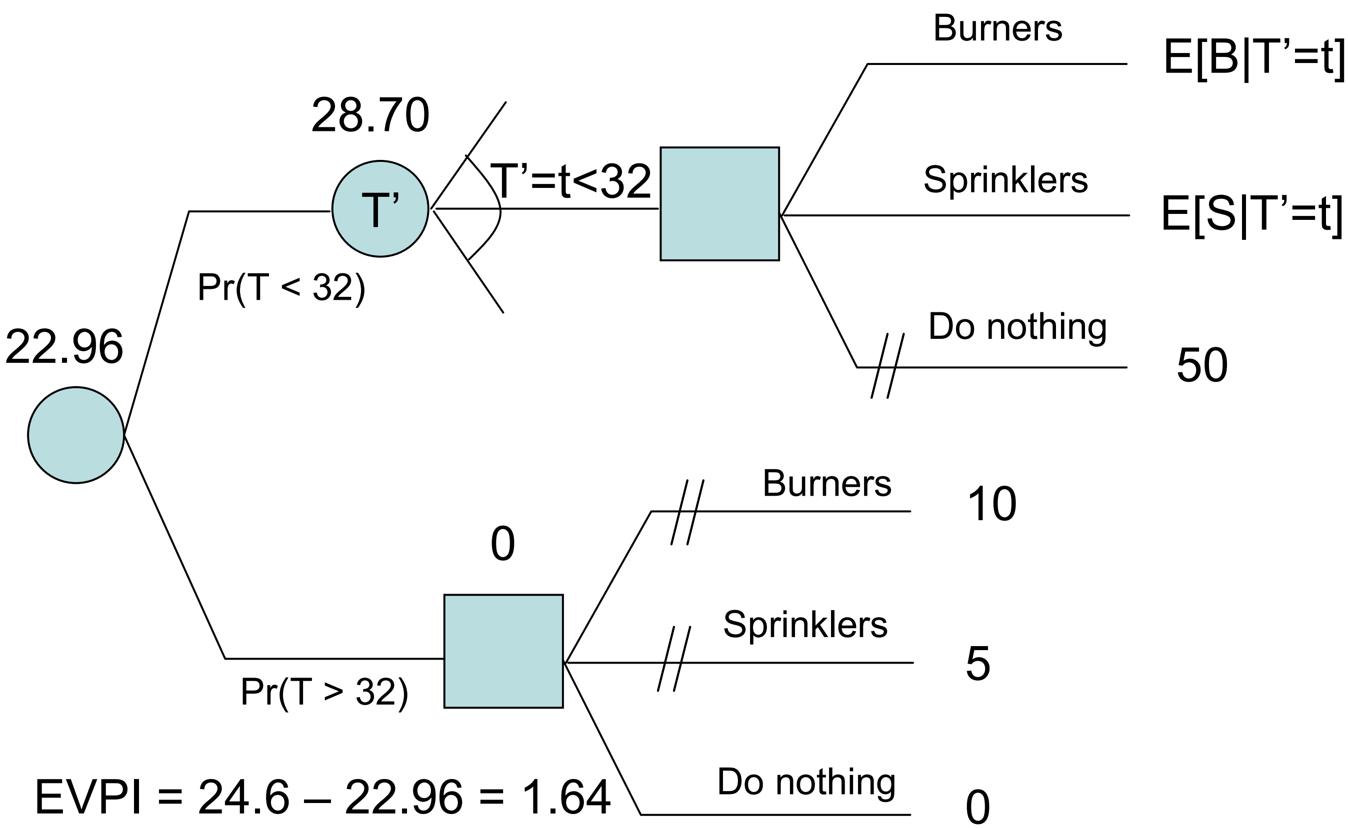
- EVPI on the temperature T from Expert B is **more complicated** since it requires **evaluation of $E[B|t]$ and $E[S|t]$**.

- Given t, we evaluate $E[B|t]$ using $s = 2500$ realizations using the steps:

  Step 1: $x = \frac{t-24}{34-24}$ (Recall, $T \sim Uniform[24, 34]$)
  Step 2: **Sample quantile levels y_i**, $i = 1, \ldots, s$ from GDB(X, Y) copula with power(n) generating density for B, $n = 1/11$.
  Step 3: $E[B|t] = \frac{1}{s}\sum_{i=1}^{s} H^{-1}(y_i)$,

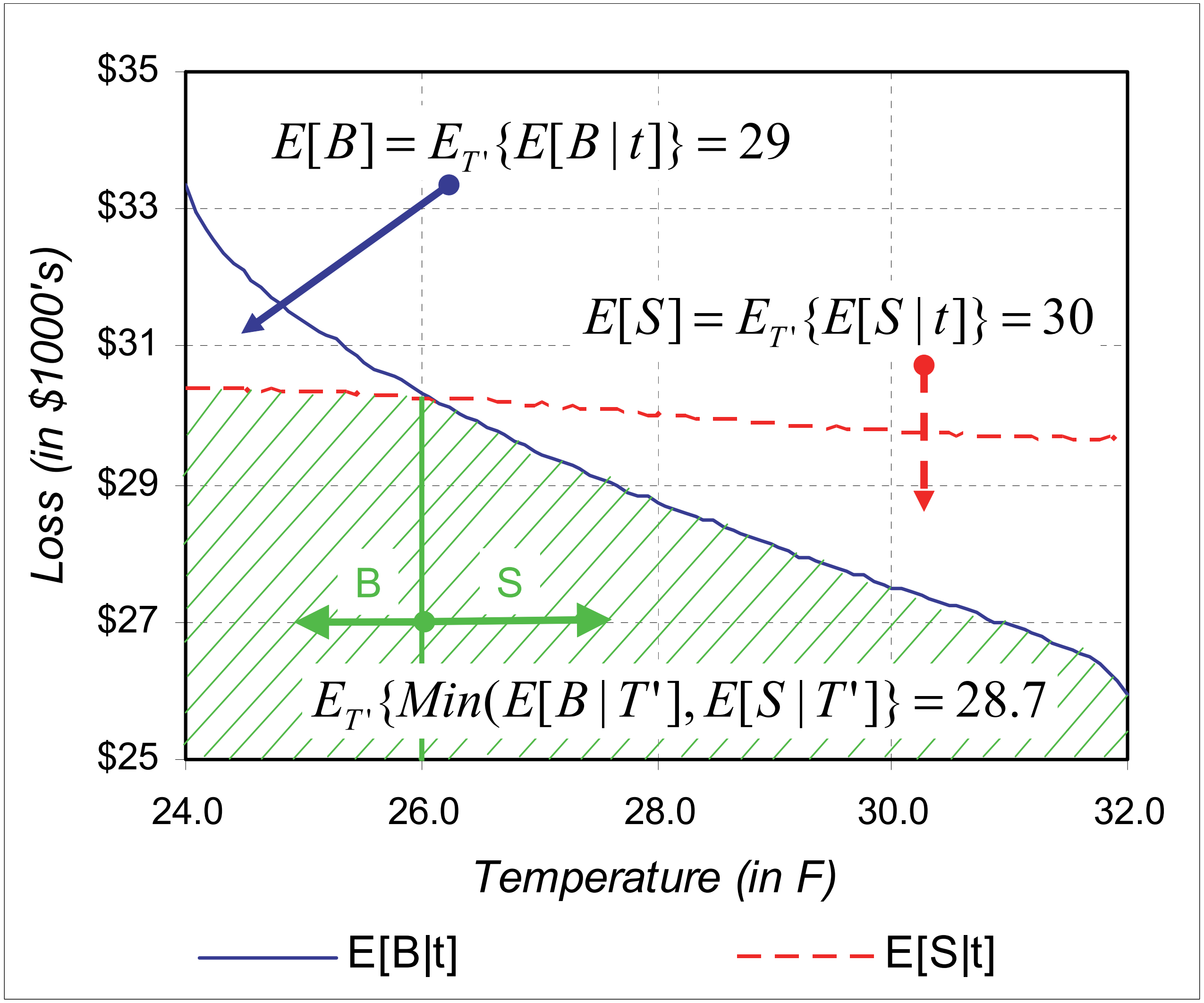
- $B \sim$ Triang($25,000; $27,000; $35,000), $H^{-1}(\cdot)$ is the inverse cdf or quantile function of B. $S \sim Triang$($28,000; $29,000; $33,000). Evaluation of $E[S|t]$ is analogous.

- $T' = (T|T < 32) \sim \mathrm{U}[24, 32]$ since $T \sim \mathrm{U}[24, 34]$.



EVPI = 24.6 – 22.96 = 1.64

EVPI Temperature T



- EVPI "freezing" $\approx$ \$1,400, EVPI "freezing" $\approx$ \$1,640
- Summarizing, the farmer is willing to pay \$240 dollars more for perfect information on the temperature T.
- Optimal decision switches to Sprinkler option when *Expert A* provides "Freezing" information.
- Optimal decision switches to Burner option when *Expert B* provides "temperature t" information, where $26 < t < 32$.
- When *Expert B* provides "temperature t" information, where $24 < t < 26$, the optimal decision remains the Sprinkler option.

QUESTIONS?

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