Two-Sided Generalized Topp and Leone (TS-GTL) distributions

"Presentation Short Course: Beyond Beta and Applications"

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1. **Introduction**
2. PDF and CDF of TS-GTL distributions
3. Moments
4. Maximum Likelihood Estimation
5. Example using $(V - I)$ indices of 80 globular clusters in the Galaxy M87
6. Concluding Remarks
7. Some References

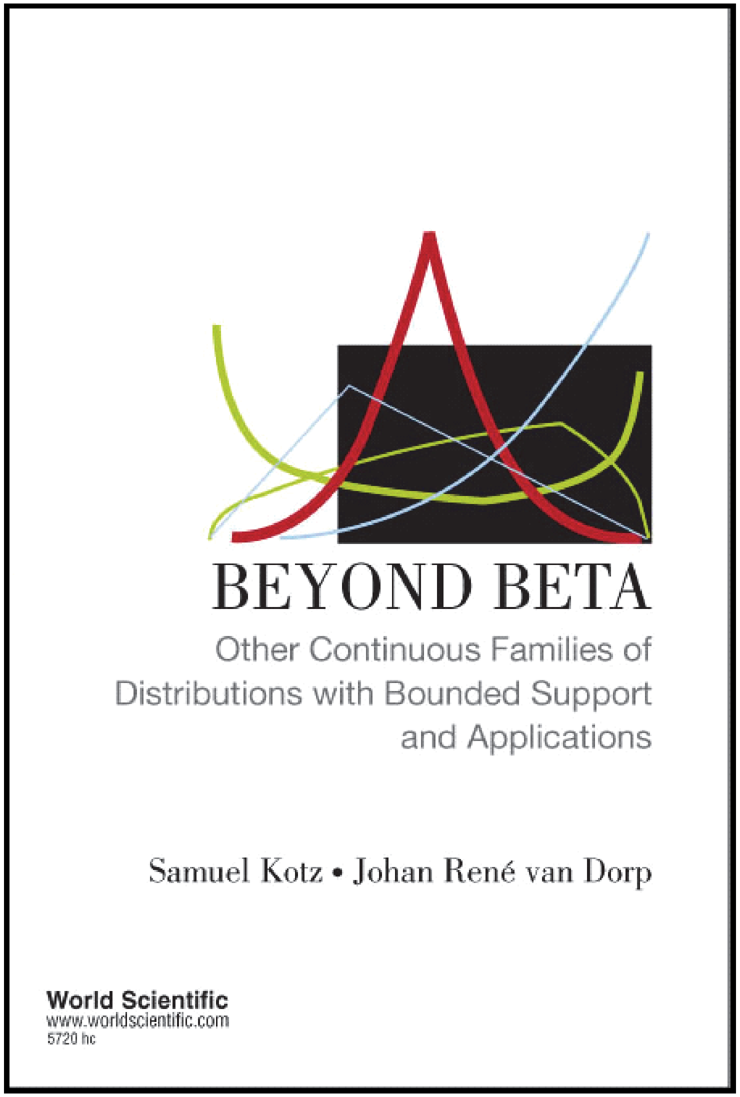
- As early as 1919, E. Pairman and K. Pearson investigated continuous distributions **on a limited range** in particular estimation of their moments.

- Nevertheless, even in the late nineties of the 20-th century **relatively few probabilistic models** of this kind were available in the literature.

- Amongst them, **the beta, uniform, triangular and Johnson S_B** distributions are the most widely explored and applied.

- **The multitude of existing unbounded continuous distributions** developed during the 20-th century **contrasts** with the scarcity of the bounded distributions.

- This motivated **Kotz and Van Dorp (2004a)** to study other constructs for **families of distributions with bounded support.**
- This resulted in publication of their 2004 monograph devoted to this topic.
- This talk and paper are **a continuation of the above** investigations.



- **Starting point:** Three-parameter **Generalized Two-Sided Power (GTSP) family of distributions** (Kotz and Van Dorp, 2004a) with the cdf

$$Pr(X \leq x) = \begin{cases} 0, & \text{for } x \leq 0 \\ p(\underset{\sim}{\Theta})\left(\frac{x}{\theta}\right)^m, & \text{for } 0 < x < \theta \\ 1 - \{1 - p(\underset{\sim}{\Theta})\}\left(1 - \frac{x-\theta}{1-\theta}\right)^n, & \text{for } \theta \leq x < 1 \\ 1, & \text{for } x \geq 1, \end{cases}$$

where the vector $\underset{\sim}{\Theta} = (\theta, m, n)$,

$$\boldsymbol{p(\underset{\sim}{\Theta}) = \frac{\theta n}{(1-\theta)m + \theta n}},$$

and $0 \leq \theta \leq 1$, $m > 0, n > 0$. The parameter θ is a ***threshold parameter*** and the parameters m and n are the evident ***power parameters***.

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- The **function x/θ** in **the first branch** is **the cumulative distribution function (cdf)** of a **uniform random variable** on $(0,\theta)$.
- The **function $1-(x-\theta)/(1-\theta)$** in **the second branch** is **the reliability function** of a **uniform random variable** on $(\theta,1)$.
- This leads to the following generalization using continuous cdf's $G(\cdot)$ and $H(\cdot)$ with the support $[0,1]$:

$$Pr(Y \le y) = \begin{cases} 0, & \text{for } y \le 0 \\ p(\underset{\sim}{\Theta})\left\{G\left(\frac{y}{\theta}\right)\right\}^m, & \text{for } 0 < y < \theta \\ 1-\{1-p(\underset{\sim}{\Theta})\}\left\{1-H\left(\frac{y-\theta}{1-\theta}\right)\right\}^n, & \text{for } \theta \le y < 1 \\ 1, & \text{for } y \ge 1. \end{cases}$$

- **The density function** corresponding to the cdf above is :

$$f_Y(y|\underset{\sim}{\Theta}) = \frac{mn}{(1-\theta)m+\theta n} \times \begin{cases} g\left(\frac{y}{\theta}\right)\left\{G\left(\frac{y}{\theta}\right)\right\}^{m-1}, & \text{for } 0 < y < \theta \\ h\left(\frac{y-\theta}{1-\theta}\right)\left\{1-H\left(\frac{y-\theta}{1-\theta}\right)\right\}^{n-1}, & \text{for } \theta \leq y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

- The density above has the following **alternative *mixture* representation**:

$$f_Y(y|\underset{\sim}{\Theta}) = p(\underset{\sim}{\Theta})\left[\frac{m}{\theta}g\left(\frac{y}{\theta}\right)\left\{G\left(\frac{y}{\theta}\right)\right\}^{m-1}\right] + \{1-p(\underset{\sim}{\Theta})\}\left[\frac{n}{1-\theta}h\left(\frac{y-\theta}{1-\theta}\right)\left\{1-H\left(\frac{y-\theta}{1-\theta}\right)\right\}^{n-1}\right],$$

where **the mixture probability $p(\underset{\sim}{\Theta})$** was defined two slides earlier.

- **For m [n] an integer,** the first member is easily recognized as **the *largest* [*smallest*] order statistic distribution** of a sample of size m [n] from the rescaled distribution $G(\cdot)$ with the support $[0, \theta]$ [support $[\theta, 1]$].

- For $n, m > 0$ and non-integer, **n and m may be interpreted** as ***virtual sample sizes***.

- **Jones (2004)** recently investigated generalizations of the distribution of order statistics of the form

$$\{B(a,b)\}^{-1} f(x) F^{a-1}(x)\{1-F(x)\}^{b-1},$$

where $B(a,b) = \Gamma(a+b)/\Gamma(a)\Gamma(b)$, $F(\cdot)$ is a particular cdf and $a, b > 0$ are not necessarily integers.

- **Our generalization of GTSP densities** may thus be viewed **along the lines** of those in Jones (2004).

- Letting $G(\cdot)$ [$H(\cdot)$] to be **a slope** [**reflected slope**] distribution on $[0, 1]$ given by:

$$\begin{cases} G(x|\alpha) = \alpha x - (\alpha - 1)x^2, \\ H(x|\beta) = 1 - \beta(1 - x) + (\beta - 1)(1 - x)^2, \end{cases}$$

where $0 \le \alpha, \beta \le 2$. We obtain the density:

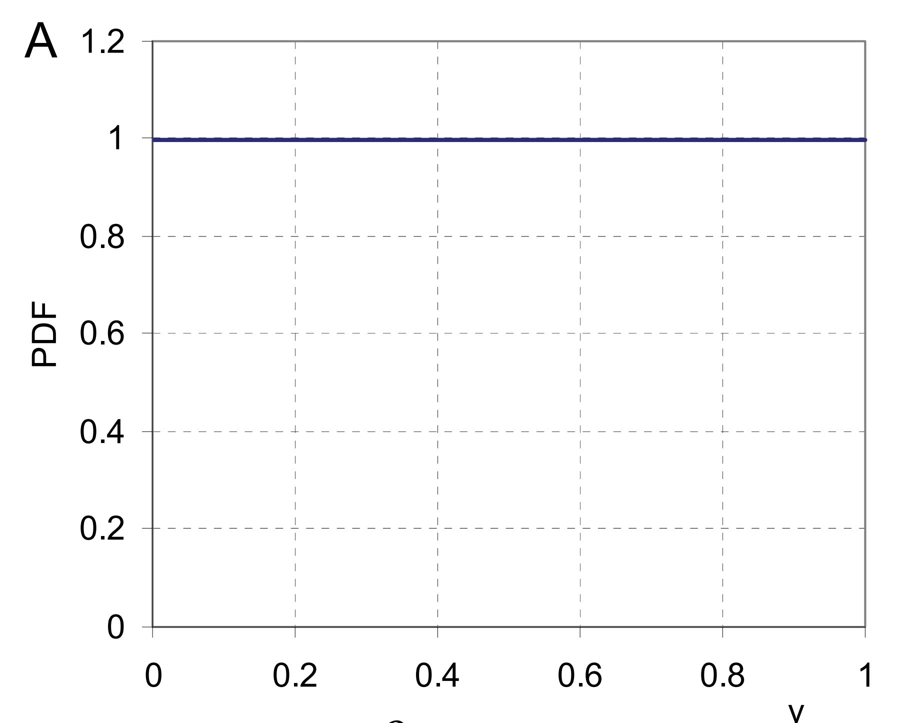
$$f_Y(y|\underset{\sim}{\Theta}, \alpha, \beta) = \frac{mn}{(1 - \theta)m + \theta n} \times$$

$$\begin{cases} \left\{\alpha - 2(\alpha - 1)\left(\frac{y}{\theta}\right)\right\}\left\{\alpha\left(\frac{y}{\theta}\right) - (\alpha - 1)\left(\frac{y}{\theta}\right)^2\right\}^{m-1}, & \text{for } 0 < y < \theta, \\ \left\{\beta - 2(\beta - 1)\left(\frac{1-y}{1-\theta}\right)\right\}\left\{\beta\left(\frac{1-y}{1-\theta}\right) - (\beta - 1)\left(\frac{1-y}{1-\theta}\right)^2\right\}^{n-1}, & \text{for } \theta \le y < 1, \\ 0, & \text{elsewhere,} \end{cases}$$

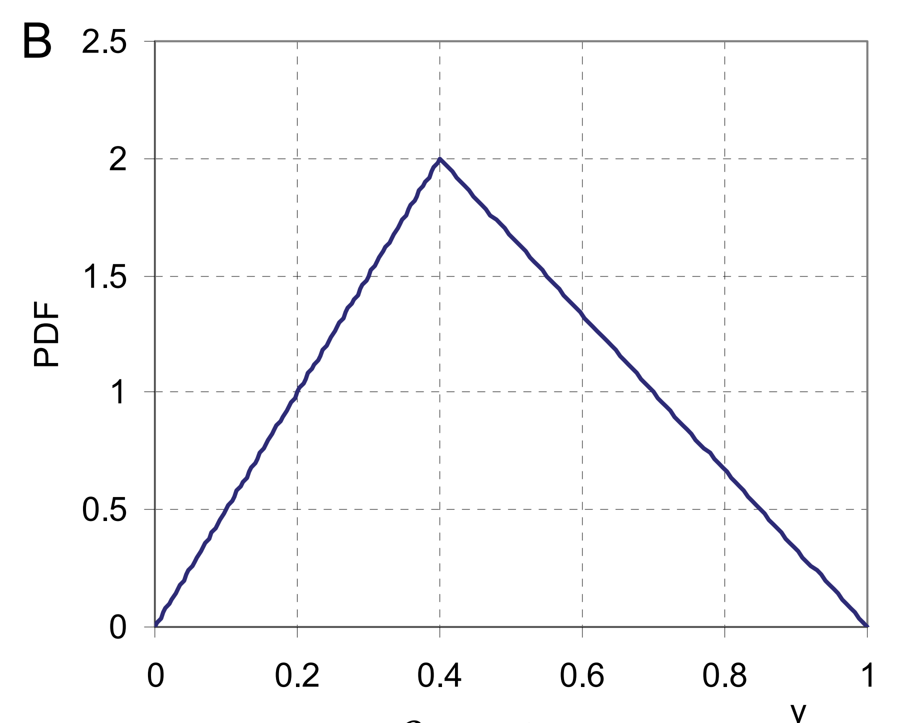
(where as above $\underset{\sim}{\Theta} = (\theta, m, n)$) with the cdf

$$F_Y(y|\underset{\sim}{\Theta},\alpha,\beta)=$$
$$\begin{cases} 0, & \text{for } y \leq 0, \\ p(\underset{\sim}{\Theta})\left\{\boldsymbol{\alpha}\left(\frac{y}{\theta}\right)-(\boldsymbol{\alpha}-\mathbf{1})\left(\frac{y}{\theta}\right)^{\mathbf{2}}\right\}^m, & \text{for } 0 < y < \theta, \\ 1-\{1-p(\underset{\sim}{\Theta})\}\left\{\boldsymbol{\beta}\left(\frac{1-y}{1-\theta}\right)-(\boldsymbol{\beta}-\mathbf{1})\left(\frac{1-y}{1-\theta}\right)^{\mathbf{2}}\right\}^n, & \text{for } \theta \leq y < 1, \\ 1, & \text{for } y \geq 1. \end{cases}$$

- Setting $n=m$ and $\alpha=\beta=2$, we arrive at the density of a **Two-Sided Topp and Leone distribution**.
- Originally, **Topp and Leone (1955)** had introduced their distribution with specific reliability applications in mind.
- We shall refer to the distribution above as the ***Two-Sided Generalized Topp and Leone* (TS-GTL) distribution**.



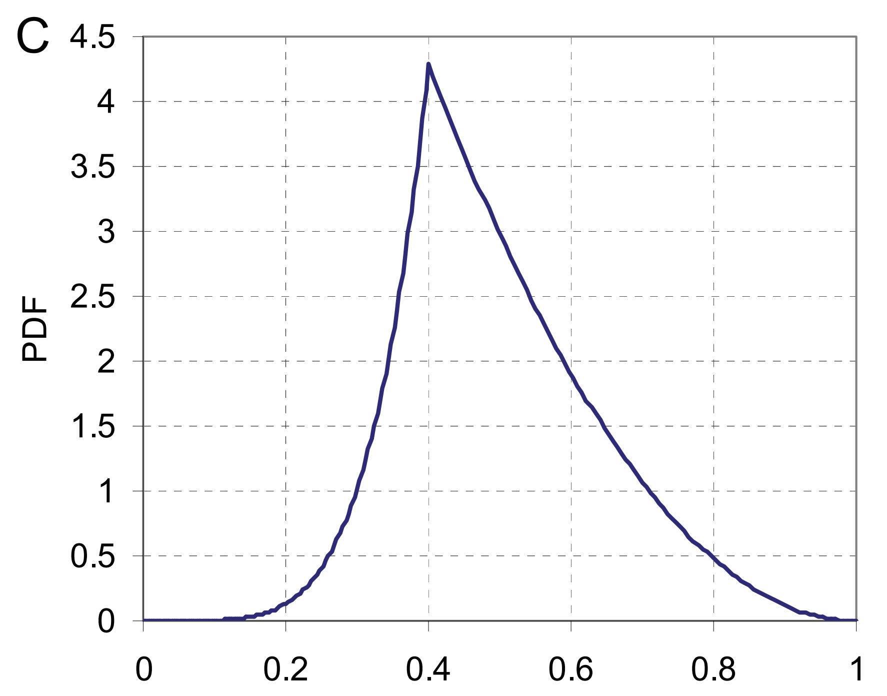
$m = 1, n = 1, \alpha = 1, \beta = 1$



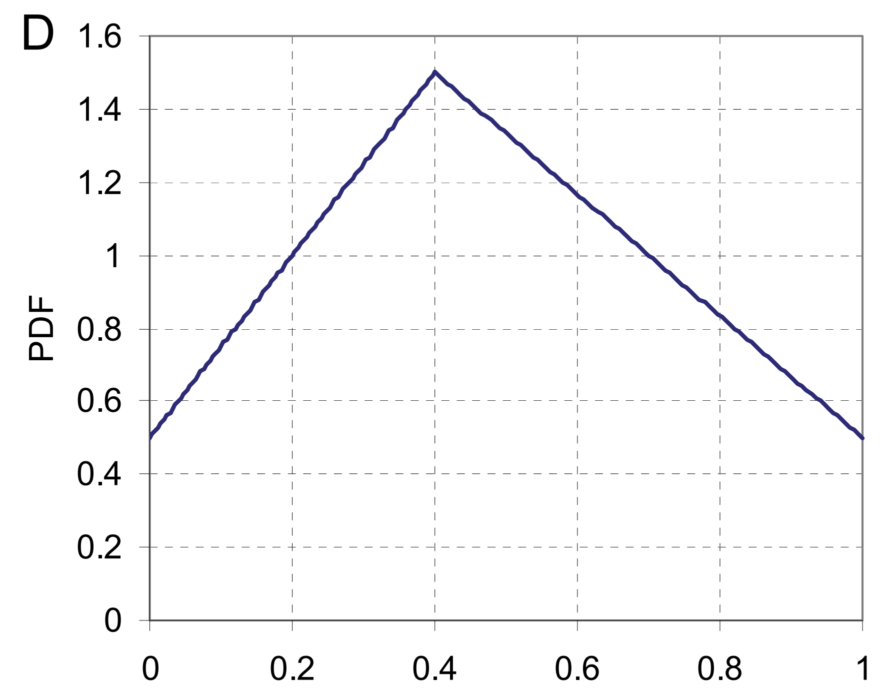
$m = 2, n = 2, \alpha = 1, \beta = 1, \theta = 0.4$

Uniform

Triangular



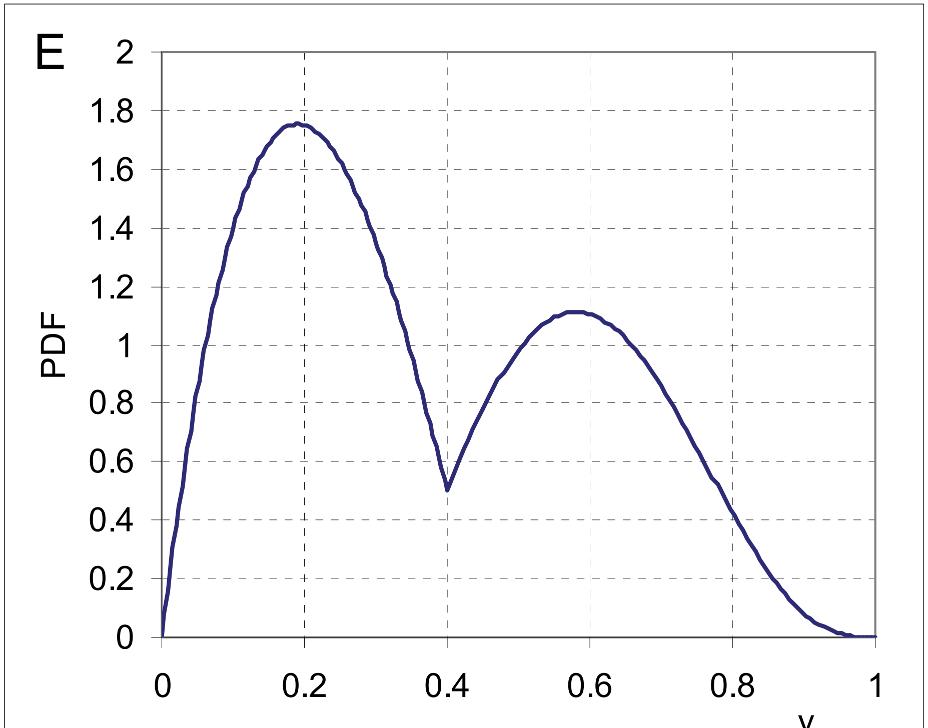
$m=6, n=3, \alpha=1, \beta=1, \theta=0.4$



$m=1, n=1, \alpha=0.5, \beta=0.5, \theta=0.4$

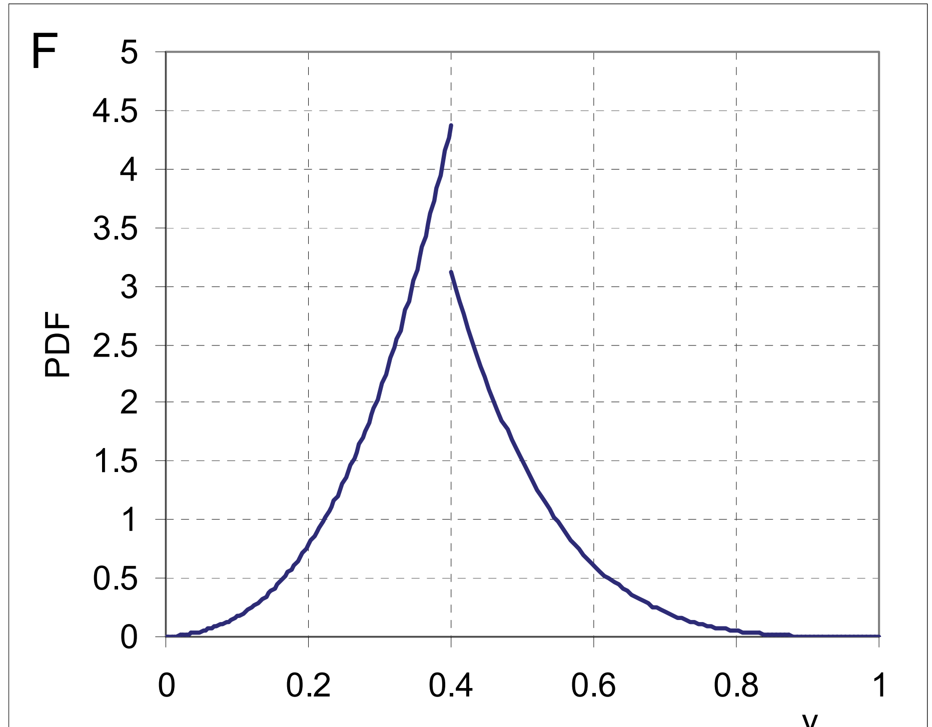
Generalized Two-Sided Power

Two-Sided Slope



$\boldsymbol{m}=2, \boldsymbol{n}=4, \alpha=1.8, \beta=1.8, \theta=0.4$

A novel bi-modal form



$\boldsymbol{m}=2, \boldsymbol{n}=4, \alpha=0.25, \beta=0.75, \theta=0.4$

A discontinuous form

- The TS-GTL density constitutes **a single framework for families of distributions** that were dispersed amongst several related but separate classes.

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- Calculate $E[Y^k|\underset{\sim}{\Theta}, \alpha, \beta]$ for the TS-GTL density via **its mixture structure**.

- Let $\boldsymbol{X_1}$ $[\boldsymbol{X_2}]$ be a random variable with density function $\boldsymbol{mg(x)\left\{G(x)\right\}^{m-1}}$ $\left[\boldsymbol{nh(x)\left\{H(x)\right\}^{n-1}}\right]$ where G and H are slope cdf's as.

- From the pdf's mixture structure we have :

$$E[Y^k|\underset{\sim}{\Theta}, \alpha, \beta] = p(\underset{\sim}{\Theta})\frac{\boldsymbol{E[X_1^k|\alpha, m]}}{\theta} + \{1 - p(\underset{\sim}{\Theta})\}\frac{\left[\boldsymbol{E[X_2^k|\beta, n]} + \theta\right]}{1-\theta}.$$

- In this expression $\boldsymbol{X_1}$ $[\boldsymbol{X_2}]$ is a [Reflected] Generalized Topp and Leone distribution with the support $[0, 1]$ **(see Kotz and Van Dorp (2004a), p. 198)**.

- Alternatively, $\boldsymbol{X_1}$ [$\boldsymbol{X_2}$] is **the largest** [**smallest**] order statistic of a random sample from a [**reflected**] **slope distribution** with parameter α [β] of virtual sample size m [n] (since $m > 0$ [$n > 0$] is not necessarily integer).

- Jones (2004) (amongst others) notes that **derivation of closed form expressions for the moments of an order statistic distribution** could be **somewhat complicated** on a case by case basis.

- The moments of X_1 and X_2 are no exception. **Nadarajah and Kotz (2003)** in a short paper derived the moment expressions for X_1 for the case for $\alpha = 2$.

- These results were further generalized by **Kotz and Van Dorp (2004a)**, to derive **the cumulative moments $\boldsymbol{M_k}$ for** ***reflected generalized Topp and Leone*** distributions for $0 \leq \beta \leq 2$ for X_2 :

$$M_k = \int_0^1 x^k (1 - H(x|\beta, n)) dx$$
$$= \sum_{i=0}^{k} \binom{k}{i} (-1)^i \beta^n \int_0^1 x^{n+i} \Big\{1 - \frac{(\beta - 1)x}{\beta}\Big\}^n dx$$

- The **moments $\mu'_k = E[X_2^k|\beta, n]$** are connected with **the cumulative moments M_k**, $k = 1, \ldots, 4$, via the well known relationship :

$$\mu_k^{'} = kM_{k-1}, k = 1, 2, 3, \ldots$$

(see, e.g., Stuart and Ord (1994)). For $\beta \in (1, 2]$ M_k reduces to

$$M_k = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \beta^n \Big\{\frac{\beta}{\beta - 1}\Big\}^{n+i+1} \frac{B(\frac{\beta-1}{\beta} \,|n + i + 1, n + 1)}{\mathbb{B}^{-1}(n + i + 1, n + 1)}$$

where as above $\mathbb{B}(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$.

- Noting that the pdf of X_1 **is a generalized Topp and Leone distribution** and that of X_2 **is a reflected one**, we have in turn the following relationship:

$$E[X_1^k|\alpha,m] = E[(1-X_2)^k|\beta,n]\Big|_{\beta=\alpha,n=m} = \sum_{i=0}^{k}\binom{k}{i}(-1)^k E[X_2^i|\alpha,m].$$

- Summarizing, the **mean, variance, skewness and kurtosis** can straightforwardly be evaluated **by means of an algorithm that utilizes expressions above** for $\alpha,\ \beta \in (1,2]$. For example, we arrive at the following expression for **the mean of** $Y \sim TS\text{-}GTL(\alpha,\beta,\theta,m,n)$:

$$E[Y|\underset{\sim}{\Theta},\alpha,\beta] = \frac{p(\underset{\sim}{\Theta})}{\theta}\left[1-\alpha^m\left\{\frac{\alpha}{\alpha-1}\right\}^{m+1}\frac{B(\frac{\alpha-1}{\alpha}\ |m+1,m+1)}{\mathbb{B}^{-1}(m+1,m+1)}\right]$$
$$\frac{1-p(\underset{\sim}{\Theta})}{1-\theta}\left[\beta^n\left\{\frac{\beta}{\beta-1}\right\}^{n+1}\frac{B(\frac{\beta-1}{\beta}\ |n+1,n+1)}{\mathbb{B}^{-1}(n+1,n+1)}+\theta\right].$$

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4. TS-GTL DISTRIBUTIONS… Maximum Likelihood

- For a **random ordered sample $\underline{X} = (X_{(1)}, \ldots, X_{(s)})$** of size s from the TS-GTL distribution, the **loglikelihood function** is, by definition,

$$Log\{L(\underline{X}, \underset{\sim}{\Theta}, \alpha, \beta)\} = sLog\{\frac{mn}{(1-\theta)m + \theta n}\} +$$

$$\sum_{i=1}^{r} Log\{g(\frac{X_{(i)}}{\theta}|\alpha)\} + (m-1)\sum_{i=1}^{r} Log\{G(\frac{X_{(i)}}{\theta}|\alpha)\} +$$

$$\sum_{i=r+1}^{s} Log\{h(\frac{X_{(i)}-\theta}{1-\theta}|\beta)\} + (n-1)\sum_{i=r+1}^{s} Log\left\{1 - H\left(\frac{X_{(i)}-\theta}{1-\theta}|\beta\right)\right\},$$

where $g(\cdot|\alpha), G(\cdot|\alpha), h(\cdot|\alpha), H(\cdot|\alpha)$ are **the pdf's and cdf's of a slope distribution** and $X_{(1)} < X_{(2)} < \ldots < X_{(s)}$.

- **Here, r is a positive integer such that**

$$X_{(r)} \leq \theta < X_{(r+1)}.$$

- We propose the following **MLE algorithm with $\boldsymbol{k}$-the iteration**:

STEP 0: Set $k = 1$, $\alpha_1 = \alpha^*$, $\beta_1 = \beta^*$, $m_1 = m^*, n_1 = n^*, \theta = \theta^*$.

STEP 1: Select $\boldsymbol{m_{k+1}} = \underset{\boldsymbol{m>0}}{argmax} \; Log\{L(\underline{X}|\boldsymbol{m}, n_k, \theta_k, \alpha_k, \beta_k,)\}$.

STEP 2: Select $\boldsymbol{n_{k+1}} = \underset{\boldsymbol{n>0}}{argmax} \; Log\{L(\underline{X}|m_{k+1}, \boldsymbol{n}, \theta_k, \alpha_k, \beta_k)\}$.

STEP 3: Select $\boldsymbol{\alpha_{k+1}} = \underset{\boldsymbol{0 \le \alpha \le 2}}{argmax} \; Log\{L(\underline{X}|m_{k+1}, n_{k+1}, \theta_k, \boldsymbol{\alpha}, \beta_k)\}$.

STEP 4: Select $\boldsymbol{\beta_{k+1}} = \underset{\boldsymbol{0 \le \beta \le 2}}{argmax} \; Log\{L(\underline{X}|m_{k+1}, n_{k+1}, \theta_k, \alpha_{k+1}, \boldsymbol{\beta})\}$.

STEP 5: Select $\boldsymbol{\theta_{k+1}} = \underset{\boldsymbol{0 \le \theta \le 1}}{argmax} \; Log\{L(\underline{X}|m_{k+1}, n_{k+1}, \boldsymbol{\theta}, \alpha_{k+1}, \beta_{k+1})\}$.

STEP 6: If $|Log\{L(\underline{X}|\underset{\sim}{\Theta}_{k+1}, \alpha_{k+1}, \beta_{k+1})\} - Log\{L(\underline{X}|\underset{\sim}{\Theta}_{k}, \alpha_k, \beta_k)\}| < \epsilon$ $STOP$ Else $k = k + 1$ and Goto Step 1.

4. TS-GTL DISTRIBUTIONS… Maximum Likelihood

- The algorithm above can be easily modified to a ML algorithm for sub-classes in the TS-GTL family.

- Omitting **Steps 1, 2 and 4** and setting $m = n = 1$, $\beta = \alpha$ results in a ML algorithm for the **TS-Slope distributions**.

- Setting $\alpha = 1$, $\beta = 1$ and **removing Steps 3 and 4**, we obtain an ML algorithm for **GTSP distributions**.

- Next, by setting $m = n$ and **removing Step 2** ($m = n = 2$ **and removing Steps 1 and 2**) it reduces to an algorithm for the **TSP** (**triangular**) **distribution**.

- Finally, **eliminating just Step 4**, while guaranteeing $\beta = \alpha$ leads to an ML algorithm for continuous TS-GTL distributions.

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4. TS-GTL DISTRIBUTIONS… Maximum Likelihood

- To obtain **an initial starting solution** $m^*, n^*, \alpha^*, \beta^*$ and θ^* in Step 0 one could, for example, **select $m^*, n^*, \alpha^*, \beta^*$ and θ^* visually to match a plot** of a TS-GTL pdf to that of an empirical pdf **or more directly use least squares estimates** for $m^*, n^*, \alpha^*, \beta^*$ and θ^*.

- Setting **the partial derivatives of log-likelihood** with respect **to left branch power parameter m equal to 0**, we obtain the following **MLE for m_{k+1} at Step 1**:

$$\widehat{m}(\theta, n) = \frac{1}{2}\frac{\theta n}{1-\theta}\Big\{-1+\sqrt{1+\mathcal{K}(\theta,n)\frac{4s(1-\theta)}{n\theta}}\Big\} > 0,$$

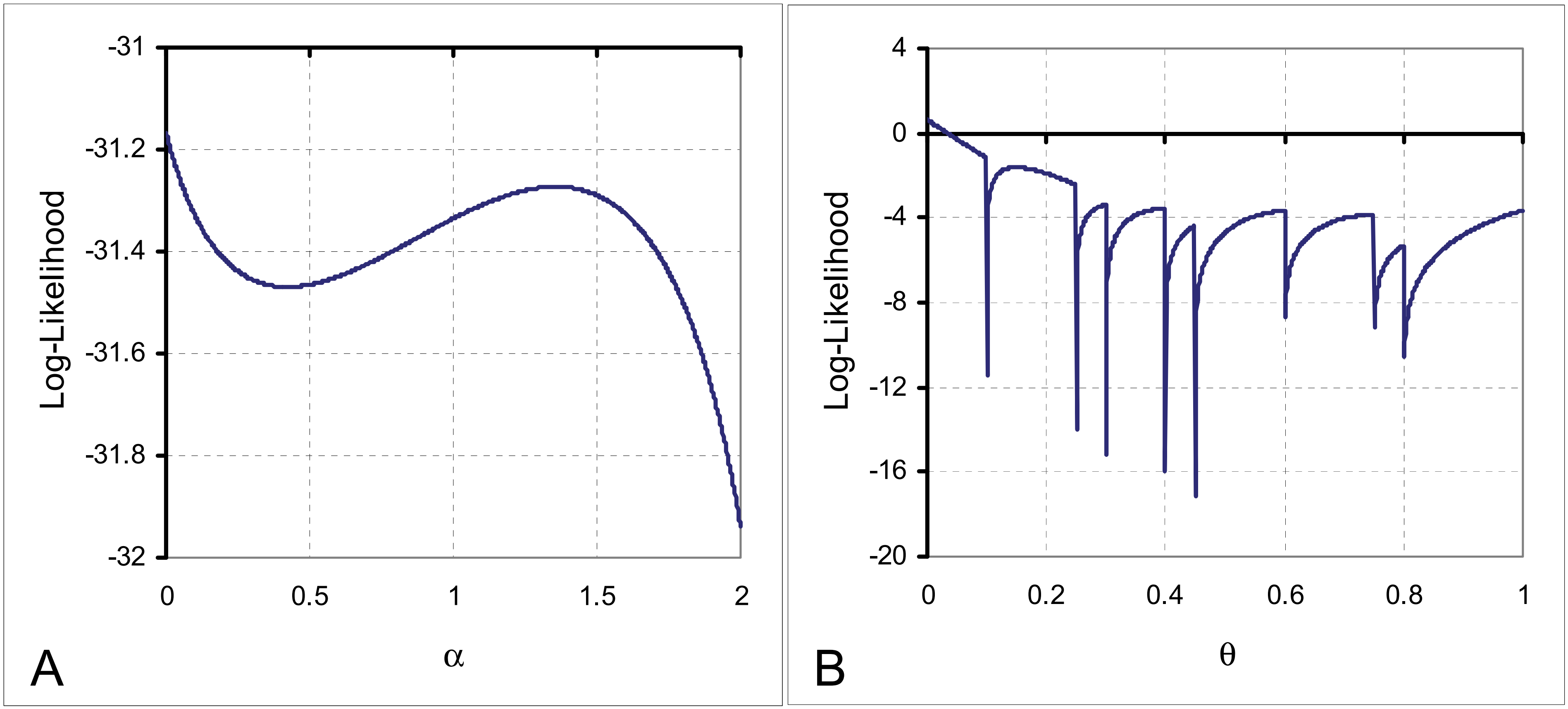
where

$$\mathcal{K}(\theta,n) = \left[-\sum_{i=1}^{r} Log\{G(\frac{X_{(i)}}{\theta}|\alpha)\}\right]^{-1} > 0.$$

- **A similar formula can be derived for $\widehat{n}(\theta, m)$ in Step 2.**

4. TS-GTL DISTRIBUTIONS… Maximum Likelihood

- Determination of the ML estimates of the log-likelihood profile as a function of **α, β or θ turn out to be more challenging.**



4. TS-GTL DISTRIBUTIONS... Maximum Likelihood

- Figure 2A plots a **log-likelihood profile as a function of α**, which indicates that **multiple stationairy points** may possibly exist, whereas the global optimum over $\alpha \in [0, 2]$ is actually attained at $\alpha = 0$.

- Figure 2B plots a **log-likelihood profile as a function of θ**, which shows (i) a global optimum at the lower bound of the range $\theta \in [0, 1]$, (ii) a **discontinuous** behavior of the log-likelihood as a function of θ over $[0, 1]$, but **continuous over each interval $[X_{(i)}, X_{(i+1)}]$**, $i = 0, \ldots, s$ (where s is the sample size) and (iii) the **existence of stationary points within the interval $[X_{(i)}, X_{(i+1)}]$**.

- Given the structure of the log-likelihood profiles as a function of α, β or θ, it would **seem reasonable (and practical)** to take advantage of **the boundedness of $\alpha, \beta \in [0, 2]$ and $\theta \in [0, 1]$** and **globally optimize** over these intervals **by discretizing at a desirable level of accuracy δ** and to evaluate the log-likelihood at all discretized points.

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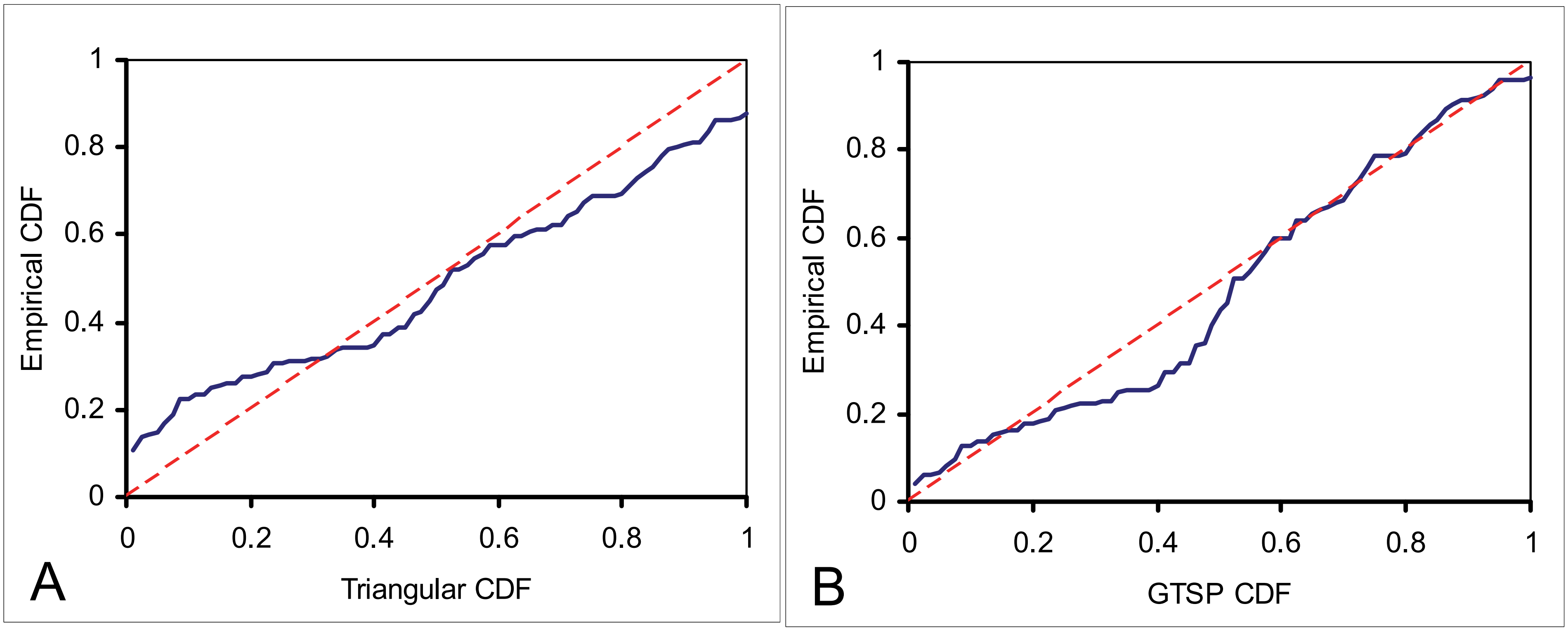
- The classical data in this example reaches the high points of our **gigantic Universe** and involves the **$(V - I)$ indices of 80 globular clusters in the Galaxy M87**. Galaxy M87, also called Virgo A, was discovered by Charles Messier, a French astronomer, in 1781 (see, e.g., Philbert (2000)).

- Davis and Brodie (2006) explain:

  "Globular clusters are **nearly spherical groups of about 10,000 to 1 million stars**. **The color** of a globular cluster **gives clues about the cluster's composition** (what kinds of elements and stars are in the cluster) and **the cluster's age**. ... V and I are different filters through which we can look at objects in the sky. Looking through a **V filter** is like looking through **a yellow pair of glasses** and looking through an **I filter** is like looking through **infrared glasses** (our eyes can't see infrared, but telescopes can)."

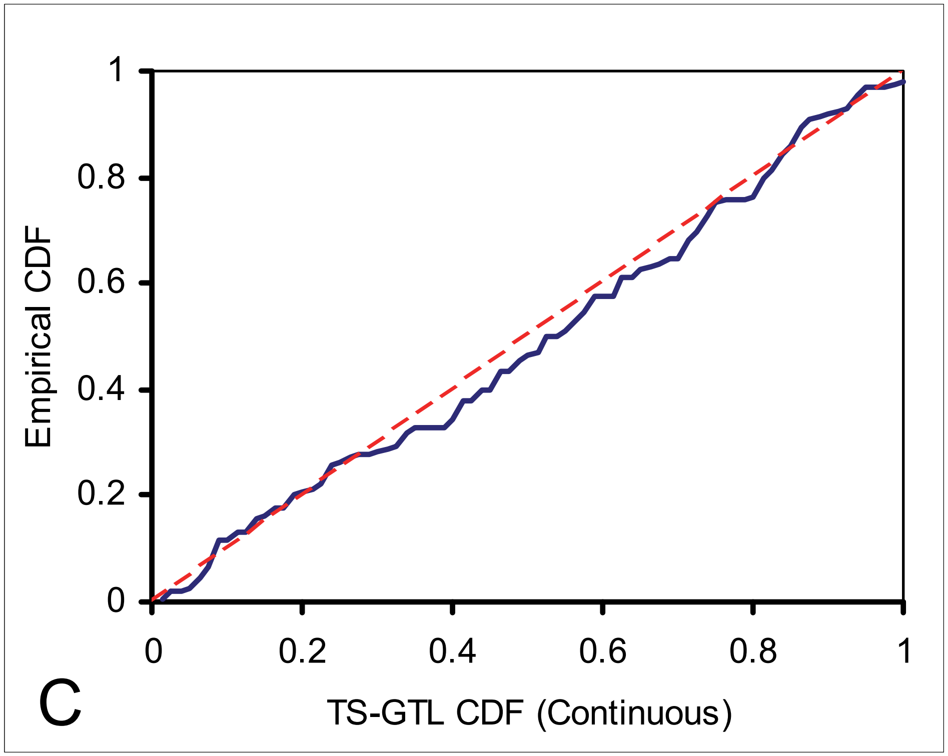
- Hence, **a $(V - I)$ index is a color measurement index**.

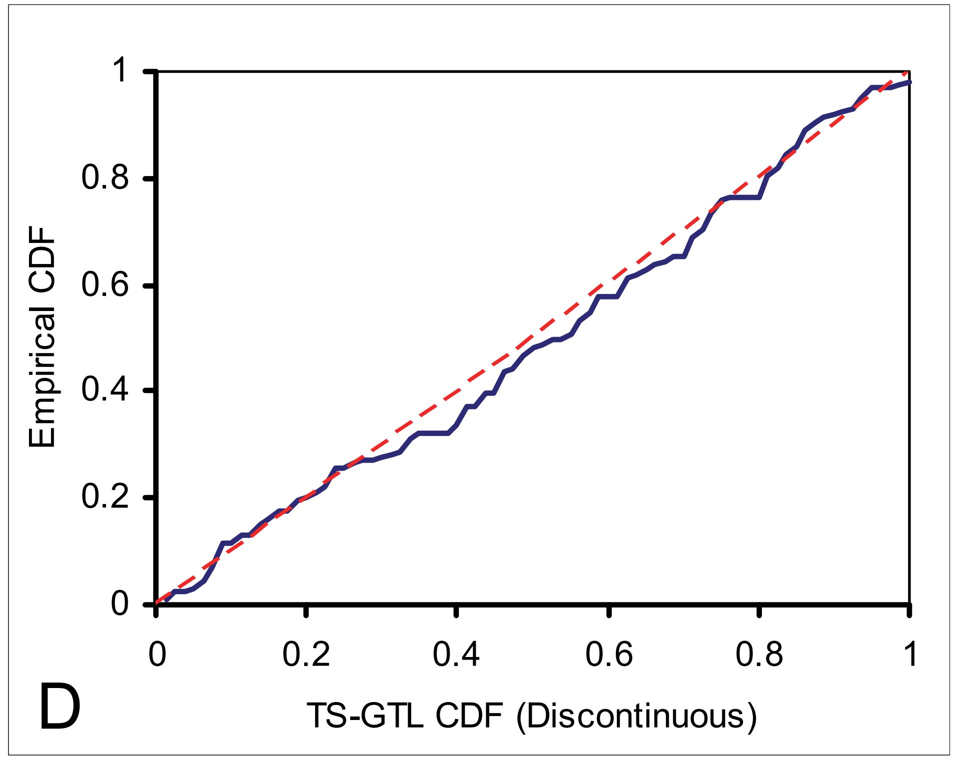
- Figure below provides **PP-plots** for the ML fitted **triangular and GTSP** distributions.



5. TS-GTL DISTRIBUTIONS… Illustrative Example

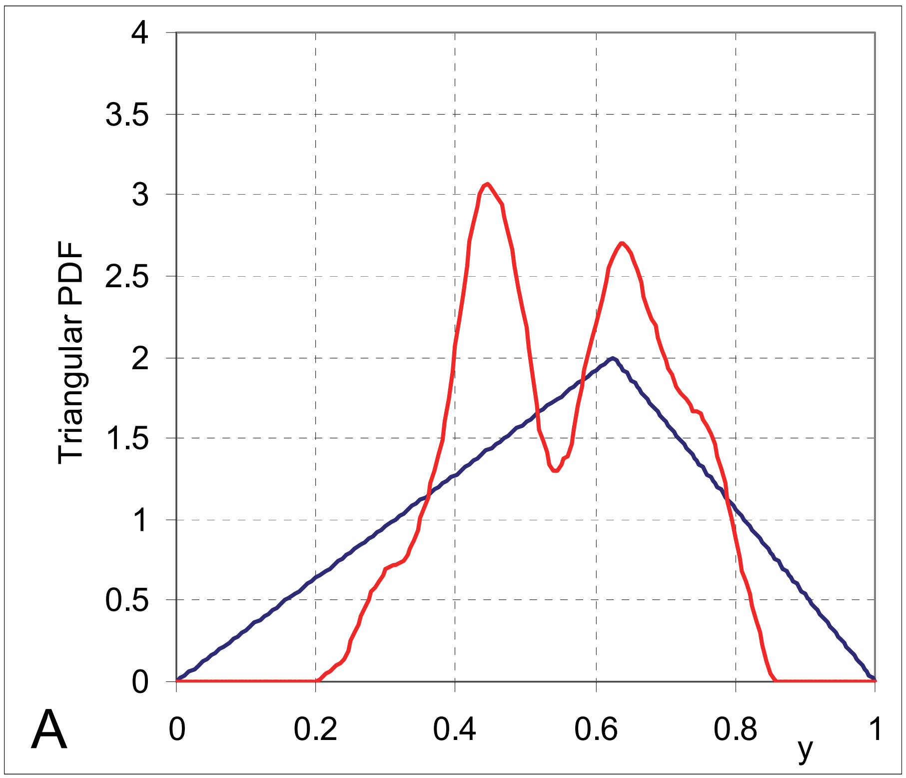
- Figure below provides **PP-plots** for the ML fitted **TS-GTL (continuous),** and **TS-GTL (discontinuous)** distributions.

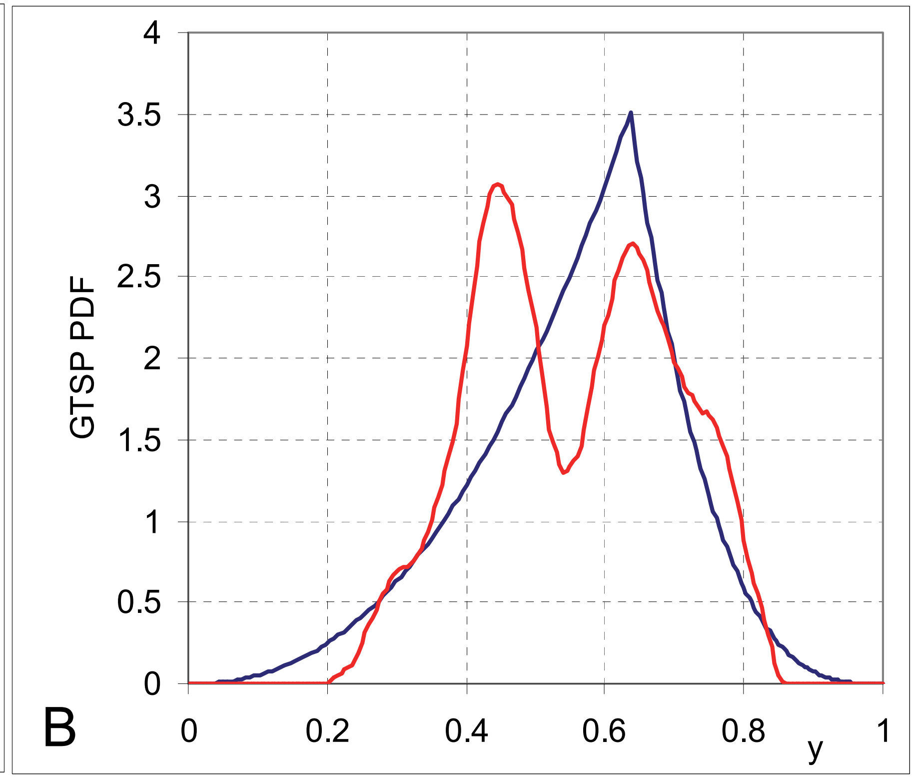




5. TS-GTL DISTRIBUTIONS… Illustrative Example

- Figure below displays an **empirical kernel density** that was generated using the **Bartlett-Epanechikov kernel** (e.g., Izenman, 1991) combined with the **over-smoothed bandwidth** (e.g., Sheather, 2004) and **ML Fitted distributions**.





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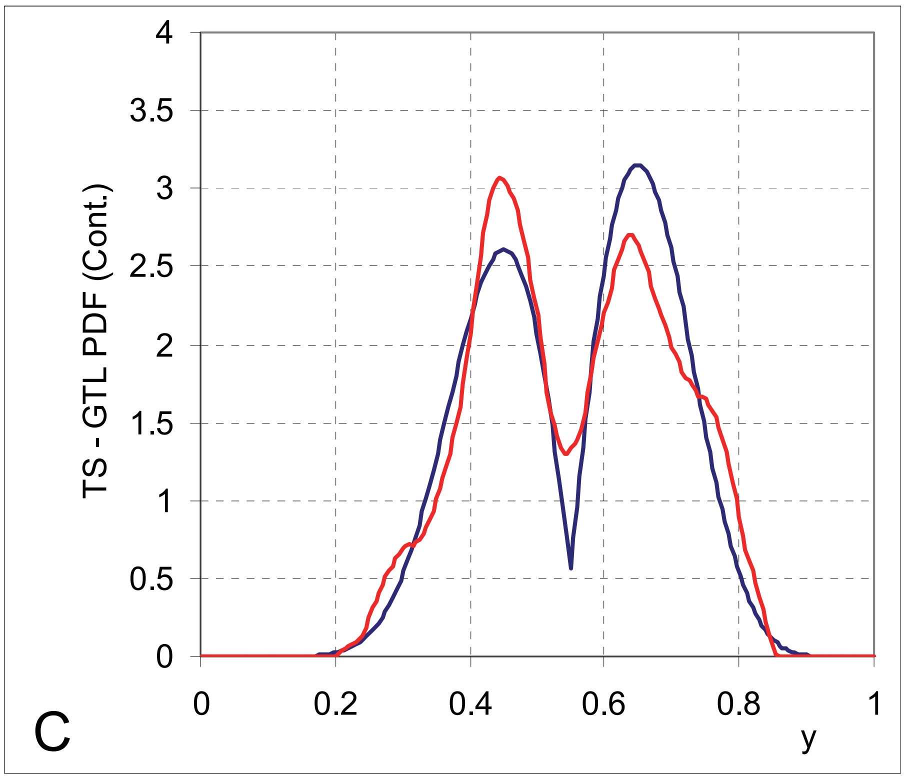
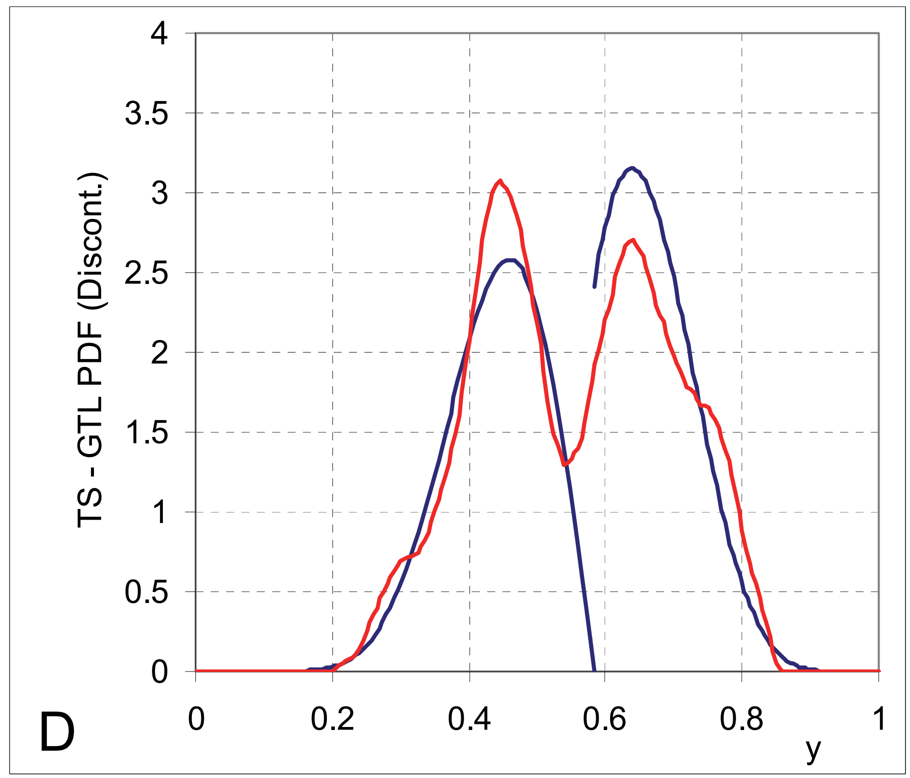
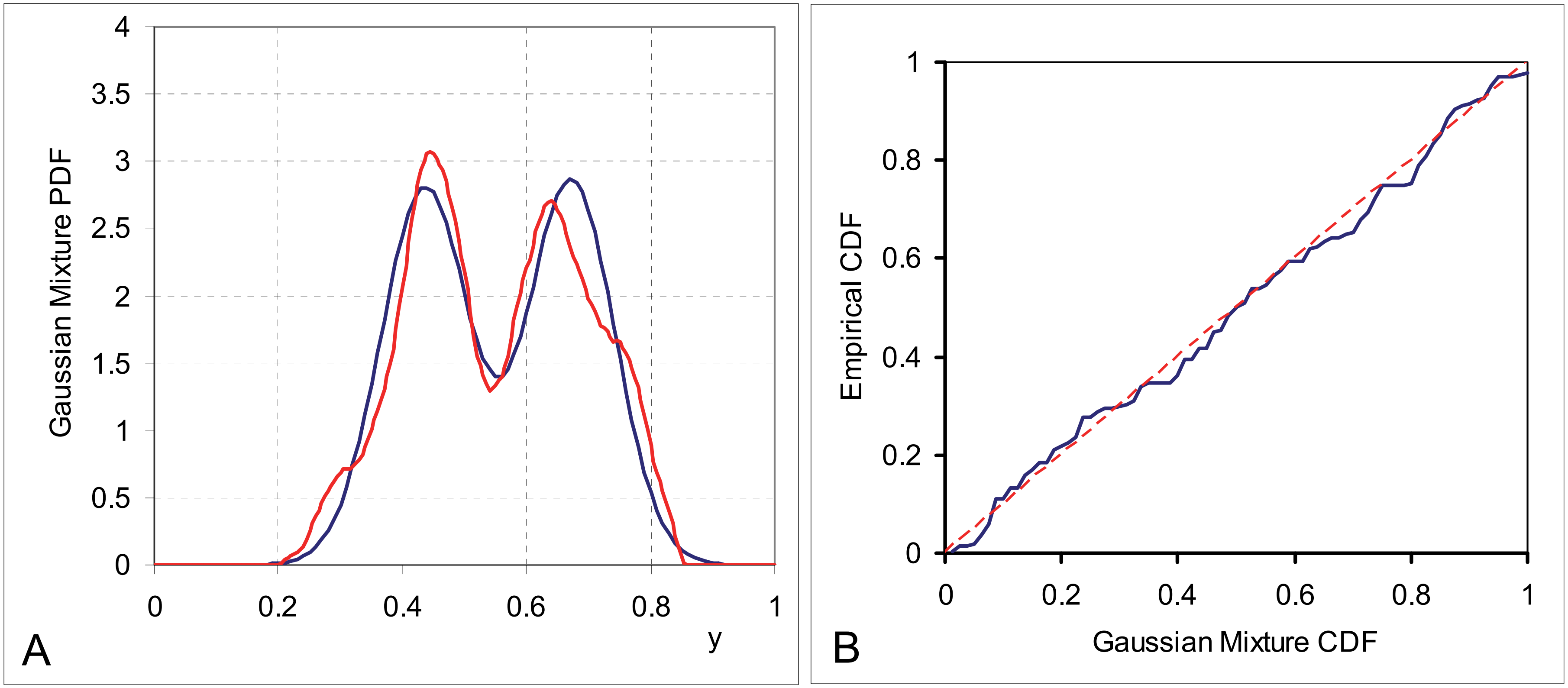




- To obtain a deeper appreciation of the data, we have decided also to fit a **Gaussian mixture model with two components** using an **Expectation Conditional Maximization (ECM) algorithm**.



				triangular	TSP	GTSP	TS - GTL (Cont.)	TS - GTL (Dis.)	Gaussian mixture
Bin	LB_i	UB_i	O_i	$(O_i-E_i)^2/E_i$	$(O_i-E_i)^2/E_i$	$(O_i-E_i)^2/E_i$	$(O_i-E_i)^2/E_i$	$(O_i-E_i)^2/E_i$	$(O_i-E_i)^2/E_i$
1	0.000	0.383	8	6.13	0.10	0.47	0.14	0.13	0.10
2	0.383	0.425	8	3.10	4.19	3.53	0.05	0.12	0.02
3	0.425	0.455	8	6.35	5.87	5.32	0.55	0.66	0.25
4	0.455	0.478	8	10.09	7.82	7.37	2.36	2.28	2.16
5	0.478	0.556	8	0.52	2.29	2.32	0.33	1.19	0.92
6	0.556	0.613	9	0.03	1.47	1.32	0.01	0.28	0.27
7	0.613	0.638	7	2.36	0.01	0.01	0.18	0.10	1.13
8	0.638	0.674	8	1.38	0.00	0.05	0.10	0.08	0.00
9	0.674	0.740	8	0.01	0.45	0.37	1.75	1.46	1.94
10	0.740	1.000	8	2.91	0.02	0.22	0.38	0.33	0.24
Chi-square statistic				32.87	22.23	20.99	5.86	6.64	7.02
Parameters				1	2	3	4	5	5
Degrees of freedom				8	7	6	5	4	4
p-value				6.5E-05	2.3E-03	1.8E-03	**0.32**	0.16	0.13
AIC - criterion				-63.14	-82.04	-80.91	-98.00	**-98.59**	-94.38
BIC - criterion				-60.76	-77.28	-73.77	**-88.48**	-86.68	-82.47

OUTLINE

1. Introduction
2. PDF and CDF of TS-GTL distributions
3. Moments
4. Maximum Likelihood Estimation
5. Example using $(V - I)$ indices of 80 globular clusters in the Galaxy M87
6. **Concluding Remarks**
7. Some References

6. TS-GTL DISTRIBUTIONS… Concluding Remarks

- This talk/paper provides **a comprehensive discussion** of a 5 parameter flexible, bounded continuous family of univariate distributions designated as **TS-GTL distributions**.

- It has been constructed from **a generalized framework of Two-Sided pdf's** which is **structurally reminiscent of Jones (2004) generalizations** of the distribution of **order statistics.**

- The TS-GTL family of distributions **combines within a single family** the subclasses of Triangular, Two-Sided Slope (TSS), Two-Sided Power (TSP) and Generalized Two-Sided Power (GTSP) distributions discussed for example in Kotz and Van Dorp (2004a).

- The **maximum likelihood procedure** was employed to fit the TS-GTL distribution to a bimodal data set. Its ML fit turns out to be superior (in this particular case) to the classical Gaussian mixture ML fit applied to this data.

7. TS-GTL DISTRIBUTIONS... Some References

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