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Steiner Trees in Industry

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1 Introduction

The Steiner tree problem is a classic intractable problem with many applications in the design of computer circuits, long-distance telephone lines, or mail rout ing, etc. The computational nature of the problem makes it a traditional research subject in theory of computing.

Given a set of points in the Euclidean plane, the shortest network interconnecting the points in the set is called the *Steiner minimum tree.* The Steiner minimum tree may contain some vertices which are not the given points. Those vertices are called *Steiner points* while the given points are called *terminals.* The shortest network for three terminals was first studied by Fermat (1601-1665). Fermat proposed the problem of finding a point to minimize the total distance from it to three terminals in the Euclidean plane. The direct generalization is to find a point to minimize the total distance from it to n terminals, which is still called the Fermat problem. The Steiner minimum tree problem is an indirect generalization. Recent research on mathematical history showed that this generalization (i.e., the Steiner minimum tree) was first studied by Gauss.

On March 19, 1836, Schumacher wrote a letter to Gauss and mentioned a paradox about Fermat problem on terminals: For four vertices of a convex quadrilateral, the solution of the Fermat problem is the intersection point of two diagonals. When two of the four vertices move to the same position, the intersection point of two diagonal would also move to this position which is not the solution of the Fermat problem for the three points resulting from the four vertices. Schumacher could not understand why this would happen. On March 21, 1836, Gauss wrote back to Schumacher and explained the paradox. In this letter, he mentioned another generation of Fermat problem, that is, aim on the network structure instead of a point position. Gauss already discussed all possible topologies of Steiner minimum trees for four points. (See Schreiber [1986].)

In the last centenary, the Steiner tree problem has been extended to various metric spaces. Among them, the Euclidean Steiner tree (i.e., the Steiner tree in Euclidean plane), the rectilinear Steiner tree (i.e., the Steiner tree in the rectilinear plane), and the network Steiner tree (i.e., the Steiner tree in an undirected network) were recognized as most important ones and received much more attentions. We call them *classic* Steiner tree problems.

It is well-known that Steiner tree problems usually are NP-hard (see Karp [1972], Garey and Johnso [1978], Garey, Graham, and Johnson [1978], Foulds and Graham [1982]), i.e., the exact optimal solutions are unlikely computable in polynomial-time. Therefore, one has to put efforts on looking for good approximation solutions. During the last ten years, great progress has been made in the study of Steiner trees, including solution of the Gilbert-Pollak conjecture on the Steiner ratio, solution of the better approximation problem, and designs of polynomial-time approximation schemes for Euclidean and rectilinear Steiner Trees.

The Gilbert-Pollak conjecture was made in 1968. There is a folklore about it (see Hwang [1990]). A large company usually has a private telephone network interconnecting its branchs. For example, University of California has nine campuses. If you call from one campus to another one through private network, it would be counted as a long distance call. The private network is not really built by the company privately. It is rent from telephone company, realized by a special phone number. For example, when I graduated from University of California at Santa Barbara and went to work in Berkeley, my advisor gave me a special phone number for calling him from Berkeley. This number is the private network of University of California. Before 1967, the charge of a private network was determined by the length of the minimum spanning tree for the destinations. The minimum spanning tree for a set of terminals is the shortest tree with edges between terminals. It is different from the Steiner tree by disallowing the existence of Steiner points. This restriction causes the minimum spanning tree possibly longer than the Steiner minimum tree for the same set of terminals. In 1967, a flight company found this fact. Therefore, they requested some new services at those Steiner points, so that the minimum spanning tree for the new set of destinations is the Steiner minimum tree for the original set of destinations, which is shorter than the minimum spanning tree for the original set of destinations. Therefore, those requests increased the service of telephone company and decreased the charge from the telephone company. With this situation, the telephone company had to change the billing base from the minimum spanning tree to the Steiner minimum tree. Therefore, the telephone company faced a problem: With this change, how much should be increased on the rate of the unit length? This motivated the study of the Steiner ratio, the ratio of lengths of the Steiner minimum tree and the minimum spanning tree for the same set of terminals. Gilbert and Pollak [1968] conjectured that the Steine ratio in the Euclidean plane is at least $\sqrt{3}/2$ which is achieved by three vertices of an equilateral triangle.

The work of Gilbert and Pollak [1968] is a turning point in the study of Steiner trees. Before this work, the Steiner tree was studied mainly due to mathematical interests and hence progress was made very slowly. Gilbert-Pollak's work brought the Steiner tree into model industries. Since then, the Steiner tree attracts more and more attentions and the number of research publications in the Steiner tree grows very fast. Through many efforts made by Graham and Hwang [1976], Pollak [1978], Chung and Hwang [1978], Du and Hwang [1983], Du, Hwang and Yao [1985], Chung and Graham [1985], Friedel and Widmayer [1989], Booth [1991], and Rubinstein and Thomas [1991], the Gilbert-Pollak conjecture was finally proved by Du and Hwang [1990,1992]. The significance of their proof stems also from the potential applications of the new approach included in the proof and hence received a lot of public recognitions (see Stewart [1991], Kolata [1990], Campbell [1992], Cipra [1990, 1991], Peterson [1990], and Sangalli [1991]).

While the Steiner minimum tree is an NP-hard problem in many metric spaces, the minimum spanning tree can be computed in at most $O(n^2)$ time. Therefore, the inverse of the Steiner ratio is actually the performance ratio of the minimum spanning tree when it is considered as a polynomial-time approximation of the Steiner minimum tree. Is there polynomial-time approximation better than the minimum spanning tree? For more than twenty year (1968-1990), many polynomial-time approximations were discovered (see Chang [1972], Korthonen [1979], Kou and Makki [1987], Smith and Liebman [1979], Smith, Lee, and Liebman [1981], Waxman and M. Imase [1988], Smith and Shor [1992]), however none of them has a performance ratio which can be proved to be better than the inverse of the Steiner ratio. This situation exists not only in the Euclidean plane, but also in any interested metric space. Therefore, a long-standing open problem was generated. In general, it was called the *better approximation problem* whether there exists a polynomial-time approximation for Steiner minimum trees in each metric space with performance ratio smaller than the inverse of the Steiner ratio. Zelikovsky [1993] made the first breakthrough. He found a polynomial-time 11/6-approximation for network Steiner minimum trees which beats the inverse of the Steiner ratio in networks, $\rho_2^{-1} = 2$. Soon later, Berman and Ramaiye [1994] gave a polynomial-time 92/72-approximation for the Steiner minimum tree in the rectilinear plane which beats the inverse of the rectilinear Steiner ratio 3/2 (see Hwang [1976]), and Du, Zhang, and Feng [1991] showed a general solution for the open problem. They showed that in any metric space, there exists a polynomial-time approximation with performance ratio better than the inverse of the Steiner ratio provided that for any set of a fixed number of points, the Steiner minimum tree is polynomial-time computable.

After the better approximation problem is settled, it is naive to ask how small performance ratio a polynomial-time approximation can achieve. Bern and Plassmann [1989] showed that the network Steiner minimum tree problem is MAX-SNP hard. Namely, it is unlikely to have a polynomial-time approximation scheme. For Euclidean and rectilinear Steiner minimum trees, Arora [1996] and Mitchell [1996] independently discovered a surprising result that there exist polynomial-time approximation schemes. Their approaches work not only in Steiner trees but also in a family of geometric optimization problems. This fact shows clearly that the Steiner tree is not an isolated research topic. It always influences and reflects the progress in the general theory of computing, especially in algorithm design and analysis.

"What could be the next major development?" Jeff Ullman asked when one of authors visited at Stanford University ten years ago. Now, many researchers think about the same problem in the area of Steiner trees. After powerful techniques have been discovered for studying Steiner trees, classic problems on approximation of Steiner trees in the Euclidean and rectilinear plane have been settled. Should we put our research interests only in network Steiner trees? If we use yahoo.com to search subject Steiner-tree, then we may found 1720 web-pages on this subject. Many of them come from industries. This suggests a wide field in Steiner trees. In fact, most major theoretical open problems in Steiner trees were initiated from industry applications. Now, it is time to look back, to find out the impact of previous theoretical development in industries, and to obtain source from industries to suppose new developments in theory. Therefore, we review some variations of Steiner tree problems and related research problems.

2 Important Techniques Discovered Previously

All three major developments described as above result from discovery of new techniques in analysis and designs of approximation algorithms.

The Gilbert-Pollak conjecture was proved with a so-called minimax approach. Indeed, when an approximation is obtained from adding restriction to the original optimization problem, the determination of the performance ratio can be transformed to a minimax problem. In this approach, the central part is a new minimax theorem about minimizing the maximum value of several concave functions over a polytope as follows:

Minimax Theorem. Let $f(x) = \max_{i \in I} g_i(x)$ where I is a finite set and $g_i(x)$ is a continuous, concave function in a polytope X. Then the minimum value of $f(x)$ over the polytope X is achieved at some critical point, namely, a point satisfying the following property:

(*) There exists an extreme subset Y of X such that $x \in Y$ and the index set $M(x)$ $(= \{i \mid f(x) = g_i(x)\})$ is maximal over Y.

The Steiner ratio problem is first transformed to such a minimax problem ($g_i(x)$= (the length of a Steiner tree) - (the Steiner ratio)·(the length of a spanning tree with graph structure i) where x is a vector whose components are edge-lengths of the Steiner tree) and the minimax theorem reduces the minimax problem to the problem of finding the minimax value of the concave functions at critical points. Then each critical point is transformed back to an input set of points with special geometric structure; it is a subset of a lattice formed by equilateral triangles. This special structure is called *critical structure* which enables us to verify the conjecture corresponding to the non-negativeness of minimax value of the concave functions.

Using the same approach, Gao, Du, and Graham [1995] proved that in any Minkowski plane (or 2-dimensional Banach space), the Steiner ratio is at least 2/3. This settles a conjecture made independently by Cieslik [1990] and Du *et al.* [1993]. The main contribution of Gao, Du, and Graham is in the study of Steiner trees for points in equilateral triangle lattice. In fact, Du and Hwang [1992] already showed that the Steiner ration in any Minkowski plane is achieved by points in an equilateral triangle lattice. Recently, Brazil et al [1996] showed a very interesting result on Steiner trees for square-lattice points. As an intermediate result, they found that every full Steiner tree for square-lattice points has linear topology, i.e. all Steiner points lay on a path in the tree. If a similar result can be proved for points on every equilateral triangle lattice (we intend to do it), then it is possible to give a new proof for the above conjecture and to solve other open problems, including one of conjectures in Minkowski planes that the Steiner ratio in any Minkowski plane equals the Steiner ratio in its dual plane. Wan, Du, and Graham [1997] showed that this conjecture is true for five points.

The excellent idea of Zelikovsky [1993] for establishing better approximations consists of two parts. The first part is design of approximation with greedy algorithm. This greedy algorithm is not directly applied to the Steiner minimum tree. It is applied to the k-size Steiner minimum tree, a restriction of the Steiner minimum tree. Let us explain it as follows.

A tree interconnecting a terminal set is called a *Steiner tree* if every leaf is a terminal. However, a terminal in a Steiner tree may not be a leaf. A Steiner tree is *full* if every terminal is a leaf. When a terminal is not a leaf, the tree can be decomposed into several small trees at this terminal. In this way, every Steiner tree can be decomposed into smaller trees in each of which every terminal is a leaf. These smaller trees are called *full components* of the tree. The *size* of a full component is the number of terminals in the full component. A k-size Steiner tree is a Steiner tree with all full components of size at most k. The k-size Steiner minimum tree is the shortest one among all k-size Steiner trees. The 2-size Steiner minimum tree is the minimum spanning tree. The k-size Steiner minimum tree for $k \geq 3$ is certainly an approximation better than the minimum spanning tree. However, the k-size Steiner minimum tree for $k \geq 4$ is NP-hard and no polynomial-time algorithm has been found for the 3-size Steiner minimum tree. Therefore, a greedy approximation is employed instead of themselves.

The second part of Zelikovsky's excellent idea is to connect the performance ratio of the greedy approximation to the k-Steiner ratio. The k-Steiner ratio in a metric space is the least ratio of lengths between the Steiner minimum tree and the k-Steiner minimum tree for the same set of terminals in the metric space. The 2-Steiner ratio is exactly the Steiner ratio. A better lower bound for the k-Steiner ratio will give a better performance ratio for approximations of Zelikovsky's type. Zelikovsky [1993] showed that the 3-Steiner ratio in graphs is at least $3/5$. Du, Zhang, and Feng [1991] showed that the k-Steiner ratio in graphs is at least $\lfloor \log_2 k \rfloor / (1 + \lfloor \log_2 k \rfloor)$. Recently, Borchers and Du [1995] completely determined the exact value of the k-Steiner ratio in graphs for all k and Borchers, Du, Gao, and Wan [1998] determined the exact value of the k-Steiner ratio in the rectilinear plane for all k. The techniques discovered in these works may improve the currently best-known approximation performance ratio for network Steiner tree problem provided by Karpiski and Zelikovsky [1997] (we intend to do it).

In 1995, S. Arora and J. Mitchell independently discovered powerful techniques to establish polynomial-time approximation schemes for geometric optimization problems, including Euclidean and rectilinear Steiner tree problems. It is quite interesting to notice that Arora [1996] appeared only a few days before Mitchell [1999]. Any way, they use very different techniques to reach the same goal. Therefore, both are very interesting.

Arora discovered a new technique about dynamic partition. The dynamic partition was first introduced to the area of Steiner trees by Jiang and Wang [1994]. They designed a polynomial-time approximation scheme for the Euclidean and rectilinear Steiner minimum tree under restriction that the ratio of lengths between the longest edge and the shortest edge in a minimum spanning tree is bounded by a constant. Arora's technique is based on recursive partition. In Jiang and Wang [1994], although partition can be moved parallelly, the size of each cell is fixed. It cannot be varied according to local information about distribution of terminals. Therefore, only in case that terminals are distributed almost evenly, the partition could work well. This is why such a condition that the ratio of lengths between the longest edge and the shortest edge in a minimum spanning tree is bounded by a constant is required. However, in Arora's recursive partition, each big cell is partitioned into small cells independently from other big cells. How to cut only depends on the situation inside of itself. This advantage enables him to discard the condition in Jiang and Wang.

Mitchell's technique was initiated from studying a minimum length rectangular partition problem. Given a rectilinear region R surrounded by a rectilinear polygon and some rectilinear holes, a *rectangular partition* of R is a set of segments in R, which divide R into small rectangles each of which does not contain any hole in its interior. The problem is to find such a rectangular partition with the minimum total length. This problem is NP-hard.

Du et al. [1986] introduced a concept of guillotine subdivision. A guillotine subdivision is a sequence of cuts performed recursively such that each cut partitions a piece ito at least two. Du et al. [1986] showed that the minimum length guillotine rectangular partition can be computed in polynomial-time. However, they were only able to show that this guillotine subdivision is a 2-approximation of the minimum length rectangular partition in a special case. Mitchell [1996] showed that this is actually true in general. He also successfully utilized this technique to obtain constant approximations for other geometric optimization problems.

Inspired by this success, Mitchell [1999] extended guillotine subdivision to m-guillotine subdivision, a rectangular polygonal subdivision such that there exists a cut whose intersection with the subdivision edges consists of a small number ($O(m)$) of connected components and the subdivisions on either side of the cut are also m-guillotine. With a minor change of the proof of Mitchell [1996], Mitchell established a polynomial-time approximation scheme for minimum length rectangular partition. Mitchell [1997] and Mitchell et al [1999] further extended this m-guillotine subdivision technique to other geometric optimization problems, including Euclidean and rectilinear Steiner tree problems.

3 Some Problems Related to Industries

Successful researches on classical Steiner tree problems encourage extensive study on variations of Steiner trees with various application in industries. Currently, they form a quite active research direction in Steiner trees.

3.1 Class Steiner Trees

In physical VLSI designs, "after the placement of components on a chip, set of pins on the component boundaries are to be connected within the remaining free chip space. For each set of pins sharing the same electrical signal (a net), a Steiner minimal tree is sought, with the pins as required vertices, and the vertices of a grid-like graph, defined by the positions of pins and component boundaries, as Steiner vertices." (See Ihler et al [1999].) Motivated from the flexibility of pin positions, Ihler et al proposed a variation of Steiner tree problem, called *class Steiner tree* problem as follows: given a connected graph with required classes of terminals, find a shortest connected subgraph that contains at least one terminal from each class.

Ihler et al showed that the class Steiner tree problem is harder than set-covering problem and hence unlikely to have a constant-bounded approximation. Therefore, they were satisfied with two approximations with pretty large performance ratio.

However, we feel that there are some important informations lost from the real world problem to the mathematical formulation:

- The Steiner tree lays in the rectilinear plane rather than an arbitrary graph.

- Each class of terminals lay on the boundary of a connected component.

The lack of the above information made the computational complexity increasing. We believe that if we consider the class Steiner tree problem with these two conditions, then there may exist much better approximation, even polynomial-time approximation schemes. We intend to study along this direction.

A similar problem appeared in designs of highway intersections: It is to construct roads of minimum total length to interconnect several highways under the constraint that the roads can intersect each highway only at one point in a designated interval which is a line-segment. Du, Hwang, and Xue [1999] presented a set of optimality conditions for the problem and showed how to construct a solution to meet this set of optimality conditions. Howeever, no bounded polynomial-time approximation has been found for this problem. We intend to study its approximation together with the above problem. We intend to study its approximation together with the above problem.

3.2 On-line and Dynamic Steiner Trees

When a new customer is out of original telephone network, the company has to build a new line to connect the customer into the network. This situation brings us an on-line Steiner tree problem as follow: Assume that a sequence of points in a metric space are given step by step. In the ith step, only locations of the first n_i points in the sequence are known. The problem is to construct a shorter network at each step based on the network constructed in previous steps. The study of on-line problems was initiated from Sleator and Tarjan [1985] and Manase, McGeoch, and Sleator [1988]. A criterion for the performance of an on-line algorithm is to compare the solution generated by the on-line algorithm with the solution of corresponding off-line problem. In the Euclidean plane, it has been known that the worst-case ratio of lengths between on-line solution and off-line solution is between $O(n \log n / \log \log n)$ and $O(n \log n)$ (see Alon and Azar [1993], Westbrook and Yan [1995], and Tsai et al [1996]).

When customers are allowed to drop from the network, it called the *dynamic Steiner tree* problem. The dynamic Steiner tree has application in network routing (see Aharoni and Cohen [1998]). For online and dynamic optimization problems, the running time is very important issue in designs of approximation algorithms. Therefore, reducing running time with preserved performance ratio is an interesting issue in the study of those problems.

3.3 Steiner Tree Packing

Given m sets of terminals, find m Steiner trees each interconnecting one set of terminals such that no cross lines exists and the total length reaches the minimum. This is called the *Steiner tree packing* problem. This problem came from VLSI design at the earlier year (see Hwang et al [1992]). Recently, the development of new technologies requires to solve some variations of Steiner tree packing problems (see Pulleyblank [1995]). For example, The edges of the Steiner trees are required to lie in channels between cells. Each channel has a capacity which tells at most how many edges can run through it. This problem is widely open.

3.4 High Dimensional Steiner Trees

Recently, the satellite communication promotes studying on Steiner trees in three-dimensional Euclidean space and the multilayer chips initiates interests in Steiner trees in three-dimensional rectilinear space. Actually, Steiner trees in high-dimensional space have been studied for many year due to some theoretical interest and its application in constructing phylogenetic trees (Cavalli-Sforza and Edwards [1967]). There are two long-standing conjectures about them.

Conjecture 3.1 (Chung-Gilbert [1976) *] The Steiner ratio in n-dimensional Euclidean space is at least $\sqrt{3}/(4-\sqrt{2})$.*

Conjecture 3.2 (Graham-Hwang [1976) *] The Steiner ratio in n-dimensional rectilinear space is $d/(2d-1)$.*

Proving these conjectures is quite challenge and would certainly need to make a great progress in analysis techniques.

It was also conjectured by Gilbert and Pollak [1968] that in any Euclidean space the Steiner ratio is achieved by the vertex set of a regular simplex. Chung and Gilbert [1976] constructed a sequence of Steiner trees on regular simplexes. The lengths of constructed Steiner trees goes decreasingly to $\sqrt{3}/(4-\sqrt{2})$. Although the constructed trees are not known to be Steiner minimum trees, Chung and Gilbert conjectured that $\sqrt{3}/(4-\sqrt{2})$ is the best lower bound for Steiner ratios in Euclidean spaces. Clearly, if $\sqrt{3}/(4-\sqrt{2})$ is the limiting Steiner ratio in d-dimensional Euclidean space as d goes to infinity, then Chung-Gilbert's conjecture is a corollary of Gilbert and Pollak's general conjecture. However, this general conjecture of Gilbert and Pollak has been disproved by Smith [1992] for dimension from three to nine and by Du and Smith [1996] for dimension larger than two. Now, interesting questions which arise in this situation are about Chung and Gilbert's conjecture. Could Chung-Gilbert's conjecture also be false? If the conjecture is not false, can we prove it by the minimax approach?

First, we claim that Chung-Gilbert's conjecture could be true. In fact, we could get rid of Gilbert-Pollak's general conjecture, and use another way to reach the conclusion that the limiting Steiner ratio for regular simplex is the best lower bound for Steiner ratios in Euclidean spaces. To support our viewpoint, let us analyze a possible proof of such a conclusion as follows.

Consider n points in $(n-1)$-dimensional Euclidean space. Then all of $n(n-1)/2$ distances between the n points are independent. Suppose that we could do a similar transformation and the minimax theorem could apply to these n points to obtain a similar result in the proof of Gilbert-Pollak's conjecture for Euclidean plane, i.e. a point set with critical geometric structure has the property that the union of all minimum spanning trees contains as many equilateral triangles as possible. Then such a critical structure must be a regular simplex.

The above observation tells us two facts:

(a) Chung-Gilbert's conjecture can follow from the following two conjectures.

- The Steiner ratio for n points in an Euclidean space is not smaller than the Steiner ratio for the vertex set of $(n-1)$-dimensional regular simplex.
- (Smith [1992]) $\sqrt{3}/(4-\sqrt{2})$ is the limiting Steiner ratio for simplex.

(b) It may be possible to prove Conjecture 1 by the minimax approach if we could find a right transformation.

One may wonder why we need to find a right transformation. What happens to the transformation used in proof of Gilbert-Pollak's conjecture in the Euclidean plane? Here, we remark that such a transformation does not work for Conjecture 1. In fact, in the Euclidean plane, with a fixed graph structure, all edge-lengths of a full Steiner tree can determine the set of original points and furthermore the length of a spanning tree for a fixed graph structure is a convex function of the edges-lengths of the Steiner tree. However, in Euclidean spaces of dimension more than two , edge-lengths of a full Steiner tree are not enough to determine the set of original points. Moreover, adding other parameters may destroy the convexity of the length of a spanning tree as a function of the parameters.

From the above, we see that proving Chung-Gilbert's conjecture requires a further development of the minimax approach.

Graham-Hwang's conjecture can be easily transferred to a minimax problem requested by our minimax approach. For example, choose lengths of all straight segments of a Steiner tree. When connection pattern of the Steiner tree is fixed, the set of original points can be determined by such segments-lengths, the length of the Steiner tree is a linear function and the length of a spanning tree is a convex function of such segment-lengths, so that g_i is a concave function of such segment-lengths. However, for this transformation, it is hard to determine the critical structure. To explain the difficulty, we notice that in general the critical points could exist in both the boundary and interior of the polytope. (See the minimax theorem.) In the proof of Gilbert-Pollak's conjecture in plane, a crucial fact is that only interior critical points need to be considered in a contradiction argument. The critical structure of interior critical points are relatively easy to be determined. However, for the current transformation on Graham-Hwang's conjecture, we have to consider some critical points on the boundary. It requires a new technique, either determine critical structure for such critical points or eliminate them from our consideration.

One possible idea is to combine the minimax approach and Hwang's method. In fact, by the minimax approach, we may get useful condition on the set of original points. With such a condition, the point set can have only certain type of full Steiner trees. This may reduce the difficulty of extending Hwang's method to high dimension.

The techniques developed for solving problems about Steiner trees in the Euclidean and rectilinear space can usually be extended to Minkowski-Banach spaces. The following open problems are also our target:

Conjecture 3.3 (Cieslik [1990) *, Du et al [1993]] In any Minkowski plane, the Steiner ratio is between 2/3 and* $\sqrt{3}/2$.

Conjecture 3.4 (Du et al [1993) *] The Steiner ratio in a Minkowski plane equals that in its dual plane.*

Conjecture 3.5 *In any infinite dimensional Banach space, the Steiner ratio is between* $1/2$ *and* $\sqrt{3}/(2-\sqrt{2})$.

Conjecture 3.6 *The Steiner ratio in any Banach space equals that in its dual space.*

3.5 Multi-weight Steiner Trees

A complicated computer network may consist of nets of different speeds. The following problem was proposed based on such a background: Consider an undirected network with multiple edge weights $(c_1(e), c_2(e), ..., c_k(e))$ $(c_1(e) > c_2(e) > \cdots > c_k(e))$. Given a subset N of vertices and a partition $\{N_1, N_2, ..., N_k\}$ of N with $|N_1| \geq 2$, find a subnetwork interconnecting N with minimum total weight such that the length of any edge e on a path between a pair of vertices in N_j is at least $c_j(e)$ (see Iwainsky [1985] and

Duin [1991]). We found that this problem can be transformed to multiphase Steiner network problem. Thus, we intend to combine researches on these two problems.

What is the multiphase Steiner network problem? Given an edge-weighed graph B with vertex set X and subsets $X_1, Y_1, \cdots, X_m, Y_m$ of X with $X_i \cap Y_i = \emptyset$, the problem is to find a minimum weighed subgraph G such that for every $i = 1, \cdots, m$, G contains a Steiner tree for X_i without using vertices not in Y_i.

A accompany of the multiphase Steiner tree problem is the multiphase spanning network problem: Given an edge-weighed complete graph with vertex set X ($|X| = n$) and subsets $X_1, \cdots, X_m$ of vertices, the problem is to find a minimum weighed subgraph G such that for every $i = 1, \cdots, m$, G contains a spanning tree for X_i. If NP $\neq$ P, then the best performance ratio of polynomial-time approximation for this problem is $O(n \log n)$. But, in unit weigh case (this case has more applications), it is unknown whether a constant-bounded polynomial-time approximation exists or not.

Both multiphase spanning network and Steiner network problems arose in communication network design (Prisner [1992]) and vacuum system design (Du and Miller [1988]). For the former one, when the solution is a forest, the system $(X_1, \cdots, X_m)$ is called *subtree hypergraph*. Such a system has various applications in computer database schemes (Beeri et al [1983]) and statistics. It is also related to chordal graphs (Duchet [1978]). Tarjan and Yannakakis [1984] gave a $O(m+n)$-time algorithm to tell whether a set system is a subtree hypergraph or not.

3.6 Steiner Arborescence

Given a weighted directed graph G, a vertex r, and a subset P of n vertices, a *Steiner arborescence* is a directed tree with root r such that for each $x \in P$ there exists a path from r to x. The shortest Steiner arborescence is also called a *minimum Steiner arborescence.* Computing minimum Steiner arborescence is an NP-hard problem. Also, one knows that if NP $\neq$ P, then the best possible performance ratio of polynomial-time approximation for this problem is $O(\log n)$. This means that although, like the minimum spanning tree, the minimum arborescence as a shortest arborescence tree without Steiner points can be computed in polynomial-time, the Steiner ratio (the maximum lower bound for the ratio of lengths between the minimum Steiner arborescence and the minimum arborescence for the same set of given points) in directed graphs is zero. Charikar et al [1999] apply

Arora's techniques to this problem and obtained the best known result that for any $\varepsilon > 0$ there exists a polynomial-time approximation with performance ratio $O(n^\varepsilon)$. An open problem remains for closing the gap between the lower bound and the upper bound for the performance ratio.

A version of this problem in the rectilinear plane has a great interest in VLSI designs and an interesting story in the literature. Given a set P of n points in the first quadrant of the rectilinear plane, a *rectilinear Steiner arborescence tree* is a directed tree rooted at the origin, consisting of all paths from the root to points in P with horizontal edges oriented in left-to-right direction and vertical edges oriented in bottom-up direction. What is the complexity of computing the minimum rectilinear arborescence? First, it was claimed that a polynomial-time algorithm was found. However, Rao, Sadayappan, Hwang, and Shor [1992] found a serious flow in this algorithm. Although they could not show the NP-completeness of the problem, they pointed out the difficulties of computing the minimum rectilinear arborescence in polynomial-time. They also showed that while the ratio of lengths between a minimum arborescence tree and a minimum Steiner tree for the same set of points tends to infinity, there is a polynomial-time approximation with performance two. Recently, Shi and Su [2000] showed that computing the minimum rectilinear arborescence is NP-hard. Lu and Ruan [2000] showed, by employing Arora's techniques, that there is a polynomial-time approximation scheme for the problem. We intend to implement this algorithm to obtain an efficient software in the real world.

3.7 Bottleneck Steiner Trees and Related Problem

In wavelength-division multiplexing (WDM) optical network design (Li et al [1994] and Ramamurthy et al [1997]), suppose we need to connect n sites located at $p_1, p_2, \cdots, p_n$ with WDM optical network. Due to the limit in transmission power, signals can only travel a limited distance (say R) for guaranteed correct transmission. If some of the inter-site distances are greater than R, we need to provide some amplifiers or receivers/transmitters at some locations in order to break it into shorter pieces. This situation requires us to consider the problem of minimizing the maximum edge-length and the number of Steiner points in design of WDM optical network. To do so, two variations of Steiner trees have been studied.

The first is to minimize the maximum edge-length under an upper bound on the number of Steiner points. That is, given a set $P = \{p_1, p_2, \ldots, p_n\}$ of n terminals and an positive integer k, we want to find a Steiner tree with at most k Steiner points such that the length of the longest edges in the tree is minimized. This is one of the bottleneck Steiner tree problems. Wang and Du [2000] showed that (a) if $NP \neq P$, then the performance ratio of any polynomial-time approximation for the problem in the Euclidean plane is at least $\sqrt{2}$; (b) if $NP \neq P$, then the performance ratio of any polynomial-time approximation for the problem in the rectilinear plane is at least two; (c) there exists a polynomial-time approximation with performance ratio two for the problem in both rectilinear and Euclidean planes.

The second is to minimize the number of Steiner points under upper bound for edge-length. That is, given a set of n terminals $X = \{p_1, p_2, \cdots, p_n\}$ in the Euclidean plane $\mathcal{R}^2$, and a positive constant R, the problem is to compute a tree T spanning a superset of X such that each edge in the tree has a length no more than R and with the minimum number $C(T)$ of points other than those in X, called *Steiner points*. This problem is called *Steiner tree problem with minimum number of Steiner points*, denoted by *STP-MSP* for short. Lin and Xue [1998] showed that the STP-MSP problem is NP-hard. They also showed that the approximation obtained from the minimum spanning tree by simply breaking each edge into small pieces within the upper bound (called steinerized spanning tree) has a worst-case performance ratio at most five. Chen et al [2000] showed that this approximation has a performance ratio exactly four. They also presented a new polynomial-time approximation with a performance ratio at most three and a polynomial-time approximation scheme under certain conditions. Lu et al [2000] studied the STP-MSP in rectilinear plane. They showed that in the rectilinear plane, the steinerized spanning tree has performance ratio exactly three and there exists a polynomial-time approximation two.

4 Conclusion

The following problems are worth studying:

- Open problems on the Steiner ratio, such as Chung-Gilbert's conjecture, Graham-Hwang's conjecture, and Cielick's conjecture, etc..
- Find better approximation for network Steiner trees and establish an explicit lower bound for the approximation performance ratio of network Steiner trees.
- Close the gap between the lower bound and the upper bound for the approximation performance ratio of Steiner minimum arborescence.

- Find more efficient approximation algorithms for on-line and dynamic Steiner minimum trees and various Steiner tree packing problems.

- Find good approximation algorithms for multi-weighted Steiner trees and multiphase Steiner trees and study close relationship between multi-weighted Steiner trees, multiphase Steiner trees, and phylogenetic trees.

- Make clear whether there exists a polynomial-time approximation scheme for class Steiner tree in the special case with the real world background and highway interconnection problem.

- Close the gap between the lower bound and the upper bound for the approximation performance ratio of bottleneck Steiner tree in the Euclidean plane and make clear whether there exists a polynomial-time approximation scheme for the Steiner tree with minimum number of Steiner points and bounded edge-length.

- Implement efficient approximation algorithms to meet the requests from industries.

We believe that to attack these new and old open problems new techniques are still required and the Steiner tree is still an attractive topic for researchers in combinatorial optimization and computer science.

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