Np-complete

↳ Introduction:

- There are two types of problems:

  ✓ Problems whose time complexity is polynomial: O(logn), O(n), O(nlogn), $O(n^2)$, $O(n^3)$
  Examples: searching, sorting, merging, MST, etc.

  ✓ Problems with exponential time complexity: $O(2^n)$, O(n!), $O(n^n)$, etc.

  Examples: TSP, n-queen, 0/1knapsack, etc.

- Two classes of algorithms:

  ✓ P: The set of all problems, which can be solved by deterministic algorithms in polynomial time.

  ✓ NP: The set of all problems which can be solved by nondeterministic algorithms in polynomial time (NP: Nondeterministic Polynomial)

↳ Non-deterministic algorithms:

- Unlike deterministic algorithms, each operation has several outcomes

- Example:
  - ✓ x = choice(1..n);
  - ✓ x may have any value between 1 and n
  - ✓ The time of this type of instruction is O(1).
  - ✓ If there is a solution, the algorithm will terminate successfully; otherwise, it will terminate unsuccessfully.

↳ Example1: Searching problem:

- input: A(1..n) and x
- Output: index j such that A(j)=x if x is in A or j=0 if x does not belong to A.

- Ndsearch(A(1..n), x)

```
Integer j;
Begin
    J=choice(1..n);
    If A(j) =x
    Then
        Print(j);
    Else
        Print(0);
    Endif;
End;
```

- ✓ The complexity is O(1);

↳Example2: clique problem

- Definition: A maximal complete subgraph of a graph G=(V,E) is a clique.

- Input: - a graph G=(V,E) and an integer k;
- Output: Determine if G has a clique of size at least k.

- Brute force approach:

  - ✓ The obvious way to solve this problem would be to subject all $\binom{|V|}{k}$ subsets of V with cardinality k to test whether there is a clique.

- Ndclique(G,k);
  Integer I; X[1..n];
  Begin
  For I=1 to k do
  X[I] = choice(1..n);
  Endfor;
  If(X[1], X[2], ,,,, X[k])) is a clique
  Then
  print ("SUCCESS");
  else
  print("FAILURE");
  endif;
  end;

↳ Example3: Satisfiability: has a special role in the theory of computation.

- Definitions:
  - ✓ A literal is a boolean variable (its value is either true or false).
  - ✓ A logical formula is an expression that can be constructed using literals and the operations AND and OR.
  - ✓ The satisfiability problem is to determine if a logical formula is true for some assignment of truth values to the variables.

- Example:

$$F = \underbrace{(x_1 \text{ or } x_2)}_{C_1} \text{ and } \underbrace{(\bar{x}_2 \text{ or } x_3)}_{C_2} \text{ and } \underbrace{(\bar{x}_1 \text{ or } \bar{x}_2)}_{C_3}$$

where $x_i \in \{0,1\}$ $1 \le i \le 3$ and
C_i are called clauses

- Is there an assignment of truth values to the variables x_i's that makes the formula F true ("Satisfies " it)?

- For n variables, one should consider 2^n possible assignments.

- Ndsatisfiability(E,n)

```
Integer i;
Begin
    For i=1 to n do
        x_i = choice(true, false);
    Endfor;
    If E(x_1 , x_1, …, x_n) is true
    Then
        Print "success";
    Else
        Print "Failure";
    Endif;
End;
```

- Let n be the number of variables and p be the number of operations ANDs and Ors, the Ndsatisfiability takes O(max(n,p))

  Since in general p>>n, we have O(p).

↳ NP-Complete problems:

- The theory of NP-completeness consists of two classes of problems:
  - ✓ NP-complete problems
  - ✓ NP-hard problems

- NP-hard problems:
  - ✓ If an NP-hard problem can be solved in polynomial time then all NP-complete problems can be solved in polynomial time.

- ✓ In other words: A problem id NP-hard if every problem in NP is transformable to it

- NP-complete problems:

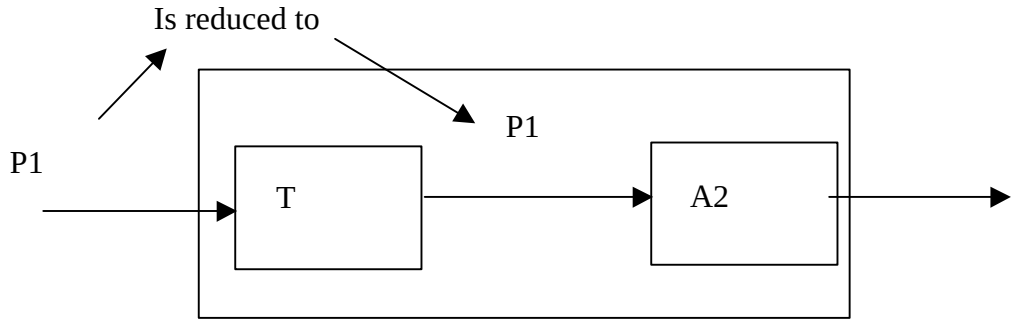
  - ✓ A problem which is NP-complete will have the property that it can be solved in polynomial time iff all other NP-complete problems can be solved in polynomial time.

  - ✓ In other words: A problem is NP-complete if it is both NP_hard and NP.

  - Note:
    - ✓ NP-complete problems are NP-hard
    - ✓ All NP-hard problems are not NP-complete.

↳ Open problem: $P \overset{?}{=} NP$

- Definition of reduction: $\propto$



A1

- ✓ A1 is defined by T and A2, where T is a polynomial transformation

- ✓ A1 ≡ (T,A2) ➔ P1 ∝ P2
  We say that P1 is reduced to P2.

- ✓ If P2 is polynomial, then P1 is also polynomial.

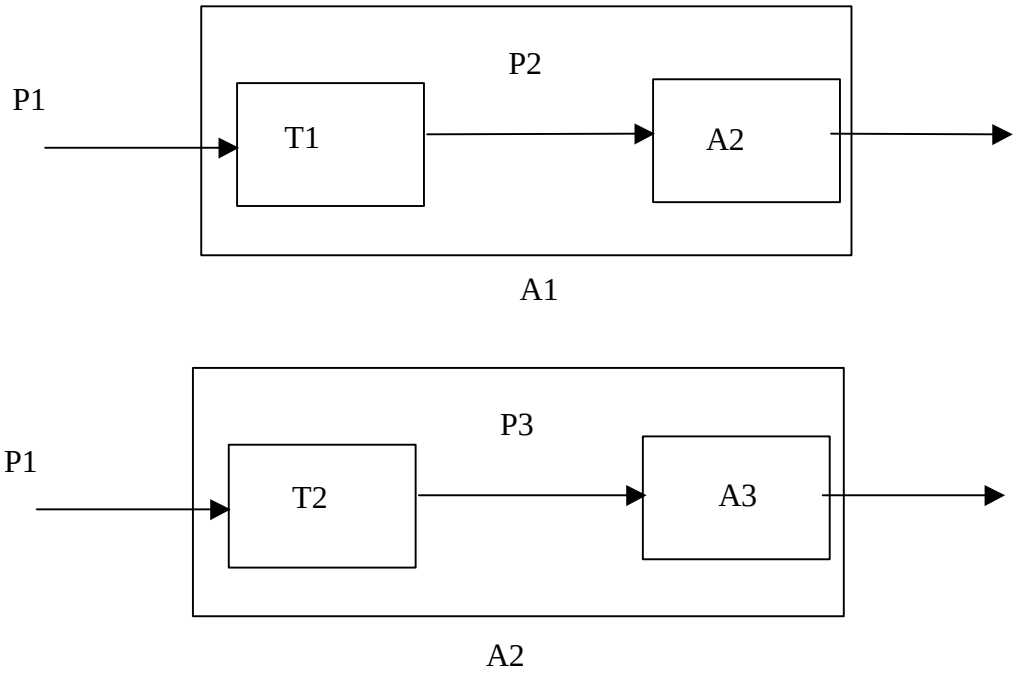
↳ NP-complete:

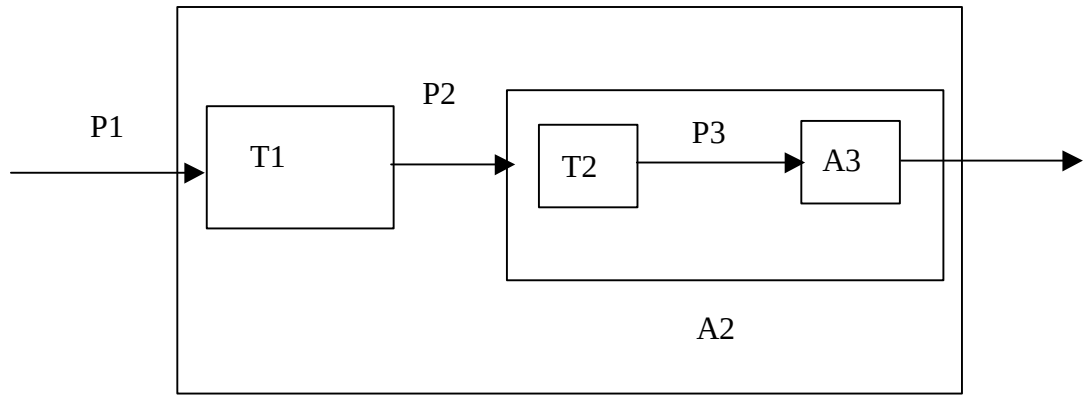
- A problem is NP-complete:
  1) if A is NP
  2) every NP problem Q: Q ∝ P

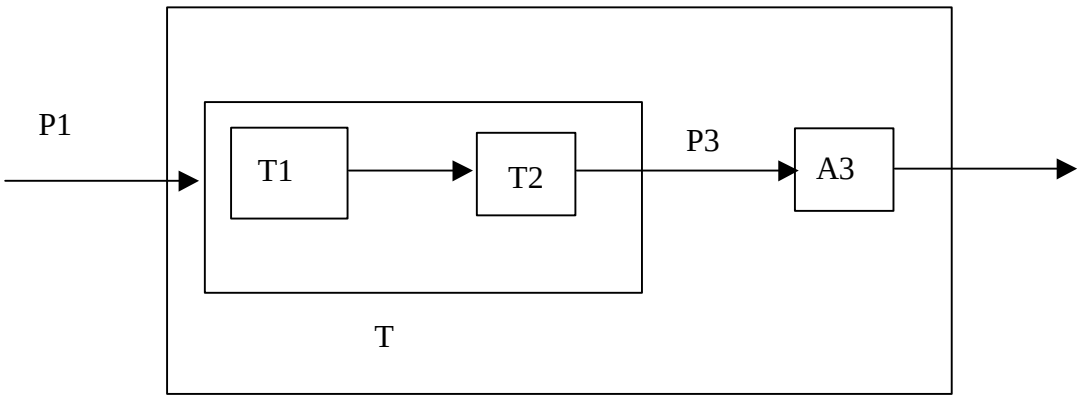
↳ Cook's theorem: Satisfiability is NP-Complete

- Theorem: If P1 ∝ P2 and P2 ∝ P3 ➔ P1∝P3

  Proof:







T is polynomial since T1 and T2 are polynomial $P1 \propto P3$.

↳ Theorem: Given a problem P, If

1) P is NP and

3) ∃ NP-complete problem Q: Q ∝P

Then P is NP-complete.

Proof:

We have to prove that P is NP and every NP problem R, R ∝ P

- P is NP be definition
- Let R be in NP

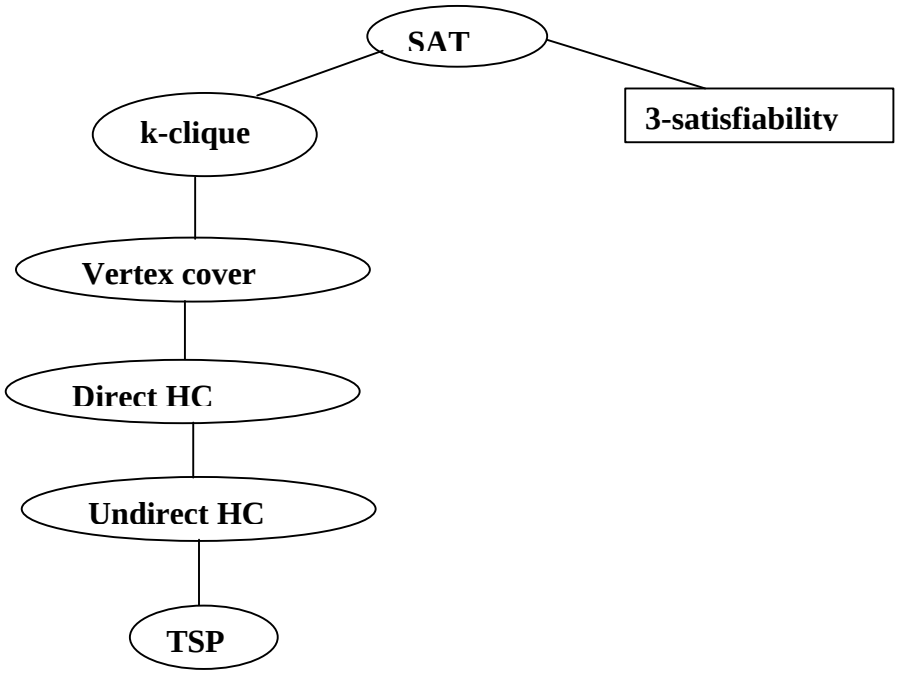
R ∝ Q since Q is NP-complete by definition

And Q∝P by definition

R ∝Q and Q ∝ P ➔ R ∝ P for every NP problem R.

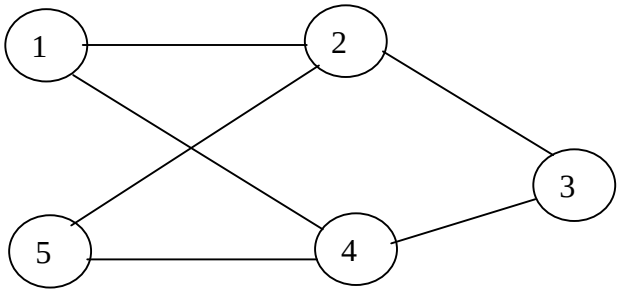
↳ Corollaire:

To prove a new problem P is NP-complete, we first prove that it is NP, and then find an NP-complete problem Q that reduces to P.



↳Example: Node cover problem:

- Definition: Given a graph G=(V,E), a subset S of V is a node
  cover of G iff all edges are incident to at least on vertex in S.



S = {1,2}