Greedy Method

➱ Objective:

➱General approach:

- Given a set of n inputs.
- Find a subset, called feasible solution, of the n inputs subject to some constraints, and satisfying a given objective function.
- If the objective function is maximized or minimized, the feasible solution is optimal.
- It is a locally optimal method.

➱Algorithm:

☞ Step 1: Choose an input from the input set, based on some criterion. If no more input exit.

☞ Step 2: Check whether the chosen input yields to a feasible solution. If no, discard the input and goto step 1.

☞ Step 3: Include the input into the solution vector and update the objective function. Goto step 1.

Optimal merge patterns

↳ Introduction:

- Merge two files each has n & m elements, respectively:
  $\Rightarrow$ takes O (n+m).

- Given n files
  What's the minimum time needed to merge all n files?

- Example:
  $(F_1, F_2, F_3, F_4, F_5) = (20, 30, 10, 5, 30)$.

  $M_1 = F_1$ & F_2 $\Rightarrow 20+30 = 50$
  $M_2 = M_1$ & F_3 $\Rightarrow 50+10 = 60$
  $M_3 = M_2$ & F_4 $\Rightarrow 60+5 = 65$
  $M_4 = M_3$ & F_5 $\Rightarrow 65+30 = 95$
  270

- **Optimal merge pattern**: Greedy method.

  Sort the list of files:

  $(5, 10, 20, 30, 30) = (F_4, F_3, F_1, F_2, F_5)$

Merge the first two files:

(5, 10, 20, 30, 30) ➔ (15, 20, 30, 30)

Merge the next two files:

(15, 20, 30, 30) ➔ (30, 30, 35)

Merge the next two files:

(30, 30, 35) ➔ (35, 60)

Merge the last two files:

(35,60) ➔ (95)

Total time: 15 + 35 + 60 + 95 = 205

⇒ This is called a 2-way merge pattern.

- **Problem:**

  ✓ Given n sorted files
  ✓ Merge n files in a minimum amount of time.

- **Algorithm**:

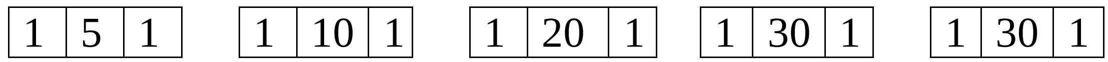
  ✓ We associate with each file a node

Left	Weight	Right

Example:

Initial

✓



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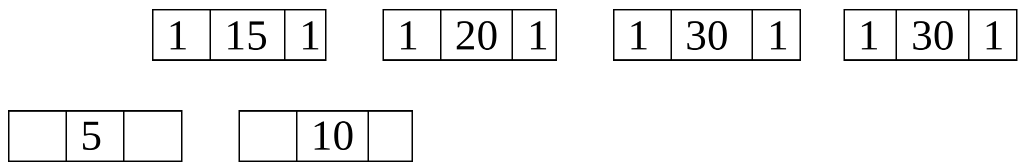

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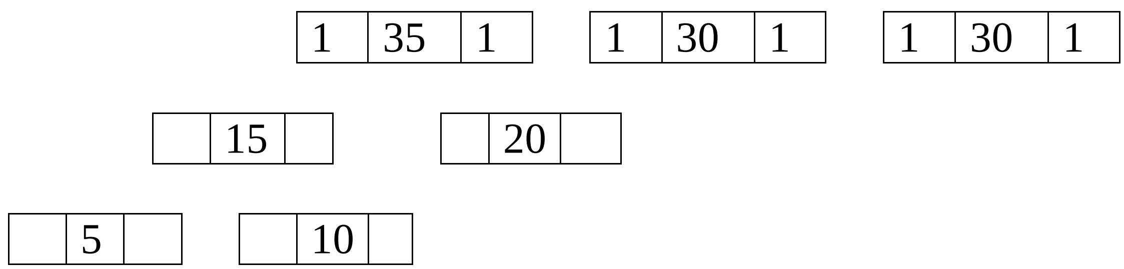

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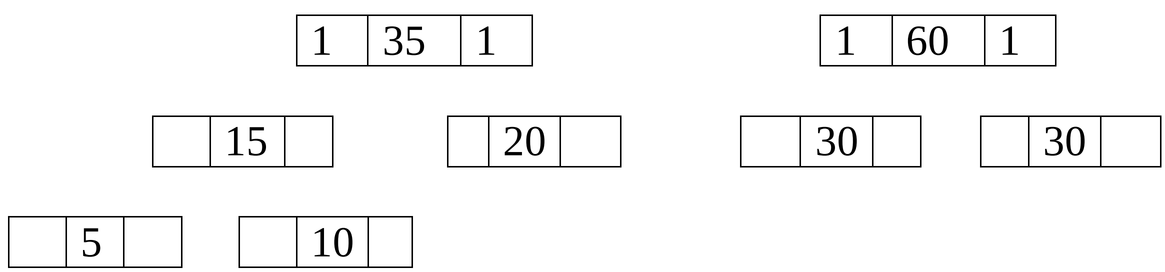

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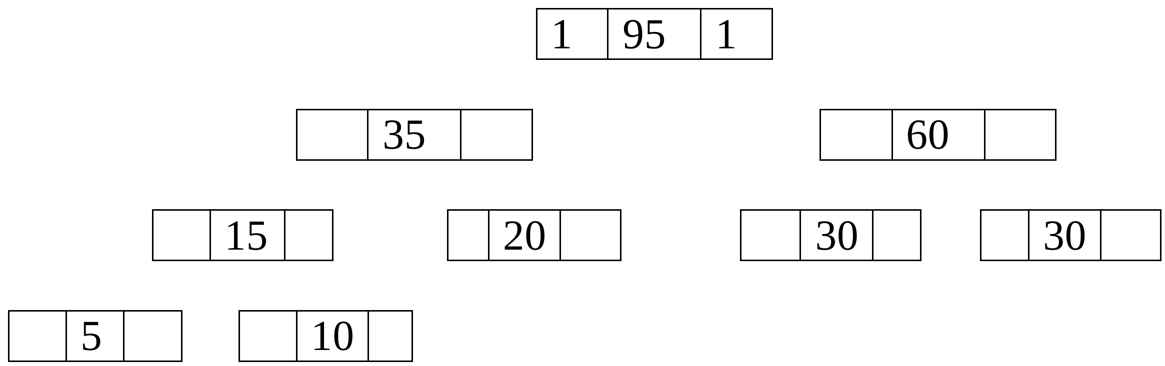


- **Algorithm:**

```
-   Least (L):  find a tree in L whose root has the smallest
                weight.
-   Function :   Tree (L,n).
      Integer i;
      Begin
      For  i=1 to n -1 do
            Get node (T)       /* create a node pointed by T  */
            Left child (T)=   Least (L)       /* first smallest */
            Right child (T)= Least (L)    /* second smallest */
            Weight (T)  = weight (left child (T))
                          + weight (right child (T))
             Insert (L,T);      /*  insert new tree with root T in L */
    End for
    Return (Least (L))   /* tree left in L */
End.
```

- **Analysis:**

$$T = O(n-1) * \max(O(\text{Least}), O(\text{Insert})).$$

- Case 1 L is not sorted.

O (Least)= O (n).
O (Insert)= O (1).

$\Rightarrow$ T= O (n_2).

- Case 2 L is sorted.

Case 2.1

O (Least)= O (1)
O (Insert)= O (n)

$\Rightarrow$ T= O (n_2)

Case 2.2

L is represented as a min-heap. Value in the root is $\leq$ the values of its children.

O (Least)= O (1)
O (Insert)= O (log n)

$\Rightarrow$ T= O (n log n).

Knapsack problem

➮ **Problem**:

- input:
  - ✓ n objects.
  - ✓ each object i has a weight w_i and a profit p_i
  - ✓ Knapsack : M
- output:
  - ✓ Fill up the Knapsack s.t. the total profit is maximized.
  - ✓ Feasible solution: $(x_1, \ldots, x_n)$.

➮ **Formally**,

✓ Let x_i be the fraction of object i placed in the Knapsack, $O \le x_i \le 1$. For $1 \le I \le n$.

✓ Then :

$$P = \sum_{1 \le i \le n} p_i x_i$$

$$\text{And} \quad \sum_{1 \le i \le n} w_i x_i \le M$$

➮**Assumptions:**

- $\sum_{i=1}^{n} w_i > M$; not all $x_i = 1$.
- $\sum_{1 \le i \le n} w_i x_i = M$

➮ **Example:**

- ✓ 3 objects (n=3).
- ✓ $(w_1,w_2,w_3)=(18,15,10)$
- ✓ $(p_1,p_2,p_3)=(25,24,15)$
- ✓ M=20

➮ **Largest-profit strategy**: (Greedy method)

- ✓ Pick always the object with largest profit.
- ✓ If the weight of the object exceeds the remaining Knapsack capacity, take a fraction of the object to fill up the Knapsack.

➮ **Example:**

- ✓ P=0 , C=M=20 /∗ remaining capacity ∗/
- ✓ Put object 1 in the Knapsack.

  P=25 Since $w_1 < M$ then $x_1=1$
  C=M-18=20-18=2

- ✓ Pick object 2

  Since $C < w_2$ then $x_2= C/w_2=2/15$.
  P=25+2/15*24 =25+3.2=28.2

- ✓ Since the Knapsack is full then $x_3=0$.

- ✓ The feasible solution is (1, 2/15,0).

➯ **Smallest-weight strategy**:

- ✓ be greedy in capacity: do not want to fill the knapsack quickly.

- ✓ Pick the object with the smallest weight.

- ✓ If the weight of the object exceeds the remaining knapsack capacity, take a fraction of the object.

➯**Example**:

- ✓ cu=M=20

- ✓ Pick object 3

  Since $w_3 < cu$ then $x_3=1$

  P= 15 cu =20-10 = 10 , x_3 =1

- ✓ Pick object 2

  Since $w_2 > cu$ then $x_2 = 10/15 = 2/3$

  P = 15+ 2/3.24

  = 15+ 16 = 31 cu= 0.

- ✓ Since cu=0 then $x_1=0$

- ✓ Feasible solution : (0,2/3,1) p=31.

➢ **Largest profit-weight ratio strategy**:

- ✓ Order profit-weight ratios of all objects.
- ✓ $P_i/w_i \geq (p_i+1)/(w_i+1)$ for $1 \leq i \leq n-1$
- ✓ Pick the object with the largest p/w
- ✓ If the weight of the object exceeds the remaining knapsack capacity, take a fraction of the object.

➢**Example:**

$P_1/w_1=25/18=1.389$
$P_2/w_2=24/15=1.6$
$P_3/w_3=15/10=1.5$

➔ $P_2/w_2>=P_2/w_2>= P_3/w_3$

Cu=20; p=0

- ✓ Pick object 2
  Since cu $\geq w_2$ then $x_2=1$
  cu=20-15=5 and p=24

- ✓ Pick object 3
  Since cu<w_3 then $x_3=cu/w_3=5/10=1/2$
  cu= 0 and P= 24+1/2.15=24+7.5=31.5

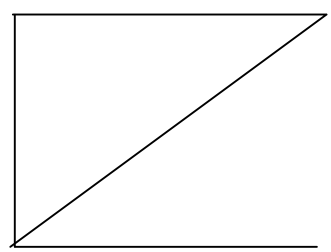
- ✓ Feasible solution (0,1,1/2) p=31.5

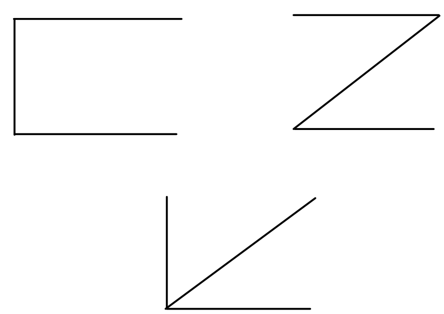
Minimum Spanning Tree.

➯**Definition**:

Let G=(V,E) be an undirected connected graph.
T=(V,E') is a spanning tree iff T is a tree.

➯**Example:**





➯**Definition**:

- If each edge of E has a weight, G is called a weighted graph.

➯**Problem:**

- Given an undirected, connected, weighted graph G=(V,E).
- We wish to find an acyclic subset $T \subseteq E$ that connects all the vertices and whose total weight:

$$w(T) = \sum_{(u,v) \in T} w(u,v) \text{ is minimized.}$$

Where w(u,v) is the weight of edge (u,v).

- T is called a minimum spanning tree of G.

↳ **Solution:**

- Using greedy method.
- Two algorithms:
  - ✓ **Prim's algorithm.**
  - ✓ **Kruskal's algorithm.**

↳ **Approach:**

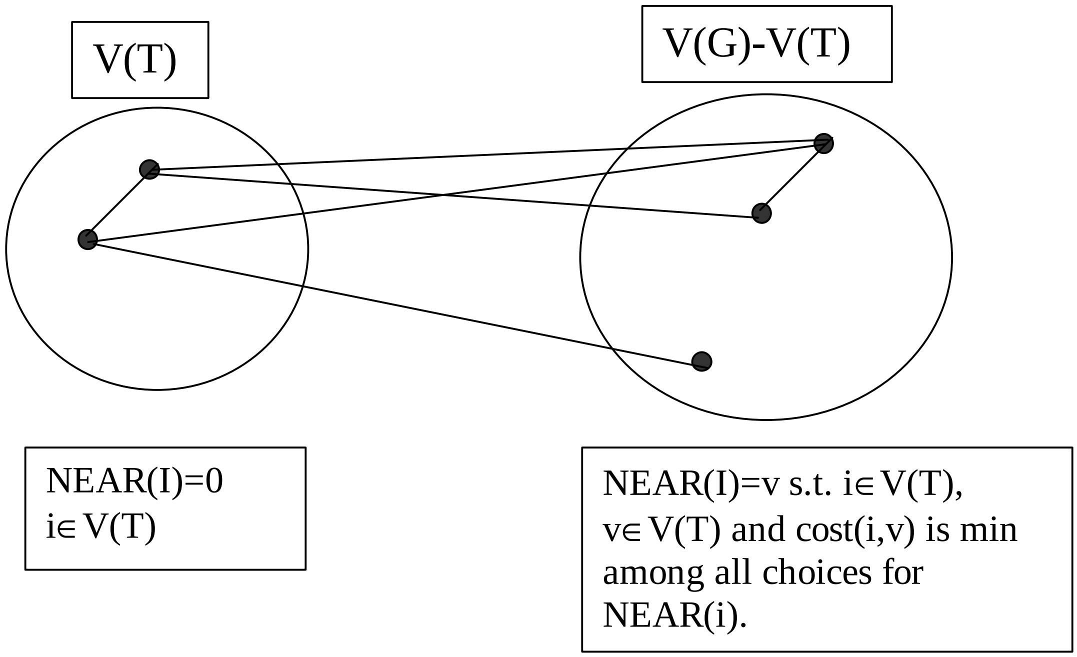
- The tree is built edge by edge.
- Let T be the set of edges selected so far.
- Each time a decision is made:
  - ∗ Include an edge e to T s.t. :
    - Cost (T)+w (e) is minimized, and
    - $T \cup \{e\}$ does not create a cycle.

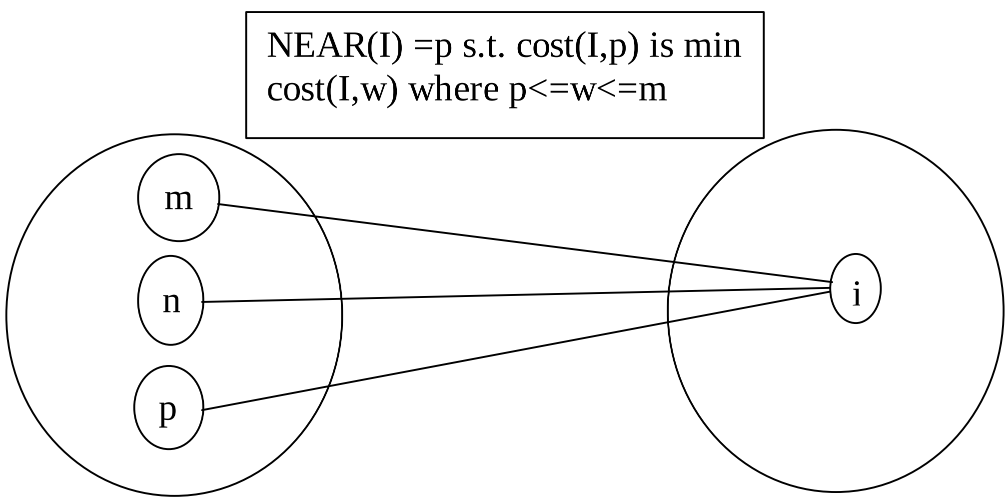
↳ **Prim's algorithm:**

- T forms a single tree.
- The edge e added to T is always least-weight edge connecting the tree, T, to a vertex not in the tree

↳ **Implementation:**

- To choose the next edge to be included in T, NEAR (i:n) array is used.


V(T)
V(G)-V(T)
NEAR(I)=0
i∈ V(T)
NEAR(I)=v s.t. i∈ V(T),
v∈ V(T) and cost(i,v) is min
among all choices for
NEAR(i).


NEAR(I) =p s.t. cost(I,p) is min
cost(I,w) where p<=w<=m
m
n
p
i

```
Procedure PRIM (G, Cost, mincost)
   /* Let n be # of vertices */
   Integer NEAR (1:n);
   Integer u,w,p,I;
1.  Begin
2.       Choose an arbitrary vertex v_o.
3.       mincost=o; NEAR (v_o)=o
4.       For each vertex w ≠ v_o do
5.            NEAR (w)=v_o;
6.       End for

7.       For I=1 to n-1 do  /* fin n-1 edges of T */
8.            Choose a vertex w s.t.
9.                 cost(w,NEAR(w) )= min (cost (u, NEAR(u)) )
10.                where NEAR (u) ≠ o
11.           mincost = mincost+ cost (w, NEAR(w));
12.           NEAR (w)=o
13.           For each vertex p do
14.               if  NEAR(p) ≠ o & cost (p, NEAR(p) ) > cost (p,w)
15.               then  NEAR (p)= w;
16.               endif
17.           end for
18.      End for
19.      End.
```

- Analysis:

    - ✓ The for loop between 4 and 6 takes O(n).
    - ✓ Lines between 8 and 10 take O(n)
    - ✓ The For loop between 13 and 17 takes O(n)
    - ✓ Finally, the main For loop that starts at line 7 takes O(n)
    - ✓ the overall algorithm takes $O(n^2)$.

➪Example:

- Let's start form v=1

Kruskal's algorithm

↳ Problem:

- T form a forest.
- The edge e added to T is always least-weight edge in the graph that connects two distinct trees of T.
- At the end of the algorithm T becomes a single tree.

↳**Example:**

```
Procedure kruskal (G, cost).
Begin
        T: forest
        T= ∅
        while |T| ≤ n-1 & E≠∅ do
                choose an edge (v,w)∈E of least weight
                delete (v,w) form E
                If (v,w) does not create a cycle in T
                then
                        add (v,w) ∈o  T
                else
                        discard (v,w);
                endif
        end while.
```

Implementation:

- Choose the edge with the smallest weight:
    - ✓ Use min-heap:
        - Get the min & read just the heap takes O (log e).
        - Construct the heap takes O (e).
- Be sure that the chosen edge does not create a cycle in the so far built forest, T:
    - ✓ Use union-find:

      Once (u,v) is selected.

      Check if Find (u) ≠ Find (v).

- Summary:
  - ✓ Min-heap on edges.
  - ✓ Union-find on vertices.
- Time complexity O (e log e).

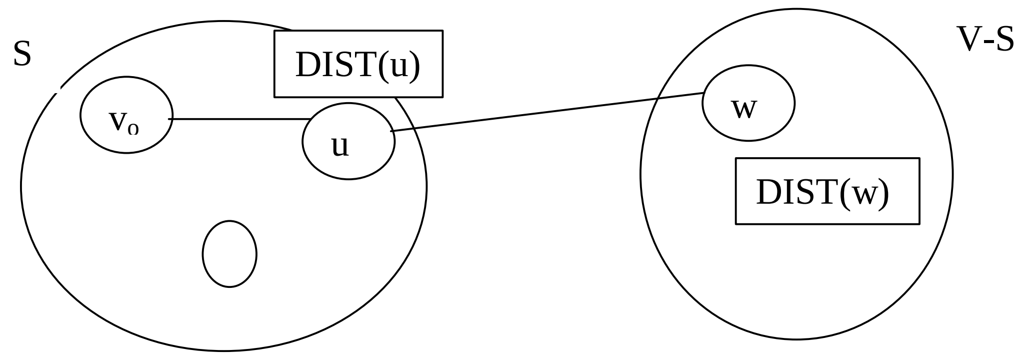
Single Source Shortest Paths.

➩ **Requirements:**

- Given a weighted digraph G= (V,E) where the weights are >0.
- A source vertex, $v_o \in V$.
- Find the shortest path from v_o to all other nodes in G.
- Shortest paths are generated in increasing order: 1,2,3,…..

➩ Algorithm Description: Dijkstra

- S: Set of vertices (including v_o) whose final shortest paths from the source v_o have already been determined.
- For each node $w \in V-S$,
  Dist (w): the length of the shortest path starting from v_o going through only vertices which are in S and ending at w.
- The next path is generated as follows:
  - It's the path of a vertex u which has Dist (u) minimum among all vertices in V-S
  - Put u in S.
- Dist (w) for w in V-S may be decreased going though u.



Compare Dist (u)+ cost (u,w) with Dist (w).

Algorithm:

```
Procedure SSSP (vo, cost, n)
    Array  S (1:n);
  Begin
      /* initialization*/
      For i=1 to n do
             S(i)=o, Dist (i)= cost (vo,i)
      End for.
      S(vo)=1, Dist (vo)=o;
      For i=1 to n-1 do.
                  Choose u  s.t.   Dist (u)=    min      {Dist (w) }
                                             S(w)=0
                  S(u)=1;
                  For all w with S(w)=o do.
                        Dist (w)= min (Dist (w), Dist (u) +
                                                  Cost (u,w) )
                  End for.
      end for.
  end.
```

✓ Time complexity: $O(n^2)$.