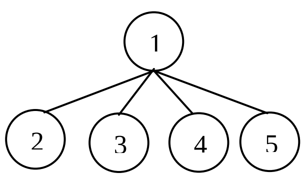
Branch and Bound

Definitions:

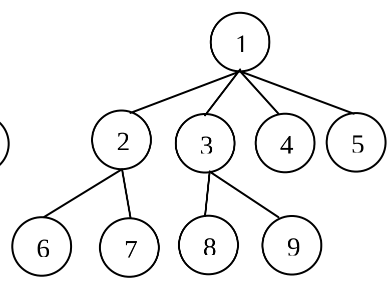
- Branch and Bound is a state space search method in which all the children of a node are generated before expanding any of its children.

- **<u>Live-node</u>**: A node that has not been expanded.

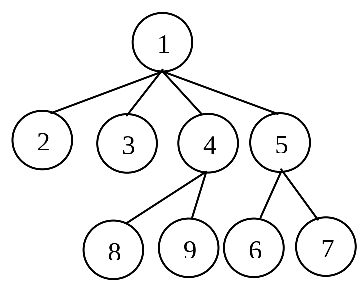
- It is similar to backtracking technique but uses BFS-like search.



Live Node: 2, 3, 4, and 5



FIFO Branch & Bound (BFS)
Children of E-node are inserted in a queue.



LIFO Branch & Bound (D-Search)
Children of E-node are inserted in a stack.

- Dead-node: A node that has been expanded
- Solution-node

LC-Search (Least Cost Search):

- The selection rule for the next E-node in FIFO or LIFO branch-and-bound is sometimes "blind". i.e. the selection rule does not give any preference to a node that has a very good chance of getting the search to an answer node quickly.

- The search for an answer node can often be speeded by using an “intelligent” ranking function, also called **an approximate cost function** $\hat{C}$

- Expanded-node (E-node): is the live node with the best $\hat{C}$ value

➯ Requirements

- Branching: A set of solutions, which is represented by a node, can be partitioned into mutually exclusive sets. Each subset in the partition is represented by a child of the original node.

- Lower bounding: An algorithm is available for calculating a lower bound on the cost of any solution in a given subset.

➯ Searching: Least-cost search (LC)

- Cost and approximation

  ✓ Each node, X, in the search tree is associated with a cost: C(X)

  ✓ C(X) = cost of reaching the current node, X (E-node), from the root + the cost of reaching an answer node from X.

$C(X) = g(X) + h(X)$

- ✓ Get an approximation of $C(x)$, $\hat{C}(x)$ such that

  $\hat{C}(x) \le C(x)$, and

  $\hat{C}(x) = C(x)$ if x is a solution-node.

- ✓ The approximation part of $\hat{C}(x)$ is

  h(x)=the cost of reaching a solution-node from X, not known.

- Least-cost search:

  The next E-node is the one with least $\hat{C}$

➩ Example: 8-puzzle

- Cost function: $\hat{C} = g(x) + h(x)$

  where

  h(x) = the number of misplaced tiles

  and g(x) = the number of moves so far

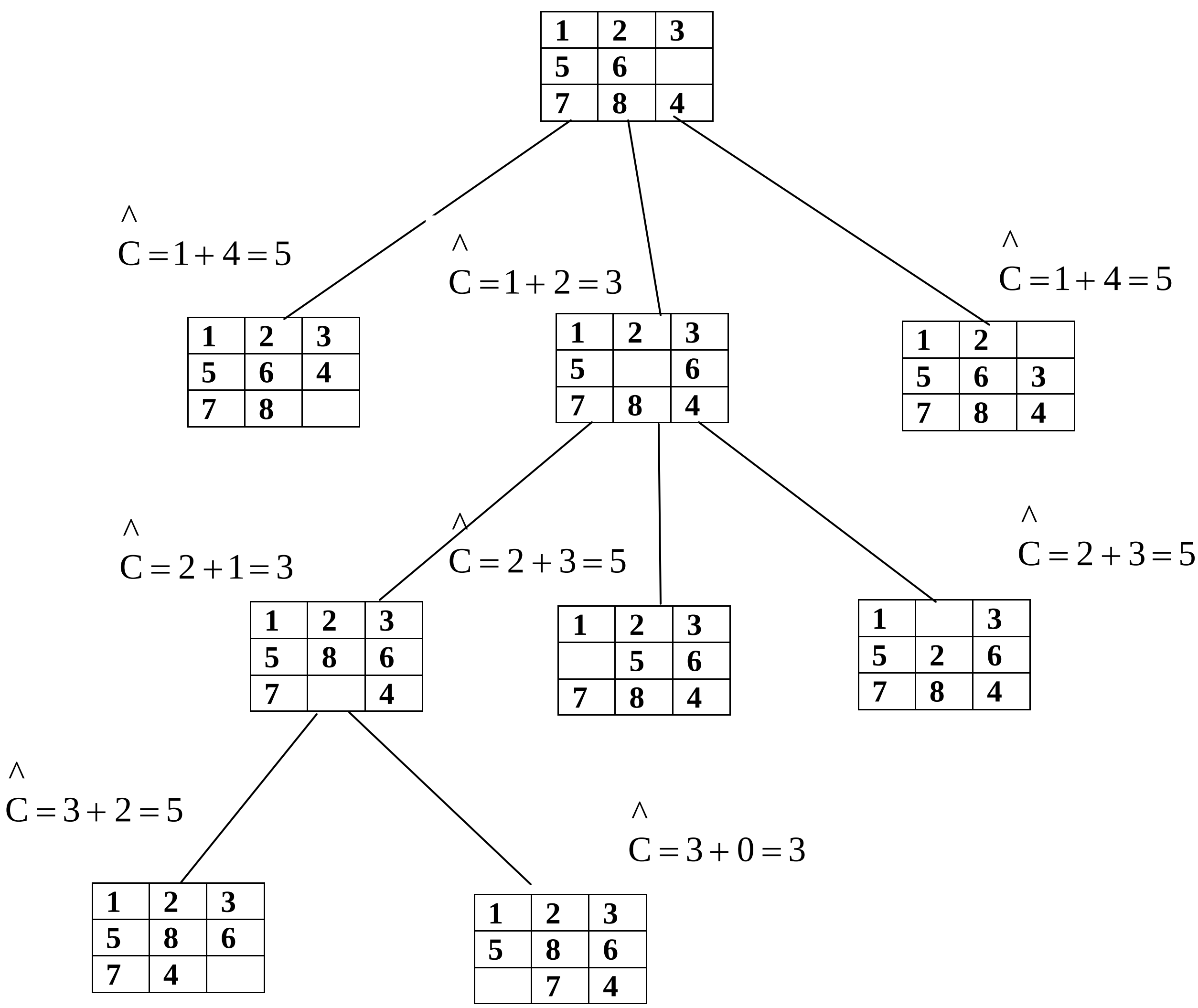
- Assumption: move one tile in any direction cost 1.

Initial State

1	2	3
5	6	
7	8	4

Final State

1	2	3
5	8	6
	7	4



Note: In case of tie, choose the leftmost node.

➯ Algorithm:

```
/* live_node_set: set to hold the live nodes at any time */
/* lowcost: variable to hold the cost of the best cost at any
given node */
Begin
    Lowcost = ∞;
    While live_node_set ≠∞ do
        -  choose a branching node, k, such that
           k ∈live_node_set; /* k is a E-node */
        -  live_node_set= live_node_set - {k};
        -  Generate the children of node k and the
           corresponding lower bounds;
           Sk={(i,zi): i is child of k and zi its lower
                bound}
        -  For each element (i,zi) in Sk do
           -  If zi > U
           -  then
              -  Kill child i; /* i is a child node */
           -  Else
                  If child i is a solution
                  Then
                      U =zi; current best = child i;
                  Else
                      Add child i to live_node_set;
                  Endif;
              Endif;
        -  Endfor;
    Endwhile;
```

➯ Travelling Salesman Problem: **A Branch and Bound algorithm**

- Definition: Find a tour of minimum cost starting from a node S going through other nodes only once and returning to the starting point S.

- Definitions:

  - ✓ A row(column) is said to be reduced iff it contains at least one zero and all remaining entries are non-negative.

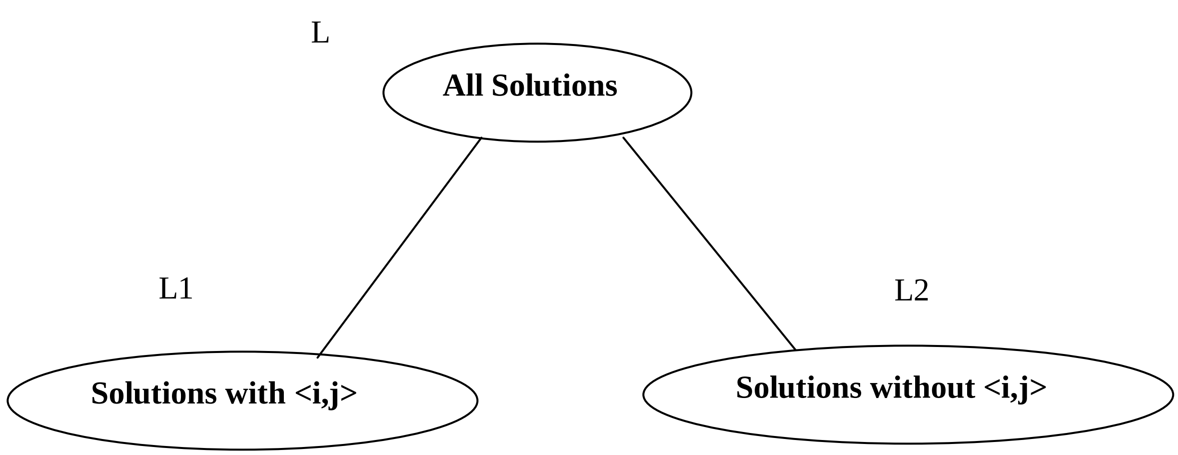
  - ✓ A matrix is reduced iff every row and column is reduced.

- **<u>Branching</u>**:

  - ✓ Each node splits the remaining solutions into two groups: those that include a particular edge and those that exclude that edge

  - ✓ Each node has a lower bound.

  - ✓ Example: Given a graph G=(V,E), let <i,j> $\in$ E,



- **<u>Bounding</u>**: How to compute the cost of each node?

  ✓ Subtract of a constant from any row and any column does not change the optimal solution (The path).

  ✓ The cost of the path changes but not the path itself.

  ✓ Let A be the cost matrix of a G=(V,E).

  ✓ The cost of each node in the search tree is computed as follows:

    - Let R be a node in the tree and A(R) its reduced matrix
    - The cost of the child (R), S:
      - Set row i and column j to infinity
      - Set A(j,1) to infinity
      - Reduced S and let RCL be the reduced cost.
      - C(S) = C(R) + RCL+A(i,j)

  ✓ Get the reduced matrix A' of A and let L be the value subtracted from A.
  ✓ L: represents the lower bound of the path solution
  ✓ The cost of the path is exactly reduced by L.

- What to determine the branching edge?

  ✓ The rule favors a solution through left subtree rather than right subtree, i.e., the matrix is reduced by a dimension.

- ✓ Note that the right subtree only sets the branching edge to infinity.

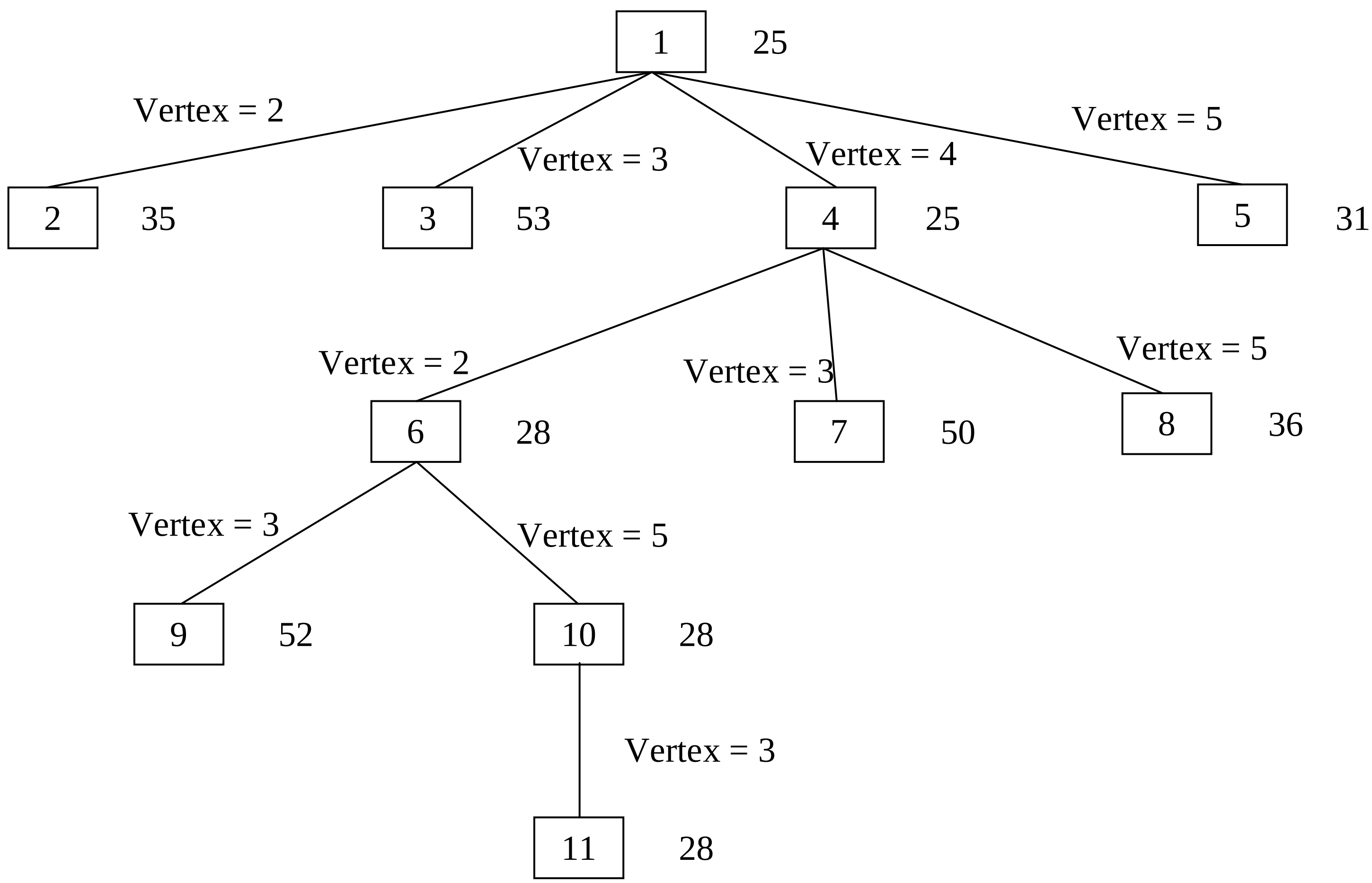
- ✓ Pick the edge that causes the greatest increase in the lower bound of the right subtree, i.e., the lower bound of the root of the right subtree is greater.

- Example:
  - The reduced cost matrix is done as follows:
    - Change all entries of row i and column j to infinity
    - Set A(j,1) to infinity (assuming the start node is 1)
    - Reduce all rows first and then column of the resulting matrix

- Given the following cost matrix:

$$\begin{bmatrix} inf & 20 & 30 & 10 & 11 \\ 15 & inf & 16 & 4 & 2 \\ 3 & 5 & inf & 2 & 4 \\ 19 & 6 & 18 & inf & 3 \\ 16 & 4 & 7 & 16 & inf \end{bmatrix}$$

- State Space Tree:



- The TSP starts from node 1: **<u>Node 1</u>**
  - Reduced Matrix: To get the lower bound of the path starting at node 1
    - Row # 1: reduce by 10

$$\begin{bmatrix} inf & 10 & 20 & 0 & 1 \\ 15 & inf & 16 & 4 & 2 \\ 3 & 5 & inf & 2 & 4 \\ 19 & 6 & 18 & inf & 3 \\ 16 & 4 & 7 & 16 & inf \end{bmatrix}$$

    - Row #2: reduce 2

$$\begin{bmatrix} inf & 10 & 20 & 0 & 1 \\ 13 & inf & 14 & 2 & 0 \\ 3 & 5 & inf & 2 & 4 \\ 19 & 6 & 18 & inf & 3 \\ 16 & 4 & 7 & 16 & inf \end{bmatrix}$$

    - Row #3: reduce by 2

$$\begin{bmatrix} inf & 10 & 20 & 0 & 1 \\ 13 & inf & 14 & 2 & 0 \\ 1 & 3 & inf & 0 & 2 \\ 19 & 6 & 18 & inf & 3 \\ 16 & 4 & 7 & 16 & inf \end{bmatrix}$$

- Row # 4: Reduce by 3:

$$\begin{bmatrix} inf & 10 & 20 & 0 & 1 \\ 13 & inf & 14 & 2 & 0 \\ 1 & 3 & inf & 0 & 2 \\ 16 & 3 & 15 & inf & 0 \\ 16 & 4 & 7 & 16 & inf \end{bmatrix}$$

- Row # 4: Reduce by 4

$$\begin{bmatrix} inf & 10 & 20 & 0 & 1 \\ 13 & inf & 14 & 2 & 0 \\ 1 & 3 & inf & 0 & 2 \\ 16 & 3 & 15 & inf & 0 \\ 12 & 0 & 3 & 12 & inf \end{bmatrix}$$

- Column 1: Reduce by 1

$$\begin{bmatrix} inf & 10 & 20 & 0 & 1 \\ 12 & inf & 14 & 2 & 0 \\ 0 & 3 & inf & 0 & 2 \\ 15 & 3 & 15 & inf & 0 \\ 11 & 0 & 3 & 12 & inf \end{bmatrix}$$

- Column 2: It is reduced.
- Column 3: Reduce by 3

$$\boxed{\begin{bmatrix} inf & 10 & 17 & 0 & 1 \\ 12 & inf & 11 & 2 & 0 \\ 0 & 3 & inf & 0 & 2 \\ 15 & 3 & 12 & inf & 0 \\ 11 & 0 & 0 & 12 & inf \end{bmatrix}}$$

- Column 4: It is reduced.
- Column 5: It is reduced.
- The reduced cost is: RCL = 25
- So the cost of node 1 is:
  - Cost(1) = 25
- The reduced matrix is:

cost(1) = 25

$$\begin{bmatrix} inf & 10 & 17 & 0 & 1 \\ 12 & inf & 11 & 2 & 0 \\ 0 & 3 & inf & 0 & 2 \\ 15 & 3 & 12 & inf & 0 \\ 11 & 0 & 0 & 12 & inf \end{bmatrix}$$

- <u>Choose to go to vertex 2: **Node 2**</u>
  - Cost of edge <1,2> is: A(1,2) = 10
  - Set row #1 = inf since we are choosing edge <1,2>
  - Set column # 2 = inf since we are choosing edge <1,2>
  - Set A(2,1) = inf
  - The resulting cost matrix is:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ inf & inf & 11 & 2 & 0 \\ 0 & inf & inf & 0 & 2 \\ 15 & inf & 12 & inf & 0 \\ 11 & inf & 0 & 12 & inf \end{bmatrix}$$

  - The matrix is reduced:
    - RCL = 0
  - The cost of node 2 (Considering vertex 2 from vertex 1) is:
    - **Cost(2) = cost(1) + A(1,2) = 25 + 10 = 35**

- <u>Choose to go to vertex 3: **Node 3**</u>
  - Cost of edge <1,3> is: A(1,3) = 17 (In the reduced matrix
  - Set row #1 = inf since we are starting from node 1
  - Set column # 3 = inf since we are choosing edge <1,3>
  - Set A(3,1) = inf
  - The resulting cost matrix is:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 12 & inf & inf & 2 & 0 \\ inf & 3 & inf & 0 & 2 \\ 15 & 3 & inf & inf & 0 \\ 11 & 0 & inf & 12 & inf \end{bmatrix}$$

- Reduce the matrix:
  - Rows are reduced
  - The columns are reduced except for column # 1:
    - Reduce column 1 by 11:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 1 & inf & inf & 2 & 0 \\ inf & 3 & inf & 0 & 2 \\ 4 & 3 & inf & inf & 0 \\ 0 & 0 & inf & 12 & inf \end{bmatrix}$$

- The lower bound is:
  - RCL = 11
- The cost of going through node 3 is:

- cost(3) = cost(1) + RCL + A(1,3) = 25 + 11 + 17 = 53

- <u>Choose to go to vertex 4: **Node 4**</u>
  - Remember that the cost matrix is the one that was reduced at the starting vertex 1
  - Cost of edge <1,4> is: A(1,4) = 0
  - Set row #1 = inf since we are starting from node 1
  - Set column # 4 = inf since we are choosing edge <1,4>
  - Set A(4,1) = inf
  - The resulting cost matrix is:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 12 & inf & 11 & inf & 0 \\ 0 & 3 & inf & inf & 2 \\ inf & 3 & 12 & inf & 0 \\ 11 & 0 & 0 & inf & inf \end{bmatrix}$$

  - Reduce the matrix:
    - Rows are reduced
    - Columns are reduced
  - The lower bound is: RCL = 0
  - The cost of going through node 4 is:
    - cost(4) = cost(1) + RCL + A(1,4) = 25 + 0 + 0 = 25

- <u>Choose to go to vertex 5: **Node 5**</u>
  - Remember that the cost matrix is the one that was reduced at starting vertex 1
  - Cost of edge <1,5> is: A(1,5) = 1
  - Set row #1 = inf since we are starting from node 1
  - Set column # 5 = inf since we are choosing edge <1,5>
  - Set A(5,1) = inf
  - The resulting cost matrix is:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 12 & inf & 11 & 2 & inf \\ 0 & 3 & inf & 0 & inf \\ 15 & 3 & 12 & inf & inf \\ inf & 0 & 0 & 12 & inf \end{bmatrix}$$

  - Reduce the matrix:
    - Reduce rows:
      - Reduce row #2: Reduce by 2

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 10 & inf & 9 & 0 & inf \\ 0 & 3 & inf & 0 & inf \\ 15 & 3 & 12 & inf & inf \\ inf & 0 & 0 & 12 & inf \end{bmatrix}$$

      - Reduce row #4: Reduce by 3

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 10 & inf & 9 & 0 & inf \\ 0 & 3 & inf & 0 & inf \\ 12 & 0 & 9 & inf & inf \\ inf & 0 & 0 & 12 & inf \end{bmatrix}$$

  - Columns are reduced

- The lower bound is:
  - RCL = 2 + 3 = 5
- The cost of going through node 5 is:
  - cost(5) = cost(1) + RCL + A(1,5) = 25 + 5 + 1 = 31

- In summary:
  - So the live nodes we have so far are:
    - 2: cost(2) = 35, path: 1->2
    - 3: cost(3) = 53, path: 1->3
    - 4: cost(4) = 25, path: 1->4
    - 5: cost(5) = 31, path: 1->5
  - Explore the node with the lowest cost: Node 4 has a cost of 25
  - Vertices to be explored from node 4: 2, 3, and 5
  - Now we are starting from the cost matrix at node 4 is:

Cost(4) = 25

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 12 & inf & 11 & inf & 0 \\ 0 & 3 & inf & inf & 2 \\ inf & 3 & 12 & inf & 0 \\ 11 & 0 & 0 & inf & inf \end{bmatrix}$$

- Choose to go to vertex 2: **Node 6** (path is 1->4->2)
    - Cost of edge <4,2> is: A(4,2) = 3
    - Set row #4 = inf since we are considering edge <4,2>
    - Set column # 2 = inf since we are considering edge <4,2>
    - Set A(2,1) = inf
    - The resulting cost matrix is:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ inf & inf & 11 & inf & 0 \\ 0 & inf & inf & inf & 2 \\ inf & inf & inf & inf & inf \\ 11 & inf & 0 & inf & inf \end{bmatrix}$$

    - Reduce the matrix:
        - Rows are reduced
        - Columns are reduced
    - The lower bound is: RCL = 0
    - The cost of going through node 2 is:
        - cost(6) = cost(4) + RCL + A(4,2) = 25 + 0 + 3 = 28

- <u>Choose to go to vertex 3: **Node 7** (</u> path is 1->4->3)
  - Cost of edge <4,3> is: A(4,3) = 12
  - Set row #4 = inf since we are considering edge <4,3>
  - Set column # 3 = inf since we are considering edge <4,3>
  - Set A(3,1) = inf
  - The resulting cost matrix is:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 12 & inf & inf & inf & 0 \\ inf & 3 & inf & inf & 2 \\ inf & inf & inf & inf & inf \\ 11 & 0 & inf & inf & inf \end{bmatrix}$$

  - Reduce the matrix:
    - Reduce row #3: by 2:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 12 & inf & inf & inf & 0 \\ inf & 1 & inf & inf & 0 \\ inf & inf & inf & inf & inf \\ 11 & 0 & inf & inf & inf \end{bmatrix}$$

    - Reduce column # 1: by 11

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 1 & inf & inf & inf & 0 \\ inf & 1 & inf & inf & 0 \\ inf & inf & inf & inf & inf \\ 0 & 0 & inf & inf & inf \end{bmatrix}$$

- The lower bound is: RCL = 13
- So the RCL of node 7 (Considering vertex 3 from vertex 4) is:
  - Cost(7) = cost(4) + RCL + A(4,3) = 25 + 13 + 12 = 50

- <u>Choose to go to vertex 5: **Node 8** (</u> path is 1->4->5)
  - Cost of edge <4,5> is: A(4,5) = 0
  - Set row #4 = inf since we are considering edge <4,5>
  - Set column # 5 = inf since we are considering edge <4,5>
  - Set A(5,1) = inf
  - The resulting cost matrix is:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 12 & inf & 11 & inf & inf \\ 0 & 3 & inf & inf & inf \\ inf & inf & inf & inf & inf \\ inf & 0 & 0 & inf & inf \end{bmatrix}$$

  - Reduce the matrix:
    - Reduced row 2: by 11

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 1 & inf & 0 & inf & inf \\ 0 & 3 & inf & inf & inf \\ inf & inf & inf & inf & inf \\ inf & 0 & 0 & inf & inf \end{bmatrix}$$

- Columns are reduced

- The lower bound is: RCL = 11
- So the cost of node 8 (Considering vertex 5 from vertex 4) is:
  - Cost(8) = cost(4) + RCL + A(4,5) = 25 + 11 + 0 = 36

- In summary:
    - So the live nodes we have so far are:
        - 2: cost(2) = 35, path: 1->2
        - 3: cost(3) = 53, path: 1->3
        - 5: cost(5) = 31, path: 1->5
        - 6: cost(6) = 28, path: 1->4->2
        - 7: cost(7) = 50, path: 1->4->3
        - 8: cost(8) = 36, path: 1->4->5

    - Explore the node with the lowest cost: Node 6 has a cost of 28
    - Vertices to be explored from node 6: 3 and 5
    - Now we are starting from the cost matrix at node 6 is:

Cost(6) = 28

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ inf & inf & 11 & inf & 0 \\ 0 & inf & inf & inf & 2 \\ inf & inf & inf & inf & inf \\ 11 & inf & 0 & inf & inf \end{bmatrix}$$

- Choose to go to vertex 3: **Node 9** (path is 1->4->2->3)
    - Cost of edge <2,3> is: A(2,3) = 11
    - Set row #2 = inf since we are considering edge <2,3>
    - Set column # 3 = inf since we are considering edge <2,3>
    - Set A(3,1) = inf
    - The resulting cost matrix is:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ inf & inf & inf & inf & 2 \\ inf & inf & inf & inf & inf \\ 11 & inf & inf & inf & inf \end{bmatrix}$$

    - Reduce the matrix:
        - Reduce row #3: by 2

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ inf & inf & inf & inf & 0 \\ inf & inf & inf & inf & inf \\ 11 & inf & inf & inf & inf \end{bmatrix}$$

        - Reduce column # 1: by 11

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ inf & inf & inf & inf & 0 \\ inf & inf & inf & inf & inf \\ 0 & inf & inf & inf & inf \end{bmatrix}$$

- The lower bound is: RCL = 2 +11 = 13
- So the cost of node 9 (Considering vertex 3 from vertex 2) is:
  - Cost(9) = cost(6) + RCL + A(2,3) = 28 + 13 + 11 = 52

- Choose to go to vertex 5: **Node 10** (path is 1->4->2->5)
  - Cost of edge <2,5> is: A(2,5) = 0
  - Set row #2 = inf since we are considering edge <2,3>
  - Set column # 3 = inf since we are considering edge <2,3>
  - Set A(5,1) = inf
  - The resulting cost matrix is:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ 0 & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ inf & inf & 0 & inf & inf \end{bmatrix}$$

- Reduce the matrix:
  - Rows reduced
  - Columns reduced
- The lower bound is: RCL = 0
- So the cost of node 10 (Considering vertex 5 from vertex 2) is:
  - Cost(10) = cost(6) + RCL + A(2,3) = 28 + 0 + 0 = 28

- In summary:
  - So the live nodes we have so far are:
    - 2: cost(2) = 35, path: 1->2
    - 3: cost(3) = 53, path: 1->3
    - 5: cost(5) = 31, path: 1->5
    - 7: cost(7) = 50, path: 1->4->3
    - 8: cost(8) = 36, path: 1->4->5
    - 9: cost(9) = 52, path: 1->4->2->3
    - 10: cost(2) = 28, path: 1->4->2->5

  - Explore the node with the lowest cost: Node 10 has a cost of 28
  - Vertices to be explored from node 10: 3
  - Now we are starting from the cost matrix at node 10 is:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ 0 & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ inf & inf & 0 & inf & inf \end{bmatrix}$$

- <u>Choose to go to vertex 3: **Node 11** (</u> path is 1->4->2->5->3)
    - Cost of edge <5,3> is: A(5,3) = 0
    - Set row #5 = inf since we are considering edge <5,3>
    - Set column # 3 = inf since we are considering edge <5,3>
    - Set A(3,1) = inf
    - The resulting cost matrix is:

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \end{bmatrix}$$

    - Reduce the matrix:
        - Rows reduced
        - Columns reduced
    - The lower bound is: RCL = 0
    - So the cost of node 11 (Considering vertex 5 from vertex 3) is:
        - Cost(11) = cost(10) + RCL + A(5,3) = 28 + 0 + 0 = 28

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