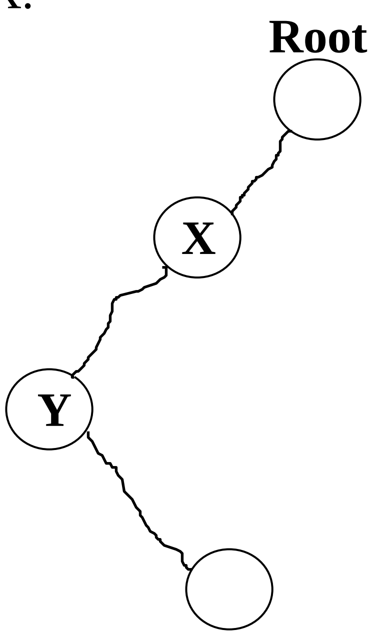
Tree Structures

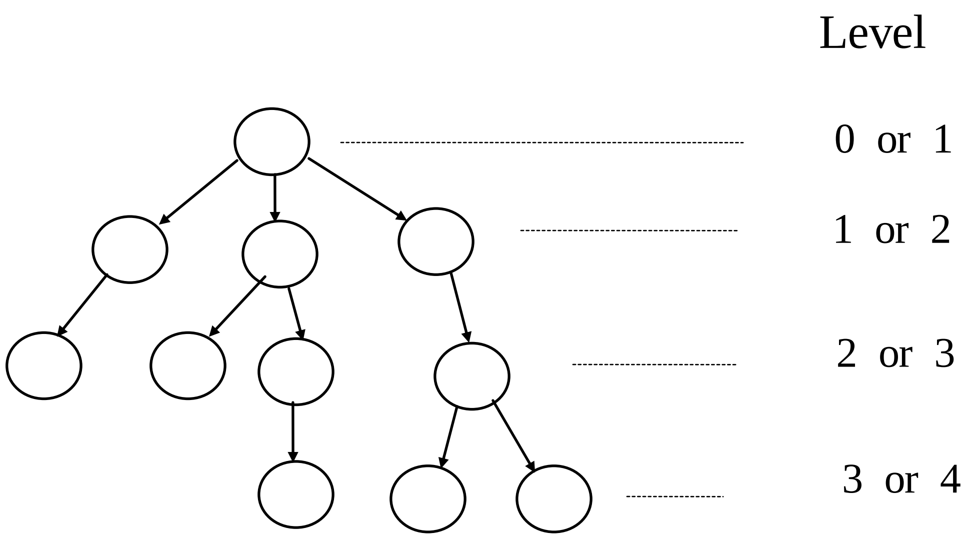
- **Definitions**:
    - A **tree** is a connected acyclic graph.
    - A disconnected acyclic graph is called a **forest**
    - A tree is a connected digraph with these properties:
        - There is exactly one node (**Root**) with in-degree=0
        - All other nodes have in-degree=1
        - A **leaf** is a node with out-degree=0
        - There is **exactly one path** from the root to any leaf
    - The **degree** of a tree is the maximum out-degree of the nodes in the tree.
    - If (X,Y) is a path:

        X is an **ancestor** of Y, and

        Y is a **descendant** of X.

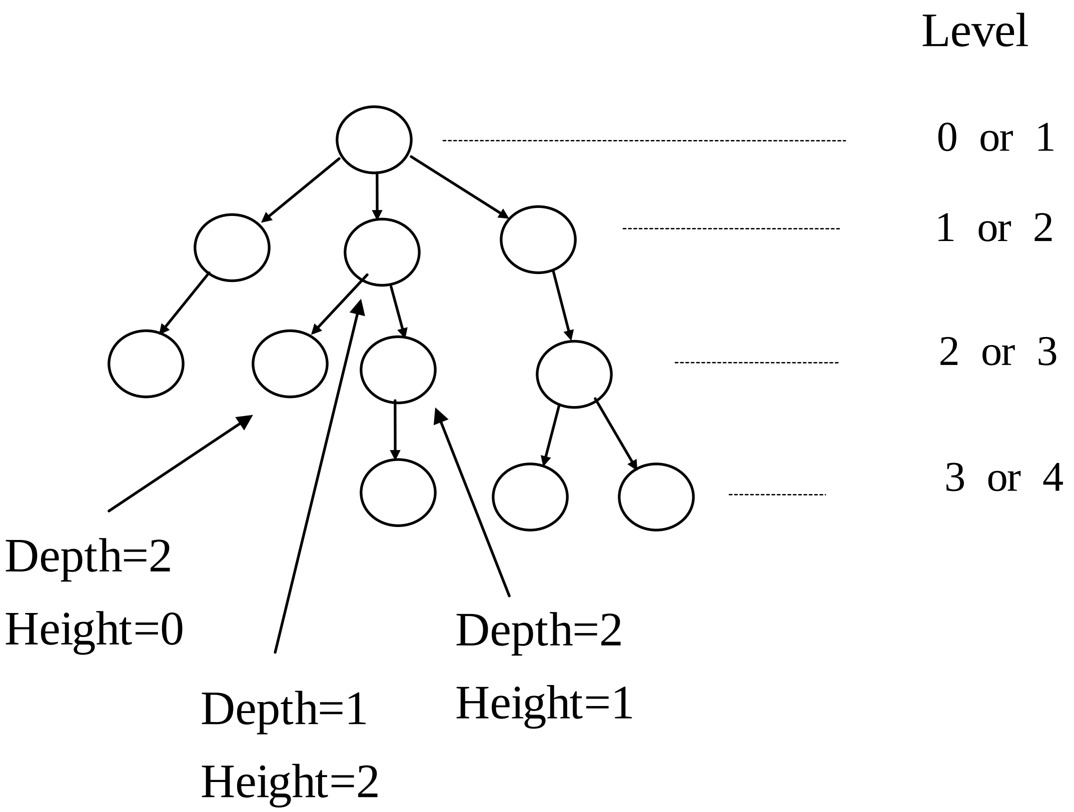


- **Level of a node:**



- **Height or depth:**
  - The depth of a node is the number of edges from the root to the node.
  - The root node has depth zero
  - The height of a node is the number of edges from the node to the deepest leaf.
  - The height of a tree is a height of the root.
  - The height of the root is the height of the tree
  - Leaf nodes have height zero
  - A tree with only a single node (hence both a root and leaf) has depth and height zero.
  - An empty tree (tree with no nodes) has depth and height −1.
  - It is the **maximum level** of any node in the tree.

- Example:



- Please note that if you label the level starting from 1, the depth (height) is level-1 (max level -1)

- **Children, Parents, and Siblings**

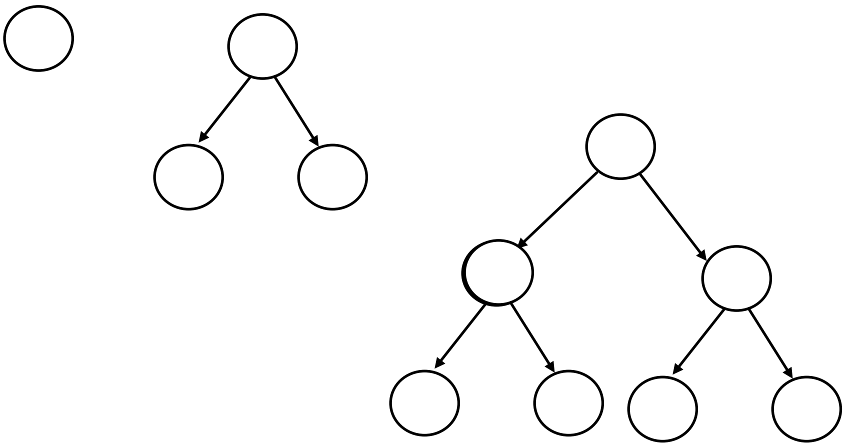
- **Subtree**

- **Properties**:

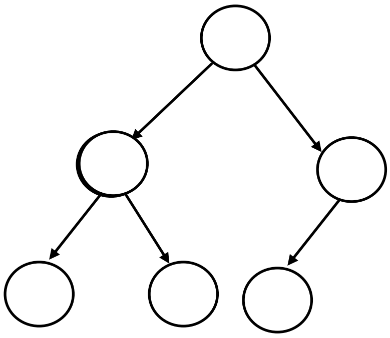
  (1) for a tree T =(V,E), where n=|*V*| and e=|*E*|, we have

$$e = n - 1$$

- **Binary Trees**
  - **Definitions**:
    - It is a tree whose **<u>degree is ≤ 2</u>**
    - The two children are called **<u>left and right</u>** children
  - **Properties**:
    - **Strictly binary**:
      - Each node has either two children or 0
    - **Full Binary** tree:
      - A tree is a full binary tree of depth h iff each node of level h is a leaf and each intermediate node has left and right children.



    - **Complete Binary** tree:
      - Every intermediate node in levels between 0 and h-2 have 2 children
      - Every node in level h-1 has either 2 children or 1 child. If there is one child, then it is a left child.



- **Balanced** Binary Tree :
  - A tree is a balanced (or height balanced) BT iff for each node X in T, the depth of the left and right subtrees of X differ by at most 1.

- **Lemma 1**:
  - The maximum number of nodes on level i of a binary tree is 2^i (starting from level 0).
  - The maximum number of nodes in a binary tree of depth k is: $2^{k+1}-1$, $k>0$ (starting from level 0).
- **Lemma 2**:
  - For any non empty binary tree, T, if n_0 is the number of leaves and n_2 is the number of nodes of degree 2, then

$$\boxed{n_0 = n_2 + 1}$$

  - Proof:
    - The total number of nodes in a BT T is: $n = n_0 + n_1 + n_2$

      n_i is the number of nodes of degree i (i children)

      for i=0, 1, and 2.
    - We have $e = n - 1$ from property 1 where e is the number of links in T.
    - The number of links e can also be computed as follows:

      n_0 contribute by 0 links

      n_1 contribute by $n_1*1 = n_1$ links

      n_2 contribute by $n_2*2 = 2n_2$ links
    - Therefore,

$$e = n_1 + 2n_2 = n - 1 = n_0 + n_1 + n_2 - 1$$

$$\Longrightarrow \quad \boxed{n_0 = n_2 + 1}$$

- **Representations**:
  - Sequential
  - Linked-list

- **Sequential representation**:
  - For a complete tree of n nodes:

    (1) The parent of a node i is:

$$\boldsymbol{Parent}(\boldsymbol{i}) = \begin{cases} \left\lfloor \frac{\boldsymbol{i}}{2} \right\rfloor & \boldsymbol{if\ i} \neq 1 \\ \boldsymbol{No\ parent} & \boldsymbol{if\ i} = 1 (\boldsymbol{i\ is\ the\ root}) \end{cases}$$

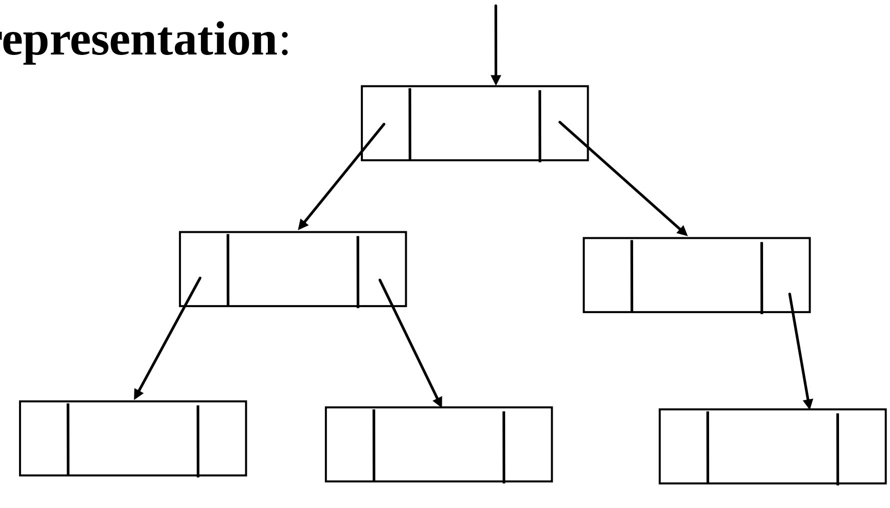
    (2) The leftchild of a node i is:

$$\boldsymbol{Leftchid}(\boldsymbol{i}) = \begin{cases} 2\boldsymbol{i} & \boldsymbol{if}\ 2\boldsymbol{i} \leq \boldsymbol{n} \\ \boldsymbol{No\ leftchild} & \boldsymbol{if}\ 2\boldsymbol{i} > \boldsymbol{n} \end{cases}$$

    (3) The rightchild of a node i is:

$$\boldsymbol{Rightchild}(\boldsymbol{i}) = \begin{cases} 2\boldsymbol{i} + 1 & \boldsymbol{if}\ 2\boldsymbol{i} + 1 \leq \boldsymbol{n} \\ \boldsymbol{No\ Rightchild} & \boldsymbol{if}\ 2\boldsymbol{i} + 1 > \boldsymbol{n} \end{cases}$$

- **Linked-list representation**:



Binary Tree Traversals

- **There are three traversals**:
  - Inorder: LNR
  - Preorder: NLR
  - Postorder: LRN

- **Inorder Traversal: LNR**

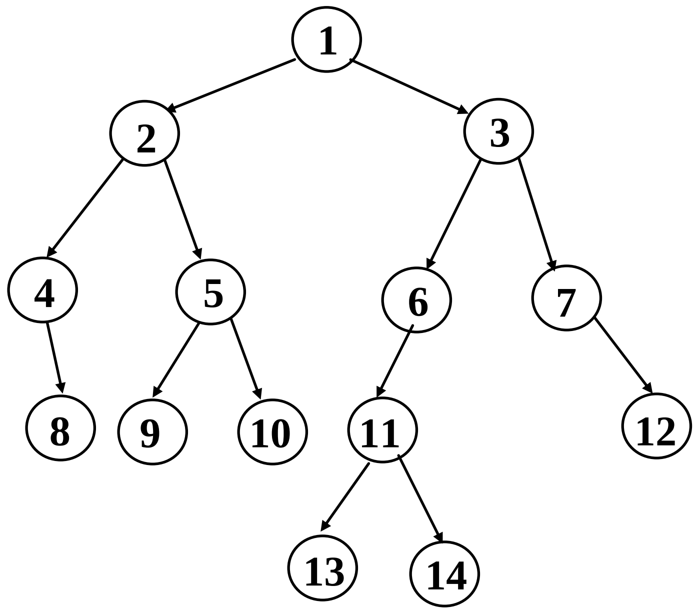
- Procedure:

```
Procedure LNR (t:tree);
Begin
    If t=null
    then return
    else  Begin
              LNR(t->left);
              visit(t-data);
              LNR(t->right);
          end;
```

- Complexity:

T(n) = O(n) where n is the number of nodes in T.

- Example:



LNR: 4-8-2-9-5-10-1-13-11-14-6-3-7-12

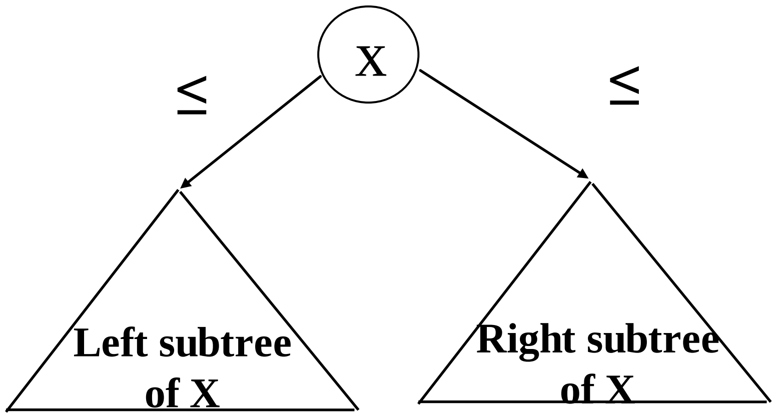
NLR: 1-2-4-8-5-9-10-3-6-11-13-14-7-12

LRN: 8-4-9-10-5-2-13-14-11-6-12-7-3-1

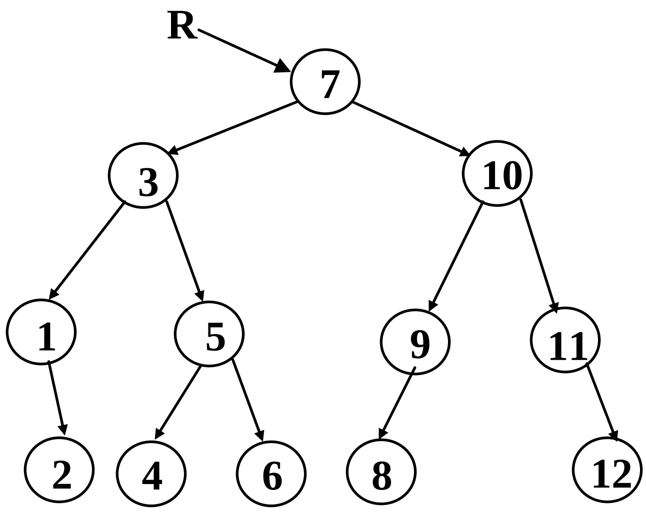
Binary Search Tree ADT

- **Objective**:
  - Insertion, deletion, and Find take O(log(n)) where n is the number of elements in the list.

- **Definition**:
  - Let us assume that every node in a binary search tree (BST) is assigned a key value X. For every node X in a BST, the keys in the left subtree of the node containing X are smaller than X and the keys in the right subtree of the node containing X are greater than X.



- Example:



- **Operations**:
  - Search or Find
  - Find_min
  - Find_max
  - Insert
  - Delete

- **Search:**

- function:

```
Node Search(Node T; int x);
Begin
    If ( T == null)
    then return(null);
    else  Begin
            If (x < T.data)
            then return(Search(T.left));
            else if (x > T.data)
                then return(Search(T.right));
                else return(T);
        End;
End;
```

- Complexity:

  O(h) where h is the depth of the tree.

- **Insertion in a BST**:
  - There are three steps:
    - create a new node for the element to be inserted
    - Search or Find the location at which the new node will be inserted
    - Insert the new node
  - Procedure Insert(Node Root; int x)

```
Begin       /* The element to be inserted is x */
        /* Create new node */
        t = create_node();   /* Allocate space for x */
        t.leftChild = null; t.rightChild = null; t.data = x;
        /* Search for the insertion location */
        p = Root; q = nil;
        While (p!=null) do Begin
                q = p;
                if p.data > x
                then p = p.left;
                else p = p.right;
        End;
        /* Insert the new element */
        If (q == null)      /* Empty tree */
        then Root = t;
        else   Begin
                        if q.data > x
                        then q.left = t;
                        else q.right = t;
                End;
End;
```

- Complexity:

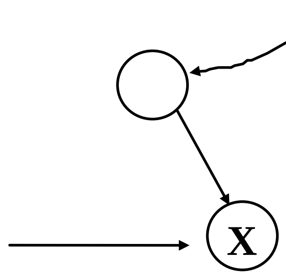
  O(h) where h is the depth of the tree.

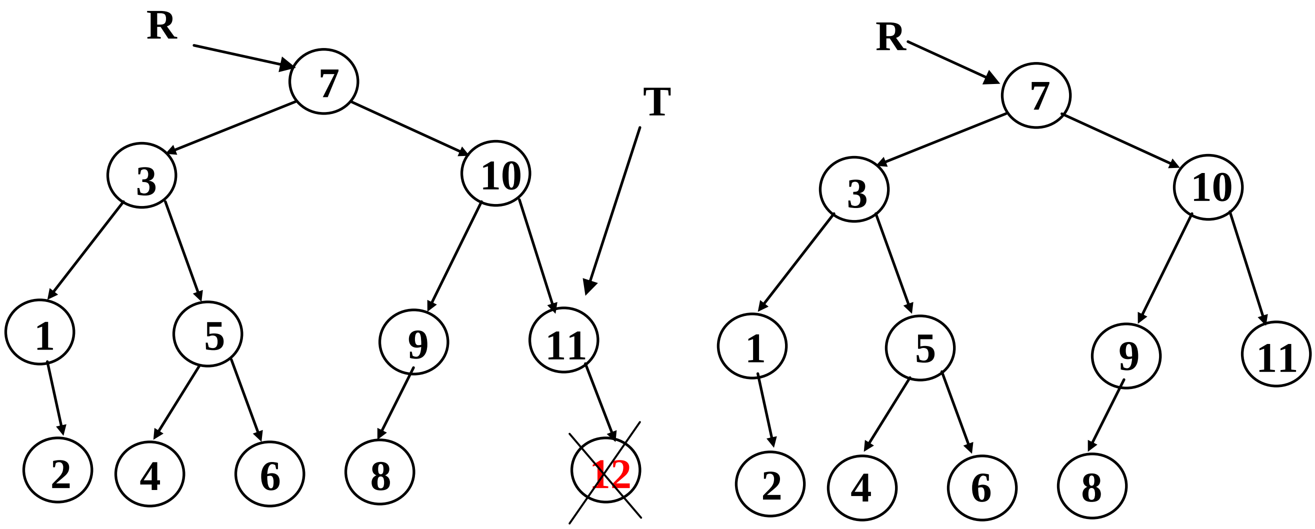
- Example:

- **Deletion in a BST**:
  - There are three cases:
    - Node to be deleted has no children (Leaf).
    - Node to be deleted has one child.
    - Node to be deleted has two children (complicated).

  - Case 1: Node to be deleted has no children (Leaf):

Node to be deleted





- Deletion steps:

  //Delete node with value 12 in a BST with root R

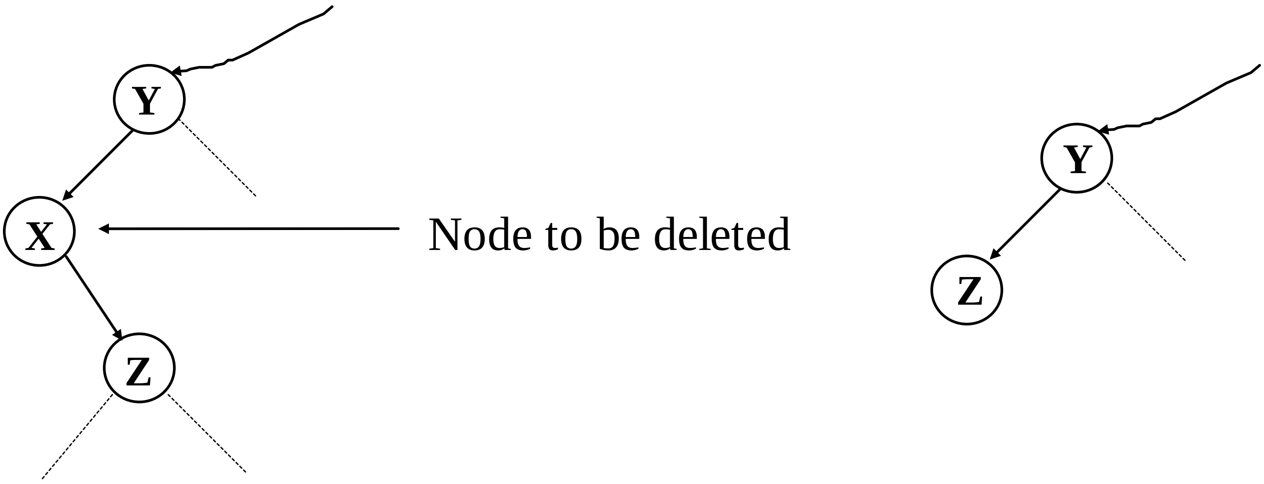
  //T is the parent of the node that contains 12

  T = findParent(R, 12)

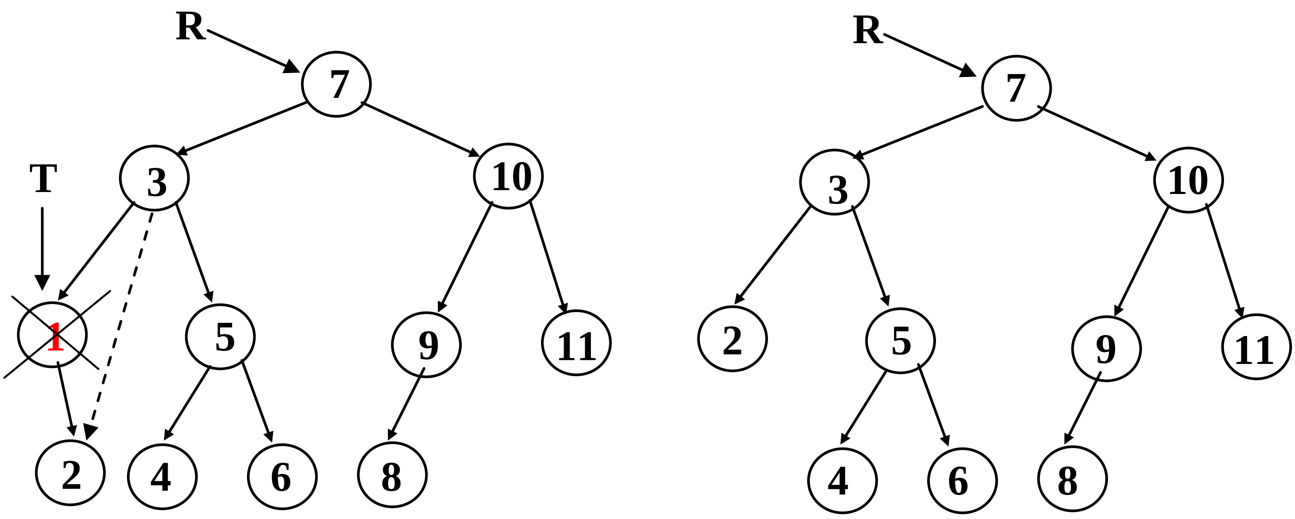
  //Delete the element.

  T.rightChild = null;

- Case 2: node to be deleted has one child



- Example:



- Deletion steps:

  //Delete node with value 1 in a BST with root R

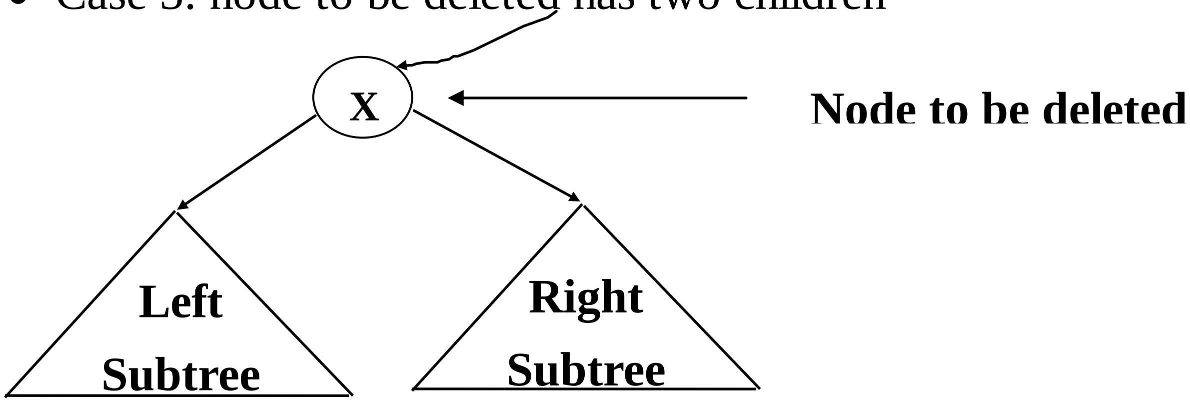
  //T is the parent of the node that contains 1

  T = findParent(R, 1)

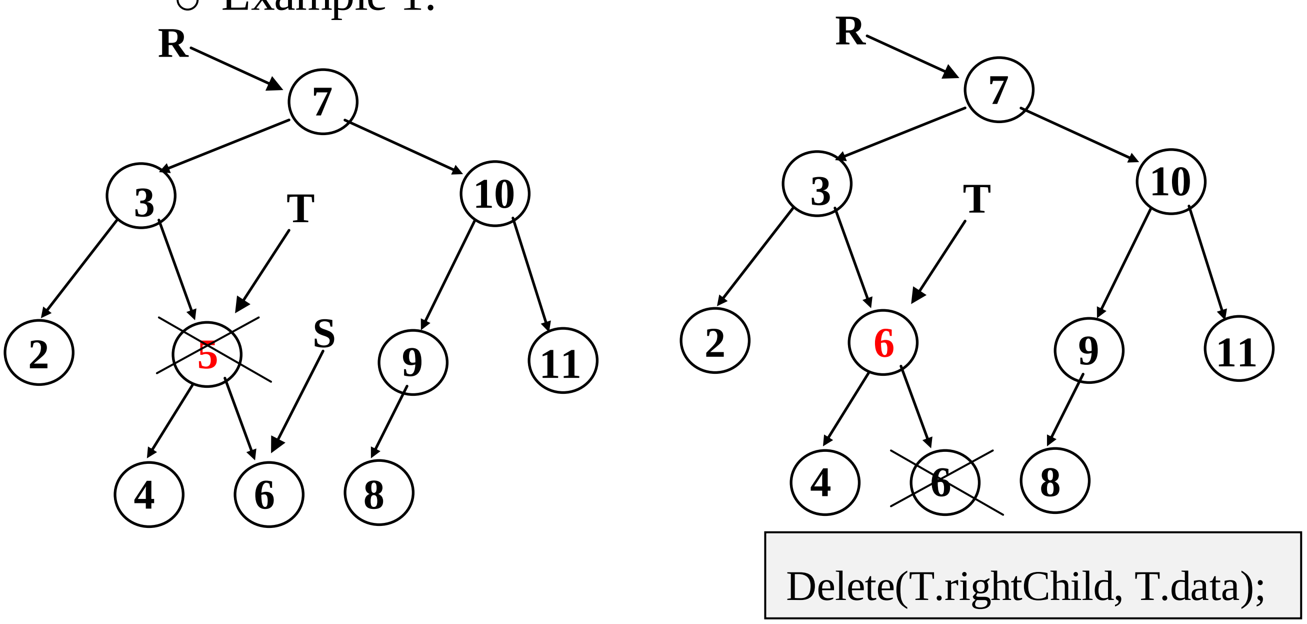
  //Delete the node

  T = T.rightChild; //Since the left child is null.

- Case 3: node to be deleted has two children



  - X must be replaced by either its:
    - **predecessor** (Max in the left subtree)
    - **successor** (Min in the right subtree)
  - Example 1:



    - Deletion steps:

//Delete node with value 5 in a BST with root R

//T is the parent of the node that contains 5

T = findParent(S, 5);

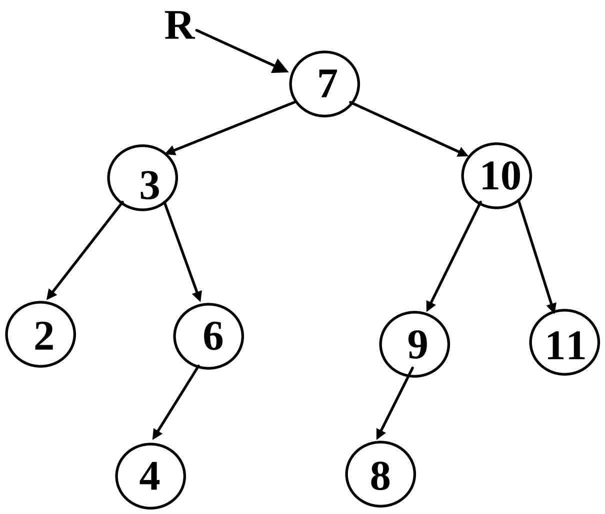
S =findSuccessor(T); //Find the min of the right subtree.

//Delete the node

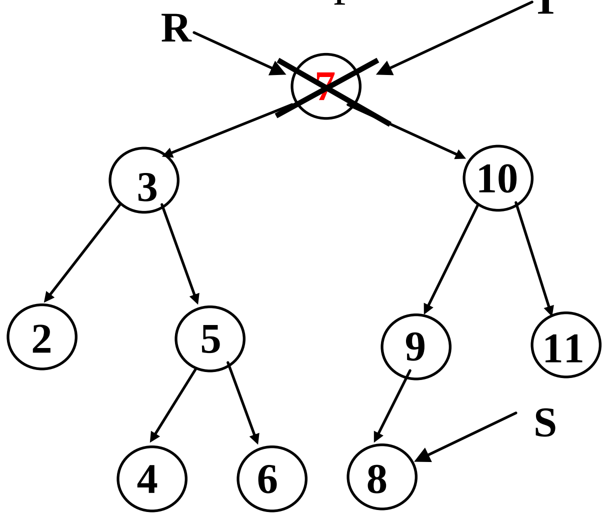
T.data = S.data;

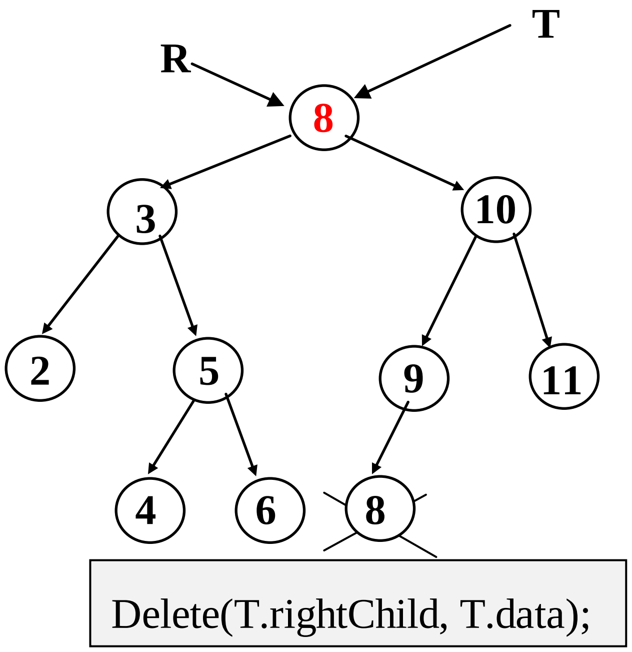
Delete(T.rightChild, T.data);

- Tree after deleting node 5:



- Example 2:





  - Deletion steps:

//Delete node with value 7 in a BST with root R

//T is the parent of the node that contains 7

T = findParent(S, 7);

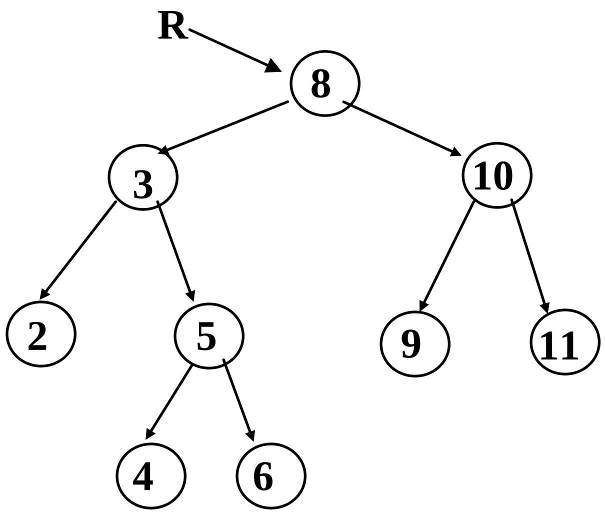
S =findSuccessor(T); //Find the min of the right subtree.

//Delete the node

T.data = S.data;

Delete(T.rightChild, T.data);

- Tree after deleting node 7:



- Procedure Delete(Node Root; int x)

```
Begin
    If (T ==null)   then print ("Sorry the element is not found ");
    else   if (x< T.data)
               then Delete(T.leftChild,x);                /* Go left */
               else if (x>T.data)
                      then Delete(T.rightChild,x)     /* Go Right */
                      else   Begin
                                 If (T.leftChild == null)                 /* only a right child or none*/
                                 then   begin
                                               temp = T; T = T.rightChild; free(temp);
                                        end;
                                 else   if (T.rightChild ==null)          /* only a left child */
                                        then   begin  temp = T; T = T.leftChild;   end;
                                        else   begin    /* Case 3: Two children. Replace with successor */
                                                      temp = Find_min(T.rightChild);
                                                      T.data = temp.data;
                                                      Delete(T.rightChild,T.data)
                                               end;
                             End;
End;
```

- **Time Complexity:**
    - If the tree is a complete binary tree with n nodes, then the worst-case time is O(log n).
    - If the tree is very unbalanced (i.e. the tree is a linear chain), the worst-case time is O(n).
    - Luckily, the expected height of a randomly built binary search tree is O(log n)
        - basic operations take time O(log n) on average.

Threaded Binary Trees

- **Motivations**:
  - To do traversal in languages that do not support recursion
  - Non- recursive traversals

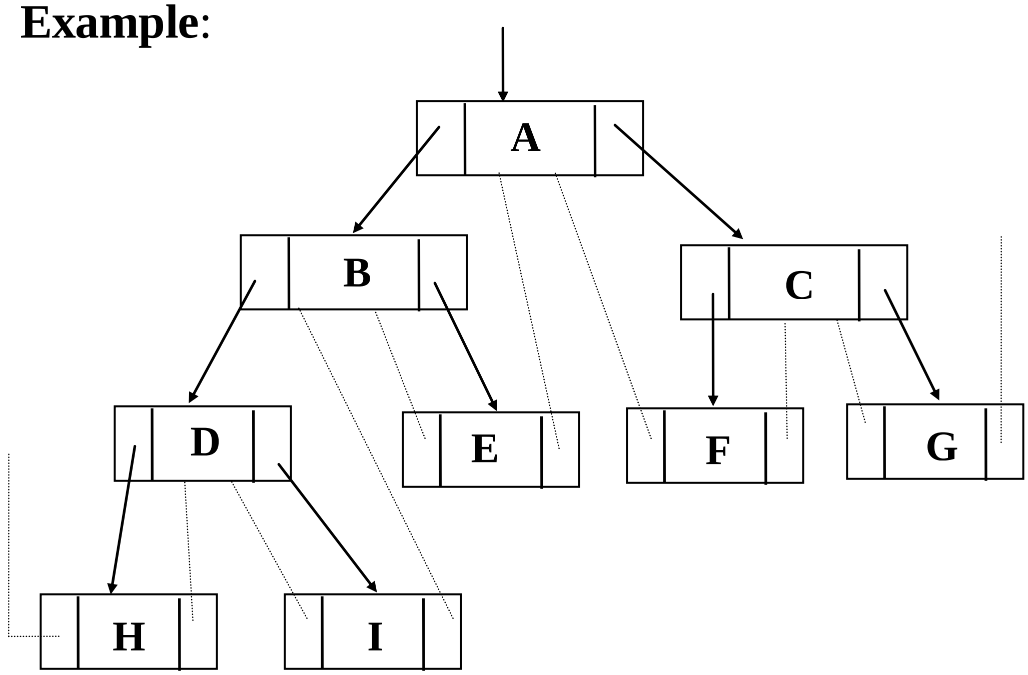
- In a binary tree of n nodes there are 2n links out of which n+1 are null links. In case of full tree of depth k, we have $n=2^{k+1}-1$. The number of leaves is $2^k = \frac{n+1}{2}$. Therefore, the number of null links is: $2*\frac{n+1}{2} = n+1$.

- **Objective**:
  - Make use of the null links (by A.J. Perlis & C. Thornton).
  - Replace null links by pointers, called threads, to other nodes in the tree.
- **Threads setup**:
  - If p->right == null

    then p->right = the node which would be printed after p (inorder successor of p) when traversing the tree in inorder.
  - If p->left == null

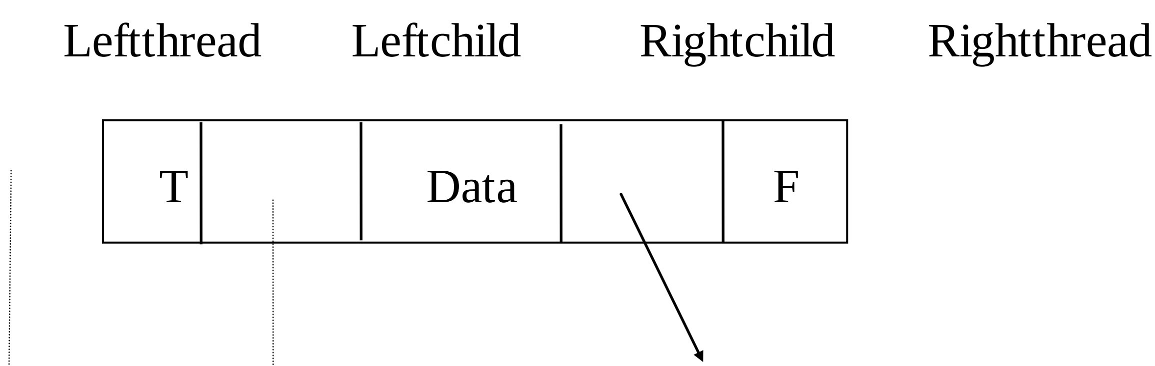
    then p->left = the node which would be printed before p (inorder predecessor of p) when traversing the tree in inorder.

- **Example**:



LNR: H D I B E A F C G

- **Implementation**:
  - How to distinguish between threads and normal pointers?



- **Application**:
  - Perform a non-recursive inorder traversal without a stack to simulate recursion.

- Code Example

```
public class BinaryTreeNode {

     private int key;
     private BinaryTreeNode leftChild;
     private BinaryTreeNode rightChild;

     public BinaryTreeNode(){
          key = 0;
          leftChild = null;
          rightChild = null;
     }
     public BinaryTreeNode(int d, BinaryTreeNode left, BinaryTreeNode
right){
          key = d;
          leftChild = left;
          rightChild = right;
     }
     public int getKey(){
          return(key);
     }
     public BinaryTreeNode getLeftChild(){
          return(leftChild);
     }
     public BinaryTreeNode getRightChild(){
          return(rightChild);
     }
     public void setLeftChild(BinaryTreeNode node){
          leftChild = node;
     }
     public void setRightChild(BinaryTreeNode node){
          rightChild = node;
     }
}

public class BinarySearchTree {

      private BinaryTreeNode root;

      public BinarySearchTree(){
             this.root = null;
      }
    public BinaryTreeNode getRoot(){
          return(root);
     }

      private void findPosition(BinaryTreeNode node, BinaryTreeNode start){
             int sKey = start.getKey();
             if (sKey>node.getKey()){
```

```
                if (start.getLeftChild() == null){
                    start.setLeftChild(node);
                }
                else{
                   findPosition(node, start.getLeftChild());
                }
            }
            else{
                if (start.getRightChild() == null){
                    start.setRightChild(node);
                }
                else{
                   findPosition(node, start.getRightChild());
                }
            }
        }

    public void insertNode(BinaryTreeNode node){
         if (root == null){
              root = node;
        }
        else{
         findPosition(node, this.root);
        }
    }

    private boolean findElement(BinaryTreeNode node, int x){
           if (node ==  null)
             return(false);
             if (x == node.getKey())
             return(true);
           else if (x < node.getKey())
                  return(findElement(node.getLeftChild(), x));
             else
                  return(findElement(node.getRightChild(), x));
    }


    public int countLeaves(BinaryTreeNode node) {
    if (node == null)
       return 0;
    else if (node.getLeftChild() == null && node.getRightChild() == null)
       return 1;
    else
       return countLeaves(node.getLeftChild()) +
countLeaves(node.getRightChild());
  }
    public int computeDepth(BinaryTreeNode node){
        if (node == null)
                  return 0;
```

```
          return (1+  Math.min(computeDepth(node.getLeftChild()),
computeDepth(node.getRightChild())));
     }
      public void inorderPrint(BinaryTreeNode node){
     }

      public void preorderPrint(BinaryTreeNode node){
     }
      public int countNodes(BinaryTreeNode node){

     }

      public int findMin(BinaryTreeNode node){

     }
      public int findMax(BinaryTreeNode node){
     }


}
```

- **Programming Assignment:**
  - Design and implement the missing operations in the Binary Search Tree ADT:
    - findMin
    - findMax
    - countNodes
    - inorderPrint
    - preorderPrint

  - Test your implementation.