Topology of Efficiently Controllable Banyan Multistage Networks

Abdou Youssef

Department of Elect. Eng. & Comput. Sci.
The George Washington University
Washington, DC 20052

Bruce Arden

College of Engineering and Applied Science
University of Rochester
Rochester, NY 14627

Abstract

Due to their unique path property, banyan multistage interconnection networks (MIN's) can be self-routed using control tags. This paper introduces a number of routing control classes of MIN's and studies their structure. These include the D-controllable networks where the control tags are the destination labels, the FD-controllable networks where the control tags are function of the destination labels, and the doubly D- or FD-controllable networks which are D- or FD-controllable forward and backward. The paper shows that all D- and FD-controllable networks have a recursive structure, and that all doubly D-controllable (resp., FD-controllable) networks are strictly (resp., widely) functionally equivalent to the baseline network. The subclass of MIN's where the interconnections are digit permute is also studied and shown to be doubly FD-controllable and hence equivalent to the baseline. Finally, the paper presents an efficient, parallel algorithm that relabels the terminals of any one network to simulate any other network in that subclass.

§1. Introduction

Banyan multistage interconnection networks (MIN's) are increasingly important in parallel computing systems. Several networks of this type have been proposed and studied, such as omega and its inverse [4], the indirect binary n-cube [7], the baseline [10], and the generalized cube network [9].

The efficiency of MIN's is critical to overall system performance, and depends on the speed of the routing control, among other things. As these networks have the unique path property, that is, each source has a unique path to each destination, they can be self-routed via control tags. The control efficiency depends then on the speed of control tag computation. If the control tags are stored, the resulting memory cost is prohibitive for large systems. Therefore, the networks whose control tags are efficiently computable and need not be stored are of special interest. The most efficiently controllable networks are clearly those whose control tags are the destination addresses. The second most efficient are those whose control tags are simple functions of the destination tags.

Although multistage networks have received a lot of attention [1–2],[5],[6], little work has been done on control categorization of MIN's. Such categorization and the understanding of the relationship between control and structure of networks are important for the design of efficiently controllable networks. Furthermore, as will be seen later, further insight into functionality and network equivalence is gained from the study of the control-structure relationship. Particularly, one of the contributions is tying together and superseding the different approaches and results in [8], [9] and [10] related to the existing multistage interconnection networks, their underlying structure, their control and their equivalence to one another.

In this paper various control schemes or classes are introduced and the structure of the networks in each class is studied. These control classes include D-controllable networks where the control tags are simply the destination tags; FD-controllable networks where the control tags are functions of the destination tags; doubly D-controllable networks which are D-controllable from input to output and from output to input; and the doubly FD-controllable networks which are FD-controllable from input to output and from output to input. The last two control classes include all existing MIN's and are useful for two-way communication needed in shared-memory systems.

The paper focuses also on the sub-class of banyan multistage networks where the interconnections between columns are bit permutations (or digit permutations in general), which are operations that permute bits in a specified manner. These networks are called digit permutation networks. The reason for studying this subclass is twofold. First, it includes all existing banyan multistage networks. Second, it turns out that all digit permutation networks are doubly D- or FD-controllable networks and therefore share the same underlying structure as the latter networks.

The main contributions of this paper are the following. First, the topological structure of the D-controllable and FD-controllable networks is determined and shown to be recursive. Second, it is shown that all doubly D-controllable (resp., FD-controllable) networks are strictly (resp., widely) functionally equivalent to the baseline network. This allows the baseline network to simulate any doubly FD-controllable network by relabeling the input and output terminals of the baseline. Third, the paper establishes necessary and sufficient conditions for a sequence of digit permutations to construct a digit permutation network with the unique path property. An optimal algorithm is developed to decide if a sequence of digit permutations

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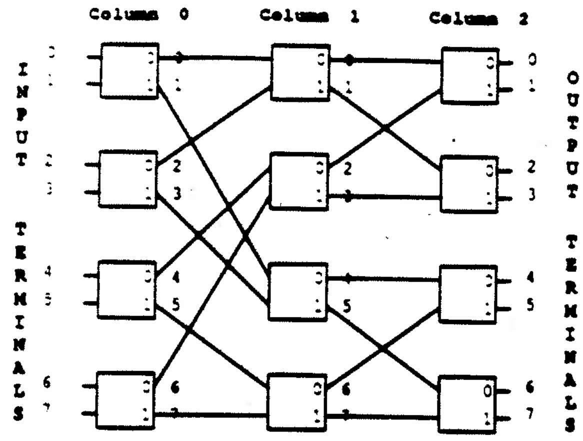
neets the aforementioned conditions. Fourth, all digit permutation networks are shown to be doubly FD-controllable and hence widely functionally equivalent to the baseline network. In addition, the control tags are shown to be digit permutations of destination tags, thus allowing for very fast routing control. Finally, an efficient, parallel algorithm that relabels the input and output terminals of one digit permutation network in order to simulate another digit permutation network is given. Such simulation is possible when two networks are widely functionally equivalent.

The paper is organized as follows. The next section gives some preliminary definitions and fundamental concepts related to network control and equivalence. Section 3 explores the structure of D-controllable networks. FD-controllable networks are treated in section 4. The functional equivalence of doubly D-controllable and doubly FD-controllable networks is established in section 5. Section 6 studies digit permutation networks. Conclusions and future directions are given in section 7.



A network in MIN(2,3)
Figure 1

§2. Definitions and Fundamental Concepts

In this section banyan multistage interconnection networks are specified, functional and topological relations among them are reviewed, their routing control is discussed and various control classes are defined.

2.1 Banyan Multistage Networks

Banyan multistage interconnection networks have N input terminals, N output terminals, and k interconnected columns of $\frac{N}{r}$ $r \times r$ crossbar switches, where $N = r^k$ and $r \geq 2$. Each $r \times r$ crossbar switch realizes all $r!$ permutations. The interconnection between every two successive columns is a permutation of $S_N = \{0, 1, ..., N-1\}$. The interconnection from the input terminals to the leftmost column is called the *left-end interconnection*, and that from the rightmost column to the output is called the *right-end interconnection*. The connectivity of these networks is such that between every input terminal i and every output terminal j there is one and only one path, denoted $i \rightarrow j$. The class of these networks is denoted MIN(r,k). Omega and its inverse [4], the indirect binary n-cube [7], and the baseline network [10] are examples of such networks where $r = 2$. Fig. 1 shows a baseline network in MIN(2,3).

For ease of reference, the input (i.e., left) and output (i.e., right) terminals of networks in MIN(r,k) are labeled 0, 1, ... , $N-1$ from top to bottom. The input and output ports of each column are similarly labeled 0, 1, ... , $N-1$. The ports are also labeled locally relative to each switch: The ports (input or output) of each switch are labeled 0, 1, ... , $r-1$, from top to bottom. The distinction between the two labels will be clear from the context. The columns are numbered 0, 1, ... , $k-1$ from left to right, and the switches of each column are labeled 0, 1, ... , $\frac{N}{r}-1$ from top to bottom. The labels of terminals and column ports are often represented in r-ary, each label having k r-ary digits. In this context, the local label of a switch port is represented by a single r-ary digit.

If W is a network of MIN(r,k) and f a permutation of S_N, f can be viewed as an interconnection and can be appended to the right end of W, forming a network denoted Wf. Another way of viewing Wf is as W except that the output terminals of W are relabeled by f, that is, output terminal j is relabeled by $f(j)$, for every $j = 0, 1, ..., N-1$. Similarly, f can be appended to the left of W, forming fW. Viewed differently, fW is the same as W except that the input terminals of W are relabeled by f^{-1}.

It should be noted that the composition of functions is taken from left to right, that is, $(x)fg = g(f(x))$. If $P(W)$ denotes the set of permutations realizable by W, then $P(gWf) = \{ghf \mid h \in P(W)\}$, which clearly follows from the definition of gWf and the left-to-right view of composition.

Finally, a network W in MIN(r,k) is said to be *left bare-ended* if its left-end interconnection is the identity permutation. It is said to be *right bare-ended* if the right-end interconnection is the identity permutation. A network is *bare-ended* if it is both left and right bare-ended.

2.2 Network Equivalence Relations

Two networks W and W' in MIN(r,k) are *strictly functionally equivalent* (denoted $W \equiv W'$) if they realize the same permutations. The two networks are *widely functionally equivalent* if they can be made to realize the same permutations by relabelling the input and/or output terminals of one of the networks, that is, if there exist two permutations g and f of S_N such that $W \equiv gW'f$.

Topological relations are defined next. To this effect, two simple operations on networks are specified. The first, called permute-links-within-switch (PL), consists of disconnecting the links connected to one side (input or output) of an $r \times r$ switch of the network and reconnecting them to different ports of the same side of the same switch. The second, called permute-switches-within-column (PS), consists of permuting the switches within a column in such a way that the links wired to a repositioned switch remain wired to it. Two networks in MIN(r,k) are *strictly topologically equivalent* if one network can be derived from the other by a sequence of PL and PS operations, and *widely topologically equivalent* if one can be derived from the other by a sequence of PL and PS operations and by relabeling the input and output terminals.

It has been shown in [12] that two networks in MIN(r,k) are strictly topologically equivalent if and only if they are strictly functionally equivalent. Similarly, they are widely topologically equivalent if and only if they are widely functionally equivalent. For that reason, we will often drop the terms topological and functional and speak merely of equivalence, strict or wide.

2.3 Control of Banyan Multistage Interconnection Networks

Due to the unique path property, the networks of MIN(r,k) can be self-routed using control tags (CT). Specifically, since there is a unique path between any input terminal i and any output terminal j in a network W in MIN(r,k), there exists a unique r-ary tag $c_{k-1}c_{k-2}...c_0$ which can be used to establish the path $i \rightarrow j$. To show this, let s_0, s_1, ... ,s_{k-1} be the consecutive switches through which the path $i \rightarrow j$ goes. The unique link between switch s_l and switch s_{l+1} that lies on the path $i \rightarrow j$ leaves switch s_l through some output port of local label b_l $(0 \le b_l \le r-1)$. Let $c_l = b_{k-1-l}$ for all $l = 0,1,...,k-1$. Now that $c_{k-1}c_{k-2}...c_0$ is defined, it can be used as a control tag as follows. Input terminal i sends $c_{k-1}c_{k-2}...c_0$ through the control lines of W, and column l uses the r-ary digit c_{k-1-l} to link the input port of switch s_l, to which the control signal comes, to output port c_{k-1-l}. Clearly then, switch s_0 uses digit c_{k-1}, s_1 uses digit c_{k-2} and so on, establishing the path to output terminal j.

The uniqueness of the path between i and j implies the uniqueness of the tag $c_{k-1}c_{k-2}...c_0$. This tag is called the control tag, denoted CT(W,i,j), or just CT(i,j) when no confusion arises.

It is clear that the control tag CT(W,i,j) for the path $i \rightarrow j$ depends on the structure of W. Storing the control tags CT(W,i,j)'s requires $N^2 k \log r$ bits which is prohibitive for large N. Therefore, it is preferable to have networks for which the CT(i,j)'s are simple to compute and need not be stored, like for instance when CT$(i,j) = j$, or when CT$(i,j) = f(j)$ for some easy-to-compute f.

A network W in MIN(r,k) is *D-controllable* (also called delta network) if CT$(W,i,j) = j$ (in r-ary) for all i and j. The baseline and omega networks are D-controllable networks. A network W in MIN(r,k) is *FD-controllable* (also called bidelta network) if there exists a permutation f of S_N such that CT$(W,i,j) = f(j)$ for all i and j. In this case, f is called the control function of W. Omega inverse is FD-controllable with control function ρ where $\rho(x_{k-1}x_{k-2}...x_0) = x_0...x_{k-2}x_{k-1}$ [4]. Note that in general, the control tags may be a function of both the input terminals and the output terminals.

In shared memory systems, where the input terminals represent processors and the output terminals memory modules, efficient communication is needed from left to right and from right to left. A network W in MIN(r,k) is *doubly D-controllable* if it is D-controllable from left to right (i.e., processor to memory) and from right to left (i.e., memory to processor). In particular, to go from output terminal j to input terminal i, the control tag needed is i. Similarly, a network is *doubly FD-controllable* if it is FD-controllable from left to right and from right to left. Double controllability can be better understood with the help of inverse networks. *The inverse* of network W, denoted W^{-1}, is the mirror image of W, where the input terminals of W are the output terminals of W^{-1} and vice versa. A network W is doubly D-controllable (resp., doubly FD-controllable) if both W and W^{-1} are D-controllable (resp., FD-controllable).

§3. Structure of D-Controllable Networks

In this section a subclass of MIN's, called Generalized Recursive Networks (GRN), will be defined recursively. Then, it will be shown that every network in GRN is D-controllable and conversely. That is, the structure of D-controllable networks is the same as GRN structure.

3.1. Definition. The class of generalized recursive networks GRN(r,k) is a subclass of MIN(r,k), defined recursively as follows. The networks of GRN$(r,1)$ are mere $r \times r$ switches with left interconnections. For $k > 1$, GRN(r,k) is the class of networks of the form $W = fT(W_0, W_1, ..., W_{r-1})$ (as shown in Figure 2 for $r = 2$) where each W_i's is a network in GRN$(r,k-1)$ such that the input terminals as well as the output terminals of W_i are labeled $ir^{k-1}, ir^{k-1}+1, ..., (i+1)r^{k-1}-1$, T is a connection (i.e permutation) that links the i-th output port of every switch in column 0 of W to an arbitrary input terminal of W_i, for $i = 0,1,...,r-1$, and f is the left-end interconnection.
If W is as above, and if the switches are permuted within columns, the resulting network is considered to be in GRN(r,k) but is said to be in a non-canonical form. The form of networks as in Figure 2 is called canonical.

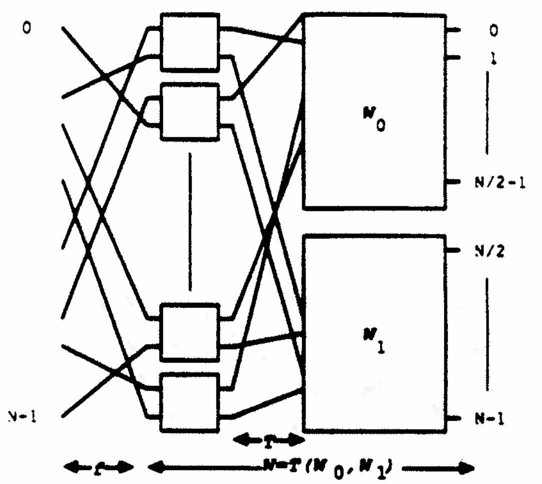
Note that the control of a GRN network is the same whether the network is in canonical form or not.

The baseline network [10] is an example of a network in GRN$(2,k)$. Particularly, if we denote by B$(2,k)$ the $2^k \times 2^k$ baseline network with 2×2 switches as building blocks, then B$(2,k) = R($B$(2,k-1)$,B$(2,k-1))$, where R is the unshuffle. The baseline can be generalized to B(r,k) such that B$(r,1)$ is a mere bare-ended $r \times r$ switch, and B(r,k) $= R($B$(r,k-1)$,...,B$(r,k-1))$, where R is the unshuffle of S_{r^k} in the system of base r, that is, $R(x_{k-1}x_{k-2}...x_0) = x_0x_{k-1}...x_1$ for every k-digit r-ary label $x_{k-1}x_{k-2}...x_0$.

The following two theorems will show that GRN is the same as the class of D-controllable networks.

3.2 Theorem. *Every network in GRN(r,k) is D-controllable.*

Proof. Let W be a network in GRN(r,k). We need to



GRN-Structure
Figure 2

show that the control tag $j_{k-1}...j_0$ establishes the path $i \to j$ in W, where $j = j_{k-1}...j_0$ in base r. The proof is by induction on $k \geq 1$.

Basis: $k = 1$. In this case W has only one $r \times r$ switch. The control tag j_0 links any input i to the output j_0, which is j.

Induction: Assume the statement is true for all networks in GRN$(r, k-1)$. It will be proved for the network W in GRN(r,k). Let $W = fT(W_0, W_1, ..., W_{r-1})$ be a network in GRN(r,k), and assume without loss of generality that it is in canonical form. The digit j_{k-1}, used to control column 0, will cause the input terminal i to link to the j_{k-1}-th output port of a switch in column 0. By definition of GRN, port j_{k-1} is linked by T to some input x of $W_{j_{k-1}}$. As $j = j_{k-1}r^{k-1} + j_{k-2}j_{k-3}...j_0$, it follows that j is the $j_{k-2}...j_0$-th output terminal of $W_{j_{k-1}}$. Since $W_{j_{k-1}}$ is in GRN$(r, k-1)$, it follows from the inductive hypothesis that the control tag $j_{k-2}...j_0$ establishes the path $x \to j_{k-2}...j_0$ in $W_{j_{k-1}}$. Consequently, the path $i \to j$, which is $i \to x \to j$, is established in W. ∎

To establish the converse of Theorem 3.2, the following lemmas are needed.

3.3 Lemma. *Let W be a D-controllable network. Then the following statements are true:*

(a) For every $l \leq \frac{N}{r} - 1$, the output terminals $r \times l$, $r \times l + 1$, ... ,$r \times l + r - 1$ are all linked to a single switch s_l in the rightmost column. Furthermore, output terminal $r \times l + t$ is linked to the output port t of s_l for $t = 0, 1, ..., r-1$.

(b) The switches of the rightmost column of W can be permuted (within column) so that W becomes right bare-ended.

Proof. (a) Fix $l = l_{k-1}...l_1$. Then $r \times l + t = l_{k-1}...l_1t$ and hence all the paths from an arbitrary source to the destinations $r \times l + t$, $t = 0, 1, 2, ..., r-1$, are identical up to the rightmost column because the $k-1$ leftmost digits of the $(r \times l + t)$'s are identical. Therefore, the terminals $r \times l$, $r \times l + 1$, ... , $r \times l + r - 1$ are all linked to the same switch (say switch s_l) in the rightmost column. In addition, as the rightmost digit t of $r \times l + t$ controls the rightmost column connecting the incoming input port of s_l to the output port t of s_l, it follows that output terminal $r \times l + t$ is linked to the output port t of s_l.

(b) Follows immediately from (a): Move switch s_l to position l for every l. ∎

In the next lemma, the switches of the rightmost $k-1$ columns in a D-controllable network W in MIN(r,k) will be partitioned into r groups, such that each group forms a D-controllable network in MIN$(r, k-1)$. To this effect, let $U_n = \{d_{k-1}d_{k-2}...d_0 \mid d_{k-1} = n\}$, for $n = 0, 1, 2, ..., r-1$. be a partition of the output terminals. Let also U_n^i be the set of switches of column i $(i \geq 1)$ that are reachable from some output terminal in U_n.

3.4 Lemma. *Let W be a D-controllable network. Then the following statements are true:*

(a) For every $n \neq m$ and for every i, $U_n^i \cap U_m^i = \emptyset$.

(b) For every i and n, the switches in U_n^i are linked forward only to switches in U_n^{i+1}.

(c) For every i and n, $|U_n^i| = r^{k-2}$.

Proof. (a) The proof is by contradiction. Let s be a switch in $U_n^i \cap U_m^i$ for some $n \neq m$. Hence, s can reach some output terminal $nd_{k-2}...d_0$ in U_n and another $md'_{k-2}...d'_0$ in U_m. There exists some input terminal p of W that can reach through s the output terminals $nd_{k-2}...d_0$ and $md'_{k-2}...d'_0$ via the control tags $nd_{k-2}...d_0$ and $md'_{k-2}...d'_0$, respectively, because W is D-controllable. Hence, the two tags must agree in the i leftmost digits, yielding $m = n$, and leading to a contradiction.

(b) If a switch s in U_n^i were linked to a switch in U_m^{i+1} for some $m \neq n$, then s could reach an output terminal in U_n and another in U_m. This would lead to the same contradiction as above.

(c) The proof is by backward induction on i.

Basis: $i = k-1$. Follows immediately from Lemma 3.3.

Induction: Assume the statement is true for all values of $i = k-1, ..., l+1$. It will be proved for $i = l$. By the inductive hypothesis, we have $|U_n^{l+1}| = r^{k-2}$. Using (b), it can be concluded that all the outgoing links from the switches in U_n^l go to switches in U_n^{l+1}, and all the incoming links to the switches of U_n^{l+1} come from switches in U_n^l. Therefore, the number of the links between the switches of U_n^l and the switches of U_n^{l+1} is equal to $r|U_n^l|$ on the one hand, and to $r|U_n^{l+1}|$ on the other hand. Hence, $|U_n^l| = |U_n^{l+1}| = r^{k-2}$. ∎

3.5 Theorem. *Every D-controllable network in MIN(r,k) is in GRN(r,k).*

Proof. Let W be a D-controllable network in MIN(r,k). It will be shown by induction on k that W is in GRN(r,k).

Basis: $k = 1$. It is obvious that W is a single switch that is right bare-ended (after Lemma 3.3-(b)), and hence in GRN$(r,1)$.

Induction: Assume the theorem is true for all values of $k < l$. It will be proved for $k = l$. By Lemma 3.4-(c), each U_n^i has r^{l-2} switches. Permute the switches of the rightmost column so that W becomes right bare-ended. For every $i = 1,2,...,l-2$ permute the switches of column i so that the labels of the switches of U_n^i are nr^{l-2}, $nr^{l-2}+1,..., (n+1)r^{l-2}-1$. Since $U_n^i \cap U_m^i = \emptyset$ for every $n \neq m$ (Lemma 3.4-(a)), and since the outgoing links from the switches of U_n^i go only to switches in U_n^{i+1} (Lemma 3.4-(b)), it follows that the subnetworks $W_n = (U_n^1, U_n^2, ..., U_n^{l-1})$, for $n = 0,1,2,...,r-1$, are disjoint. Hence, W can be put in the form $W = fT(W_0, W_1, ..., W_{r-1})$ for some f and T. It can be seen that each W_n is D-controllable. By the inductive hypothesis, each W_n is in GRN$(r, l-1)$. Consequently, W is in GRN(r,l). ∎

Therefore, the topologically defined class GRN is identical to the control-characterized class of D-controllable networks. This topological characterization will be used to show that all doubly D-controllable networks are topologically and functionally equivalent. It can also be used to develop an optimal $O(N \log_r N)$ algorithm to decide if a MIN is D-controllable [11].

§4. Structure of FD-Controllable Networks

The class GRN is extended in this section and shown to be identical to the class of FD-controllable networks. Recall that if W is an FD-controllable network in MIN(r,k), there is a permutation f, called the control function, such that the control tag CT(i,j) to establish the path $i \to j$ is $f(j)$.

4.1. Definition. *The extended GRN is the class of networks of the form Wf where W is in GRN(r,k) and f is a permutation of S_{r^k}.*

The following lemma relates the networks of the extended GRN to the FD-controllable networks and their control functions.

4.2 Lemma. *(a) If W is in GRN(r,k) and f a permutation of S_{r^k}, then Wf is FD-controllable and the control tag CT(Wf,i,j) for the path $i \to j$ is $f^{-1}(j)$.*
(b) If W is FD-controllable with control function g, then Wg is D-controllable.
(c) If W' is FD-controllable, then there exist a network W in GRN(r,k) and a permutation f such that $W' = Wf$.

Proof. (a) Going from i to j in Wf is the same as going from i to $f^{-1}(j)$ in W. Therefore, CT(Wf,i,j) = CT$(W,i,f^{-1}(j))$ which is equal to $f^{-1}(j)$ because W is in GRN and hence D-controllable. It follows that Wf is FD-controllable and its control function is f^{-1}.
(b) CT(Wg,i,j) = CT$(W,i,g^{-1}(j))$ = $g(g^{-1}(j)$ = j. Therefore, W is D-controllable.
(c) Let W' be an FD-controllable network, and g its control function. Let also $W = W'g$ and $f = g^{-1}$. After (b), W is D-controllable and hence in GRN. Clearly, $Wf = (W'g)g^{-1} = W'(gg^{-1}) = W'$. ∎

4.3 Theorem. *A network in MIN(r,k) is FD-controllable if and only if it is in the extended GRN(r,k).*
Proof. It follows from Lemma 4.2-(a,c). ∎

§5. Doubly Controllable Networks

As pointed out earlier, it is of theoretical as well as practical interest to study the networks that are efficiently controllable not just from left terminals to right terminals, but also from right terminals to left terminals

Recall that by ***right-to-left D-controllability*** it is meant that if right terminal j needs to communicate to left terminal $i = i_{k-1}i_{k-2}...i_0$ in a network W, then the control tag $i_{k-1}i_{k-2}...i_0$ is used to establish the path as follows. The digit i_{k-1} controls the switches of column $k-1$ (i.e., the rightmost column), i_{k-2} controls the switches of column $k-2$, and so on. ***Right-to-left FD-controllability*** is defined similarly, where the control tag (denoted CT$_{R-L}(i,j)$) to establish the path from right terminal j to left terminal i is $f(i)$. The permutation f is then called the ***right-to-left control function***.

So for doubly controllable networks, we make a distinction between the left-to-right control tags CT$_{L-R}$ (the old sense of CT) and the right-to-left control tags CT$_{R-L}$. We also make a distinction between left-to-right control functions and right-to-left control functions, which do not have to be identical.

Note that most existing networks have been shown to be doubly D-controllable or doubly FD-controllable. In this section, this will be generalized. It will be shown that all doubly D-controllable networks are strictly functionally equivalent and all doubly FD-controllable networks are widely functionally equivalent to the baseline.

The proof of the equivalence of the doubly D-controllable networks with B(r,k) involves the following steps. First, it will be established that if W is a doubly D-controllable network, then the switches of its leftmost and rightmost columns can be repositioned so that W becomes bare-ended of the form $W = T(W_0, W_1, ..., W_{r-1})$. Afterwards, it will be shown that the switches in column 1 can be repositioned so that T becomes identical to the corresponding interconnection R in B(r,k) and all the W_i's become doubly D-controllable. Finally, the proof will proceed by induction on k. These steps will be made precise next.

5.1 Lemma. *Let W be a doubly D-controllable network in MIN(r,k). Then the following statements are true:*
(a) The switches of column 0 and column k-1 of W can be repositioned so that W becomes bare-ended.
(b) Assume that W is bare-ended, and in canonical form $W = T(W_0, W_1, ..., W_{r-1})$. Then the switches of the leftmost column of each W_l can be repositioned so that T becomes identical to the corresponding interconnection R in the baseline B(r,k).
(c) Assume W is as in (b) and $T = R$. Then W_i is doubly D-controllable for every $i = 0,1,...,r-1$.

Proof. (a) This follows by applying Lemma 3.3-(b) to W W^{-1}.

(b) Consider W_l for some arbitrary l, and let $i_{k-1}...i_2$ be a $(k-2)$-digit r-ary label. Using the double D-controllabilty of W, it can be shown that the right ports $(i_{k-1}...i_2tl)_{0\leq t\leq r-1}$ of column 0 of W are linked to a single switch (say $s_{i_{k-1}...i_2}$) in column 1 of W, such that the right port $i_{k-1}...i_2tl$ is linked to the t-th left port of switch $s_{i_{k-1}...i_2}$. Now if switch $s_{i_{k-1}...i_2}$ is moved to position $li_{k-1}...i_2$ of column 1 of W for every $i_{k-1}...i_2$, its left ports get the global labels $(li_{k-1}...i_2t)_{0\leq t\leq r-1}$ which are respectively equal to the unshuffle of the labels $(i_{k-1}...i_2tl)_{0\leq t\leq r-1}$. In all, the interconnection from column 0 to column 1 of W becomes identical with the unshuffle R.

(c) As each W_l is D-controllable, it suffices to show that each W_l is D-controllable from right to left. Let $i = i_{k-1}i_{k-2}...i_1$ and j be an input terminal and an output terminal of W_l, respectively, for some arbitrary l. The equivalent label of i in W is $i' = li_{k-1}i_{k-1}...i_1$. As $T = R$, i' is linked to the right port $R^{-1}(i')$ of column 0 of W. The path $R^{-1}(i') \leftarrow j$ from output terminal j to input terminal $R^{-1}(i')$ in W has to go through i'. As W is D-controllable from right to left, the control tag for this path is $R^{-1}(i') = i_{k-1}i_{k-1}...i_1l$. Hence, the sub-tag $i_{k-1}...i_1$ establishes the sub-path from j to i'. Since i' is the same port as i and $i = i_{k-1}...i_1$, it follows that the control tag for the path from j to i in W_l is i. Consequently, W_l is D-controllable from right to left. ∎

5.2 Theorem. *All doubly D-controllable networks in $MIN(r,k)$ are strictly equivalent to the baseline network $B(r,k)$.*

Proof. Let W be a doubly D-controllable network. Due to the previous lemma, we can assume that $W = R(W_0, ..., W_{r-1})$ where each W_i is doubly D-controllable. As $\mathrm{B}(r,k) = R(\mathrm{B}(r,k-1), ..., \mathrm{B}(r,k-1))$, a simple induction on k establishes the theorem. ∎

The next lemma will relate doubly FD-controllable networks, the baseline network and the control functions. The proof is straightforward and hence omitted.

5.3 Lemma. *(a) If $W = gB(r,k)f$, then $CT_{L\to R}(W,i,j) = f^{-1}(j)$ and $CT_{R\to L}(W,i,j) = g(i)$.*
(b) If W is doubly FD-controllable, and if $CT_{L\to R}(W,i,j) = f(j)$ and $CT_{R\to L}(W,i,j) = g(i)$, then W is strictly topologically equivalent to $gB(r,k)f^{-1}$.

5.4 Theorem. *All doubly FD-controllable networks in $MIN(r,k)$ are widely equivalent to the baseline network $B(r,k)$.*

Proof. Follows from the previous lemma. ∎

Lemma 5.3 has other implications, some of which have been proved elsewhere through other methods [10] about the relations among some of the existing networks. The following theorem rediscovers these relations and generalizes them for arbitrary switch sizes $r \geq 2$.

5.5 Theorem. *$B^{-1}(r,k) \equiv B(r,k)$, $\Omega(r,k) \equiv \rho B(r,k)$ and $\Omega^{-1}(r,k) \equiv B(r,k)\rho$ where ρ is the digit reversal $(\rho(x_{k-1}...x_1x_0) = x_0x_1...x_{k-1})$.*

Proof. Since $\mathrm{B}(r,k)$ is doubly D-controllable, it follows that its inverse is doubly D-controllable. Consequently, $\mathrm{B}^{-1}(r,k) \equiv \mathrm{B}(r,k)$, after Theorem 5.2. $\Omega(r,k)$ is doubly FD-controllable, its left-to-right control function is the identity permutation and its right-to-left control function is ρ [4]. After Lemma 5.3-(b), we have $\Omega(r,k) \equiv \rho\mathrm{B}(r,k)$. Furthermore, $\Omega^{-1}(r,k) \equiv (\rho\mathrm{B}(r,k))^{-1} \equiv \mathrm{B}^{-1}(r,k)\rho^{-1} \equiv \mathrm{B}(r,k)\rho$ because $\rho^{-1} = \rho$. ∎

The equivalence among existing MIN's is therefore no coincidence since they are doubly D- or FD-controllable. In the remaining part of the paper, the double controllability of these networks will be shown to be a result of their inter-column interconnections, namely bit manipulation permutations, which are operations that permute the bits of binary labels in a specified manner.

§6. Digit Permutation Networks

As was just pointed out, one common feature in the definitions of the existing multistage interconnection networks is that the interconnections between columns are bit permutations. The well-known shuffle interconnection is an example. Some of the reasons for using these permutations as interconnections are their regularity, rich structure and ease of analysis. In this section, the whole class of digit permutation networks will be studied using the concepts of D-control, FD-control and double FD-control, and taking advantage of the equivalence among all doubly FD-controllable networks.

The approach is algebraic. A relation will be derived relating $\mathrm{CT}(i,j)$ with i and j. This relation will be used to find necessary and sufficient conditions for $k+1$ digit permutations to construct a MIN that has the unique path property. Later the control tags in digit permutation networks are shown to be functions of the destination tags only. This makes them FD-controllable. Making use of the fact that the inverse of a digit permutation network is a digit permutation network, it will be concluded that digit permutation networks are doubly FD-controllable and hence widely equivalent to the baseline network.

6.1. Definition. A permutation f of $S_N = \{0,1,...,N-1\}$, where $N = r^k$, is a digit permutation in the system of base r if there exists a permutation π of $S_k = \{0,1,...,k-1\}$ such that $f(x_{k-1}...x_1x_0) = x_{\pi(k-1)}...x_{\pi(1)}x_{\pi(0)}$, where $x_{k-1}...x_1x_0$ is an arbitrary k-digit r-ary label. In this case, f is denoted f_π and π is called the kernel of f_π.

6.2. Definition. A digit permutation network, denoted $\mathrm{DPN}(f_0, f_1, ...f_k)$, is a network in $\mathrm{MIN}(r,k)$ where the leftmost interconnection is f_0, the rightmost interconnection is f_k, the interconnection from column $i-1$ to column i is f_i, for $i = 1, ..., k-1$, and all the f_j's are digit permutations of S_{r^k} in the system of base r.

Denote by E_a^i, where a is an r-ary digit and $i = 0,1,...,k-1$, the following mapping from S_N to S_N:

$$E_a^i(x_{k-1}...x_0) = x_{k-1}...x_{i+1}ax_{i-1}...x_0$$

that is, E_a^i replaces the i-th digit by a.

Next, the relation between an arbitrary input terminal s, an arbitrary output terminal d and the control tag $c = c_{k-1}c_{k-2}...c_0$ for the path $s \rightarrow d$ in a digit permutation network DPN$(f_{\pi_0}, f_{\pi_1}, ..., f_{\pi_k})$ will be drived. Recall that the digit c_{k-1-i} controls column i for $i = 0, 1, ..., k-1$. Note that if the path $s \rightarrow d$ enters column i through input port $x_{k-1}...x_1x_0$, it exits that column through the output port $x_{k-1}...x_1c_{k-1-i}$ which is equal to $E^0_{c_{k-1-i}}(x_{k-1}...x_1x_0)$. Note also that if the path exits column $i-1$ through some output port y, it then enters the next column, that is, column i, through input port $f_{\pi_i}(y)$ because the interconnection between column $i-1$ and column i is f_{π_i}. We have thus proved the following lemma:

6.3 Lemma. *In a digit permutation network DPN$(f_{\pi_0}, f_{\pi_1}, ..., f_{\pi_k})$ an arbitrary output terminal d is related to an arbitrary input terminal s and the control tag $c = c_{k-1}c_{k-2}...c_0$ for the path $s \rightarrow d$ by the following relation: $d = (s)f_{\pi_0}E^0_{c_{k-1}}f_{\pi_1}E^0_{c_{k-2}}f_{\pi_2}E^0_{c_{k-3}}...E^0_{c_0}f_{\pi_k}$.*

The E's will be "filtered" out to the right of the f's in the relation above. To that effect, the following lemma is needed. The proof is straightforward and hence omitted.

6.4 Lemma. $f_\alpha f_\beta = f_{\beta\alpha}$ and $E^i_a f_\pi = f_\pi E_a^{\pi^{-1}(i)}$.

6.5 Lemma. *Under the assumptions of Lemma 6.3 we have $d = (s)f_{\beta_0}E^{\beta_1^{-1}(0)}_{c_{k-1}}E^{\beta_2^{-1}(0)}_{c_{k-2}}...E^{\beta_k^{-1}(0)}_{c_0}$ where $\beta_i = \pi_k\pi_{k-1}...\pi_i$.*

Proof. Let $g = f_{\pi_0}E^0_{c_{k-1}}f_{\pi_1}E^0_{c_{k-2}}f_{\pi_2}E^0_{c_{k-3}}...E^0_{c_0}f_{\pi_k}$

By making repeated use of Lemma 6.4 on the expression of g (from right to left) we have

$g = f_{\beta_0}E^{\beta_1^{-1}(0)}_{c_{k-1}}E^{\beta_2^{-1}(0)}_{c_{k-2}}...E^{\beta_k^{-1}(0)}_{c_0}$

As $d = g(s)$ (from Lemma 6.3), the lemma follows. ∎

The necessary and sufficient conditions as well as the relation between the control tag and the output terminal can now be easily derived as follows.

6.6 Theorem. *Let f_{π_0}, f_{π_1}, ... , and f_{π_k} be $k+1$ digit permutations, and $\beta_i = \pi_k\pi_{k-1}...\pi_i$.*
(a) The digit permutation network DPN$(f_{\pi_0}, f_{\pi_1}, ..., f_{\pi_k})$ is in MIN(r, k), that is, it has the unique path property, if and only if $\beta_1^{-1}(0)$, $\beta_2^{-1}(0)$, ... , $\beta_k^{-1}(0)$ are pairwise distinct .
(b) The control tag $c = c_{k-1}c_{k-2}...c_0$ for a path $s \rightarrow d$ in a digit permutation network DPN$(f_{\pi_0}, f_{\pi_1}, ..., f_{\pi_k})$ which has the unique path property is $c = f_\gamma(d)$, where $\gamma(i) = \beta^{-1}_{k-i}(0)$.

Proof. (a) Let s be an input terminal, d an output terminal and $c = c_0c_1...c_{k-1}$ be the control tag that establishes the path $s \rightarrow d$. Let $s' = f_{\beta_0}(s)$. Using the previous lemma, we have

$$d = (s')E^{\beta_1^{-1}(0)}_{c_{k-1}}E^{\beta_2^{-1}(0)}_{c_{k-2}}...E^{\beta_k^{-1}(0)}_{c_0}$$

It can be easily seen that the effect of each $E^{\beta_i^{-1}(0)}_{c_{k-i}}$ is to replace the digit in position $\beta_i^{-1}(0)$ of s' by c_{k-i}.

Assume first that the network has the unique path property. If $\beta_1^{-1}(0)$, $\beta_2^{-1}(0)$, ... , $\beta_k^{-1}(0)$ are not pairwise distinct, then $\{\beta_1^{-1}(0), \beta_2^{-1}(0), ..., \beta_k^{-1}(0)\}$ is a proper subset of $\{0, 1, ..., k-1\}$, and therefore, there exists some j in $\{0, 1, ..., k-1\}$ such that $j \neq \beta_i^{-1}(0)$ for all i. Consequently, the digits in the j-th digit position of s' and d must always agree. It follows that for a fixed s, and thus fixed s', no matter what control tag we use, we can never reach any output terminal d whose j-th digit differs from that of s'. This contradicts the unique path property.

Conversely, if $\beta_1^{-1}(0)$, $\beta_2^{-1}(0)$, ... , $\beta_k^{-1}(0)$ are pairwise distinct, then

$$\{\beta_1^{-1}(0), \beta_2^{-1}(0), ..., \beta_k^{-1}(0)\} = \{0, 1, ..., k-1\},$$

and therefore the mapping γ where $\gamma(i) = \beta^{-1}_{k-i}(0)$ is a permutation of $\{0, 1, ..., k-1\}$. Furthermore,

$$d = (s')E^{\gamma(k-1)}_{c_{k-1}}E^{\gamma(k-2)}_{c_{k-2}}...E^{\gamma(0)}_{c_0}$$

implying that the digit in position $\gamma(i)$ of d is c_i, that is, $d_{\gamma(k-1)}d_{\gamma(k-2)}...d_{\gamma(0)} = c_{k-1}c_{k-2}...c_0$. Therefore, $c = f_\gamma(d)$ and $d = f_{\gamma^{-1}}(c)$. As $f_{\gamma^{-1}}$ is a permutation (a digit permutation) of S_N, it follows that for a fixed input terminal s there corresponds to every control tag c one and only one output terminal. Therefore, the network has the unique path property.
(b) The relation $c = f_\gamma(d)$ has just been proved in (b). ∎

Part (a) of the previous theorem leads to a simple algorithm to determine if a sequence of $k+1$ digit permutations construct a digit permutation network that has the unique path property. The algorithm takes as input a sequence of $k+1$ kernels π_0, π_1, ... , π_k , computes the β_i^{-1}'s (noting that $\beta_i = \beta_{i+1}\pi_i$ and hence $\beta_i^{-1} = \pi_i^{-1}\beta_{i+1}^{-1}$), then computes the mapping γ such that $\gamma(i) = \beta^{-1}_{k-i}(0)$ as defined in the previous theorem. Finally it checks if γ is a permutation of $\{0, 1, ..., k-1\}$ by doing a bucket sort on $\gamma(0)$, $\gamma(1)$, ... , $\gamma(k-1)$ and checking if any of the "buckets" 0, 1, , ... , $k-1$ is empty. If no bucket is empty, then γ is a permutation, that is, the network has the unique path property; otherwise, the network does not have the unique path property. The time complexity can be easily seen to be $O(k^2) = O(\log_r^2 N)$.

6.1 Control of Digit Permutation Networks

Part (b) of the last theorem shows that digit permutation networks are FD-controllable and that their control functions are digit permutations (f_γ) which can be easily derived from the constituent digit permutations.

One consequence of the fact that the control function is a digit permutation f_γ is the increased control efficiency. One way of controlling a DPN with control function f_γ is to design the switch so that the destination labels can be used to set the switches as follows: the switches in column i use digit $\gamma(k-i+1)$ of the destination label as control digit, for $i = 0, 1, ..., k-1$. Another way is to provide some additional hardware in the input terminals to permute the digits of the output terminals according to γ and produce the control tag. As k is relatively small in practice, the additional amount of hardware is small. A third way is to compute $f_\gamma(d)$ in software every time a path $s \rightarrow d$ is to be established. This requires a small amount of memory to store γ (not f_γ) at every input terminal. This software control, though a little slower, is cheaper than the previous two ways and is more flexible in that it allows one network to simulate an equivalent network as will be seen shortly.

6.3 Equivalence of Digit Permutation Networks

As $(f_\pi)^{-1} = f_{\pi^{-1}}$, it follows that the inverse of a DPN is a DPN. Thus all DPN's in MIN(r,k) are doubly FD-controllable and hence widely equivalent to the baseline $B(r,k)$, by Theorem 5.2. This result is a generalization of the equivalence among the existing networks (in [10]).

6.4 Simulation Among DPN's

If two networks in MIN(r,k) are widely equivalent, then the terminals of one can be relabeled so that it realizes the permutations of the other. The problem is how to relabel the terminals.

We have shown in Lemma 5.3 that if a network W in MIN(r,k) is doubly FD-controllable with left-to-right control function f and right-to-left control function g, then W is strictly equivalent to $gB(r,k)f^{-1}$. Hence, the baseline network can simulate W by relabeling the input terminals of $B(r,k)$ by g^{-1} and the output terminals by f^{-1}. Generally, if W' is another doubly FD-controllable network in MIN(r,k) with left-to-right control function f' and right-to-left-control function g', then W' is strictly equivalent to $g'B(r,k)f'^{-1}$, and consequently, W' is strictly equivalent to $g'g^{-1}Wff'^{-1}$. Therefore, W can simulate W' by relabeling the input terminals of W by gg'^{-1} and the output terminals of W by ff'^{-1}. Therefore, the problem of relabeling the terminals of one network to simulate another network reduces to finding the left-to-right and right-to-left control functions of both networks. As the control function of digit permutation networks are digit permutations f_γ's, it is enough to find the γ's of the control functions.

The following algorithm takes as input two digit permutation networks W and W' represented by the sequences of their defining kernals $\pi_{0..k}$ and $\pi'_{0..k}$, respectively, and relabels the terminals of W to simulate W'. It computes γ and τ such that f_γ and f_τ are the left-to-right and right-to-left control function of W, respectively. It also computes the corresponding γ' and τ' of W'. As the input terminals of W have to be relabeled with $f_\tau f_{\tau'^{-1}}$, which is equal to to $f_{\tau'^{-1}\tau}$, the algorithm computes $\tau'^{-1}\tau$. Similarly, as the output terminals of W have to be relabeled with $f_\gamma f_{\gamma'^{-1}}$, which is equal to $f_{\gamma'^{-1}\gamma}$, the algorithm computes $\gamma'^{-1}\gamma$. Finally, the algorithm does the relabeling. Note that f_τ is the left-to-right control function of $W^{-1} = DPN(f_{\pi_k^{-1}}, f_{\pi_{k-1}^{-1}}, ..., f_{\pi_0^{-1}})$. Therefore, τ is computed in the same way as γ. The same applies to τ'.

Procedure Simulate (W, W')
(1) Compute γ, τ, γ', τ' and then $\gamma'^{-1}\gamma$ and $\tau'^{-1}\tau$;
(2) Broadcast $\gamma'^{-1}\gamma$ to all outputs and $\tau'^{-1}\tau$ to all inputs;
(3) **for** $i = 0$ **to** $N-1$ **do in parallel**
(4) relabel input terminal i of W by $(i)f_{\tau'^{-1}\tau}$;
(5) relabel output terminal i of W by $(i)f_{\gamma'^{-1}\gamma}$;

Time Complexity: Step 1 takes $O(k^2)$. Steps 2, 4 and 5 take $O(k)$ each. Thus, the procedure takes $O(\log_r^2 N)$.

§7. Conclusions

This paper examined several control classes of banyan MIN's, namely, D-control, FD-control, double D-control and double FD-control, and showed that the first two classes have a recursive structure and the last two are equivalent to the baseline. The digit permutation networks, where the interconnections are bit (digit)·permutes, were also studied and shown to be doubly FD-controllable and hence equivalent to the baseline, thus generalizing the results about the equivalence among existing networks. An optimal algorithm to simulate any DPN by any other DPN was also presented. Future work will examine the structure and functionality of MIN's whose control tags are easy-to-compute functions of both source and destination tags.

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