Solution to Homework 5

Problem 1: (20 points)

a)

$\times_7$	**0**	**1**	**2**	**3**	**4**	**5**	**6**
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Table 1 Multiplication table of mod 7

$\times_{10}$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	0	2	4	6	8
3	0	3	6	9	2	5	8	1	4	7
4	0	4	8	2	6	0	4	8	2	6
5	0	5	0	5	0	5	0	5	0	5
6	0	6	2	8	4	0	6	2	8	4
7	0	7	4	1	8	5	2	9	6	3
8	0	8	6	4	2	0	8	6	4	2
9	0	9	8	7	6	5	4	3	2	1

Table 2 Multiplication table of mod 10

b)

0: None

1: 1

2: 4

3: 5

4: 2

5: 3

6: 6

c)

0, 2, 4, 5, 6, and 8.

d)

$$5^1 \; mod \; 7 = 5$$
$$5^2 \; mod \; 7 = (5^1 \times 5) \; mod \; 7 = 4$$
$$5^3 \; mod \; 7 = (5^2 \times 5) \; mod \; 7 = (4 \times 5) \; mod \; 7 = -1$$

$$5^{612} \; mod \; 7 = (5^3)^{204} \; mod \; 7 = (-1)^{204} \; mod \; 7 = 1$$

Problem 2: (20 points)

a)

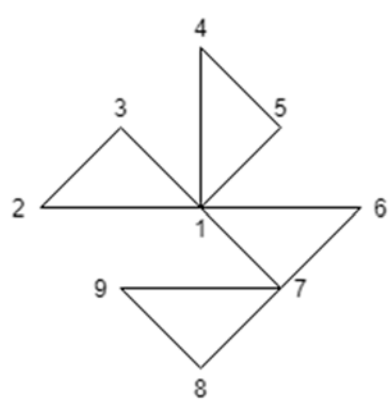

Figure 1 Undirected G

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	0	1	1	1	1	1	1	0	0
2	1	0	1	0	0	0	0	0	0
3	1	1	0	0	0	0	0	0	0
4	1	0	0	0	1	0	0	0	0
5	1	0	0	1	0	0	0	0	0
6	1	0	0	0	0	0	1	0	0
7	1	0	0	0	0	1	0	1	1
8	0	0	0	0	0	0	1	0	1
9	0	0	0	0	0	0	1	1	0

Table 3 Adjacency matrix of undirected G

b)

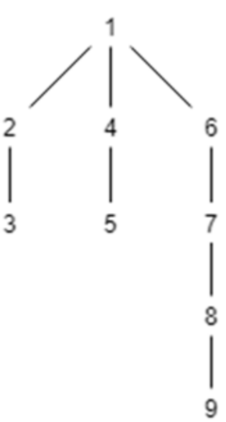

Figure 2 DFS

Figure 3 BFS

c) Yes, it is connected because all nodes are included in a DFS/BFS tree.

d) The articulation points: 1 and 7.

e) Yes. $(2,3), (3,1), (1,4), (4,5), (5,1), (1,6), (6,7), (7,8), (8,9), (9,7), (7,1), (1,2)$.

f) No, since there are articulation points.

Problem 3: (20 points)

a)

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	0	1	1	1	1	1	1	2	2
2	1	0	1	2	2	2	2	3	3
3	1	1	0	2	2	2	2	3	3
4	1	2	2	0	1	2	2	3	3
5	1	2	2	1	0	2	2	3	3
6	1	2	2	2	2	0	1	2	2
7	1	2	2	2	2	1	0	1	1
8	2	3	3	3	3	2	1	0	1
9	2	3	3	3	3	2	1	1	0

Table 4 Distance matrix

b) The diameter of G is 3, which is the largest value in the distance matrix above.

c)

1: 2

2: 3

3: 3

4: 3

5: 3

6: 2

7: 2

8: 3

9: 3

d) The centers of G are 1, 6, and 7, because their corresponding radii are the smallest (2)

e)

$$1: \frac{10}{9}$$
$$2: \frac{16}{9}$$
$$3: \frac{16}{9}$$
$$4: \frac{16}{9}$$
$$5: \frac{16}{9}$$
$$6: \frac{14}{9}$$
$$7: \frac{4}{3}$$
$$8: 2$$
$$9: 2$$

1 has the smallest average radius.

Problem 4: (10 points)

a)

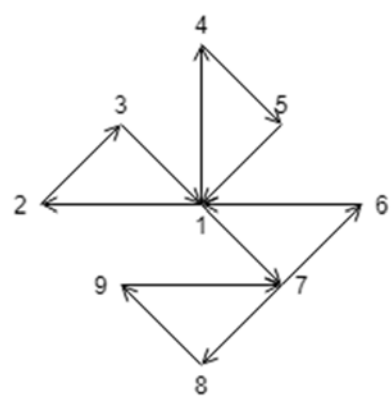

Figure 4 Directed G

	1	2	3	4	5	6	7	8	9
1	0	1	0	1	0	0	1	0	0
2	0	0	1	0	0	0	0	0	0
3	1	0	0	0	0	0	0	0	0
4	0	0	0	0	1	0	0	0	0
5	1	0	0	0	0	0	0	0	0
6	1	0	0	0	0	0	0	0	0
7	0	0	0	0	0	1	0	1	0
8	0	0	0	0	0	0	0	0	1
9	0	0	0	0	0	0	1	0	0

Table 5 Adjacency matrix of directed G

b)

	In	Out
1	3	3
2	1	1
3	1	1
4	1	1
5	1	1
6	1	1
7	2	2
8	1	1
9	1	1

Table 6 Indegree and outdegree

c)

1: 1, 2, 3, 4, 5, 7, 6, 8, 9

2: 2, 3, 1, 4, 5, 7, 6, 8, 9

3: 3, 1, 2, 4, 5, 7, 6, 8, 9

4: 4, 5, 1, 2, 3, 7, 6, 8, 9

5: 5, 1, 2, 3, 4, 7, 6, 8, 9

6: 6, 1, 2, 3, 4, 5, 7, 8, 9

7: 7, 8, 9, 6, 1, 2, 3, 4, 5

8: 8, 9, 7, 6, 1, 2, 3, 4, 5

9: 9, 7, 8, 6, 1, 2, 3, 4, 5

G is strongly connected because every node is reachable from every other node.

Problem 5: (20 points)

a)

Prove by induction on the height h that ${L_T}^{(h)} \leq {L_P}^{(h)}$, where the superscript is added to indicate that the corresponding tree is of height h.

Basis, $h = 0$: ${L_T}^{(0)} = {L_P}^{(0)} = 1 \Rightarrow {L_T}^{(0)} \leq {L_P}^{(0)}$

Induction step: Assume ${L_T}^{(h-1)} \leq {L_P}^{(h-1)}$, prove ${L_T}^{(h)} \leq {L_P}^{(h)}$.

${L_T}^{(h)} \leq 2{L_T}^{(h-1)}$ by definition of binary trees

${L_P}^{(h)} = 2{L_P}^{(h-1)}$ by definition of perfect binary trees

$L_T \leq 2{L_T}^{(h-1)} \leq 2{L_P}^{(h-1)} = L_P$

$\therefore {L_T}^{(h)} \leq {L_P}^{(h)}$

Prove by induction the height h that ${N_T}^{(h)} \leq {N_P}^{(h)}$, where here again the superscript is added to indicate that the corresponding tree is of height h.

Basis, $h = 0$: ${N_T}^{(0)} = {N_P}^{(0)} = 1 \Rightarrow {N_T}^{(0)} \leq {N_P}^{(0)}$

Assume ${N_T}^{(h-1)} \leq {N_P}^{(h-1)}$, prove ${N_T}^{(h)} \leq {N_P}^{(h)}$.

${N_T}^{(h)} = {N_T}^{(h-1)} + {L_T}^{(h)} \leq {N_T}^{(h-1)} + 2{L_T}^{(h-1)}$ by definition of binary trees

${N_P}^{(h)} = {N_P}^{(h-1)} + 2{L_P}^{(h-1)}$ by definition of perfect binary trees

In addition to the induction hypothesis ${N_T}^{(h-1)} \leq {N_P}^{(h-1)}$, we also know that ${L_T}^{(h-1)} \leq {L_P}^{(h-1)}$ which was proven above.

$\therefore {N_T}^{(h)} \leq {N_T}^{(h-1)} + 2{L_T}^{(h-1)} \leq {N_P}^{(h-1)} + 2{L_P}^{(h-1)} = {N_P}^{(h)}$

$\therefore {N_T}^{(h)} \leq {N_P}^{(h)}$

b)

${L_P}^{(h)} = 2^1{L_P}^{(h-1)} = 2^2{L_P}^{(h-2)} = \cdots = 2^h{L_P}^{(0)} = 2^h$

${L_T}^{(h)} \leq {L_P}^{(h)} \Rightarrow {L_T}^{(h)} \leq 2^h$

$N_P{}^{(h)} = N_P{}^{(h-1)} + 2^1 L_P{}^{(h-1)}$

$= N_P{}^{(h-1)} + 2 \times 2^{h-1}$

$= N_P{}^{(h-1)} + 2^h$

$= N_P{}^{(h-2)} + 2^{h-1} + 2^h$

$= \cdots$

$= N_P{}^{(0)} + 2^1 + 2^2 + \cdots + 2^h$

$= 1 + 2^1 + 2^2 + \cdots + 2^h$

$= 2^{h+1} - 1$

$N_T{}^{(h)} \leq N_P{}^{(h)} \Rightarrow N_T{}^{(h)} \leq 2^{h+1} - 1.$

c)

$$L_T \leq 2^h$$
$$\Rightarrow \log_2 2^h \geq \log_2 L_T$$
$$\Rightarrow h \geq \log_2 L_T$$

$$N_T \leq 2^{h+1} - 1$$
$$\Rightarrow 2^{h+1} \geq N_T + 1$$
$$\Rightarrow \log_2\left(2^{h+1}\right) \geq \log_2(N_T + 1)$$
$$\Rightarrow h + 1 \geq \log_2(N_T + 1)$$
$$\Rightarrow h \geq \log_2(N_T + 1) - 1$$
$$\Rightarrow h \geq \log_2(N_T + 1) - \log_2 2$$
$$\Rightarrow h \geq \log_2\left(\frac{N_T + 1}{2}\right)$$

Problem 6: (10 points)

Basis: When $n = 1$, there is only one node in the BST. It has the minimum key and is leftmost.

Assume the minimum key is leftmost when there are $n - 1$ nodes in an arbitrary BST.

When a new node is added to the BST, consider the two cases:

1. Its key is larger than the current minimum: Then it can be inserted anywhere but the left child of the current minimum. Thus, the minimum key is still leftmost.

2. Its key is less than or equal to the current minimum: Then it will be inserted to the left child of the current minimum and become the new leftmost node of the tree. Thus, the new minimum key becomes leftmost.

Since in both cases the minimum key is leftmost, when there are n nodes, the minimum key of every BST is leftmost.

Bonus problem: (5 points)

By contradiction: Assume that after removing a single edge (x,y), the tree T is still connected.

This is equivalent to: After removing (x,y) from T, the resulting graph (call it T') is still a tree, and thus there is a path from x to y in T'. Let's call that path x, x_2, x_3, …, x_k, y.

Therefore, in the original tree T, we have a cycle: x, x_2, x_3, …, x_k, y, x, which is a contradiction because in a tree there are no cycles.