CS 1311 Spring 2019
Youssef

Solution to Homework 4

1. (a)

	reflexive	symmetric	antisymmetric	transitive	equivalence	partial order
$p \implies q$	✓			✓		
$p \iff q$	✓	✓		✓	✓	
$p \iff \neg q$		✓				
$p \not\Leftrightarrow \neg q$	✓	✓				

(b)
   i. Equivalence relation, not antisymmetric. An equivalence class is a set of people who have the same hobbies.
   ii. Partial order, not symmetric. The minimum is the unique youngest person if exists. The maximum is the unique oldest person if exists. The minimals are the youngest people. The maximals are the oldest people.
   iii. Not reflexive, symmetric, not antisymmetric, transitive, and therefore not an equivalence relation and not a partial order.
   iv. Reflexive, not symmetric, not antisymmetric, not transitive, not an equivalence relation, not a partial order.

2. (a) For every x, x and x have the same number of 1's; R is reflexive. If x and y have the same number of 1's, then y and x have the same number of 1's; R is symmetric. If x and y have the same number of 1's and y and z have the same number of 1's, then x and z have the same number of ones; R is transitive. Therefore, R is an equivalence relation. The equivalences classes are:

$$\begin{aligned}\{\{0000\},\\ \{0001, 0010, 0100, 1000\},\\ \{0011, 0101, 0110, 1001, 1010, 1100\},\\ \{0111, 1011, 1101, 1110\},\\ \{1111\}\}\end{aligned}$$

(b) $H(0000, 0101) = 2, H(1101, 0110) = 3, H(1001, 0110) = 4.$

(c) For every x, $H(x, x) = 0$ is even; R is reflexive. If $H(x, y)$ is even, then $H(y, x) = H(x, y)$ is even; R is symmetric. For transitivity, suppose $H(x, y)$ is even and $H(y, z)$ is even. Define $A = \{i \mid x_i \neq y_i\}$, $B = \{i \mid y_i \neq z_i\}$, and $C = \{i \mid x_i \neq z_i\}$. Then,

$$C = (A - A \cap B) \cup (B - A \cap B).$$

And since $A - A \cap B$ and $B - A \cap B$ are disjoint, $A \cap B$ is a subset of A, and $A \cap B$ is a subset of B, it follows that

$$\begin{aligned} |C| &= |A - A \cap B| + |B - A \cap B| \\ &= |A| - |A \cap B| + |B| - |A \cap B| \\ &= |A| + |B| - 2|A \cap B| \\ &= H(x, y) + H(y, z) - 2|A \cap B|, \end{aligned}$$

which is even. Therefore, $H(x, z) = |C|$ is even; R is transitive. The equivalence classes are:

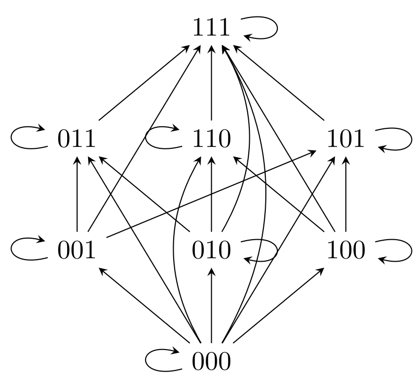
$$\{0000, 0011, 0101, 1001, 0110, 1010, 1100, 1111\},$$
$$\{0001, 0010, 0100, 1000, 0111, 1011, 1101, 1110\}.$$

3. (a) For every x, $x_i \vee x_i = x_i$, and hence $x \oplus x = x$; R is reflexive. If $x \oplus y = y$ and $y \oplus x = x$, then $x = y \oplus x = x \oplus y = y$; R is antisymmetric. For transitivity, Assume $x\ R\ y$ and $y\ R\ z$. Need to prove that $x\ R\ z$, that is we need to prove that $x \oplus z = z$.

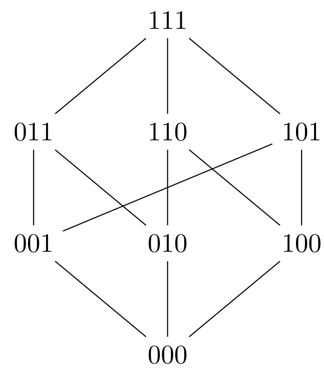
$$\begin{aligned} x \oplus z &= x \oplus (y \oplus z) && \textit{because } y\ R\ z \textit{ and thus } y \oplus z = z \\ &= (x \oplus y) \oplus z && \textit{because "or" is associative} \\ &= y \oplus z && \textit{because } x\ R\ y \textit{ and thus } x \oplus y = y \\ &= z && \textit{because } y \oplus z = z \textit{ as we saw above.} \end{aligned}$$

(b) R is not total. Let $x = 100$ and $y = 001$. Then, $x \oplus y \neq y$ and $y \oplus x \neq x$.

(c) Graph:



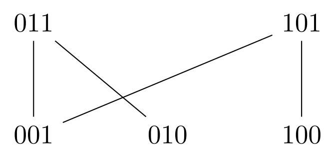
Hess diagram:



(d) The maximum is 111.

(e) The minimum is 000.

(f) Hess diagram:



The maximum and minimum don't exist. The maximals are 011 and 101. The minimals are 001, 010, and 100.

4. (a) If R is reflexive, then for every x, since xRx, we have xSx.

   (b) $xSy \implies xRy$ or $yRx \implies yRx$ or $xRy \implies ySx$.

   (c) Let R be the subset relation, i.e. xRy if and only if $x \subseteq y$. Then, $\{1\}S\{1,2\}$ and $\{1,2\}S\{2\}$, but $\{1\}S\{2\}$ is false.

   (d) Using the same relation R, $\{1\}S\{1,2\}$ and $\{1,2\}S\{1\}$, but $\{1\} \neq \{1,2\}$; S is not antisymmetric.

   (e) Suppose R is an equivalence relation. Then R is symmetric, i.e., $x\ R\ y \Leftrightarrow y\ R\ x$, and that implies that $S = R$ because

   $$x\ S\ y \Leftrightarrow (x\ R\ y\ or\ y\ R\ x) \Leftrightarrow (x\ R\ y\ or\ x\ R\ y)\ due\ to\ symmetry \Leftrightarrow x\ R\ y$$

   Since $S = R$ and R is an equivalence relation, then S is an equivalence relation.

   (f) By parts a, c, and d, not necessarily.

5. (a) If R is reflexive, then for every x, since xRx, we have xSx.

   (b) $xSy \implies xRy \wedge yRx \implies yRx \wedge xRy \implies ySx$.

(c) Suppose R is transitive. Then,

$$\begin{aligned} xSy \wedge ySz &\implies (xRy \wedge yRx) \wedge (yRz \wedge zRy) \\ &\implies (xRy \wedge yRz) \wedge (zRy \wedge yRz) \\ &\implies xRz \wedge zRx \\ &\implies xSz. \end{aligned}$$

(d) If R is antisymmetric, then $xSy \wedge ySx \implies xRy \wedge yRx \implies x = y$, i.e. S is antisymmetric.

(e) By parts a, b, and c, yes.

(f) By parts a, c, and d, yes.

6. (a) For any x, xRx xor xRx is false, and therefore, xSx is false.

(b) $xSy \implies xRy$ xor $yRx \implies yRx$ xor $xRy \implies ySx$.

(c) Let R be the subset relation as before. Then, $\{1\}S\{1,2\}$ and $\{1,2\}S\{2\}$, but $\{1\}S\{2\}$ is false.

Bonus. (a) For every $(a,b) \in \mathbb{Z} \times \mathbb{Z}^*$, $ab = ba$ because number multiplication is commutative, and therefore $(a,b)R(a,b)$; R is reflexive. For symmetry,

$$(a,b)R(c,d) \implies ad = bc \implies cb = bc = ad = da \implies cb = da \implies (c,d)R(a,b).$$

For transitivity, let $(a,b)R(c,d)$ and $(c,d)R(x,y)$, that is, $ad = bc$ and $cy = dx$. Then,
$d(ay - bx) = ady - bdx = (ad)y - b(dx) = (bc)y - b(cy) = bcy - bcy = 0$; now since $d(ay - bx) = 0$ and $d \neq 0$, we must have $ay - bx = 0$. Therefore, $ay = bx$ and thus $(a,b)R(x,y)$.

(b) First, f is a well-defined function because if $[a,b] = [c,d]$, that is $ad - bc = 0$, then $a/b = c/d$. The function f is one-to-one because $f([a,b]) = f([c,d]) \implies a/b = c/d \implies ad = bc \implies (a,b)R(c,d) \implies [a,b] = [c,d]$. And f is onto because for any $a/b \in \mathbb{Q}$, there exists $[a,b] \in E$ where $f([a,b]) = a/b$.