CS 1311
Solution to Homework 3

Spring 2019

1. (a) $x_n = (x_0 - \frac{b}{1-a})6^n + \frac{b}{1-a}$, where $a = 6, b = 10$, and $x_0 = 1$. Therefore, $x_0 = 3 \times 6^n - 2$

   (b) The characteristic polynomial is $s^2 - 6s + 8 = (s-2)(s-4)$. Thus, $s_1 = 2$ and $s_2 = 4$. The solution therefore is $x_n = A \cdot 2^n + B \cdot 4^n$ for some constants A, B. Using the initial conditions, we get the system of linear equations:

$$\begin{cases} A + B & = x_0 = 1 \\ 2A + 4B & = x_1 = 3 \end{cases}.$$

   Solving for A and B gives $A = B = 1/2$. So $x_n = 2^{n-1} + 2^{2n-1}$.

   (c) We have $a = 7, b = -12$, and $c = 1$. Therefore, $s_1 = \frac{a+\sqrt{a^2+4b}}{2} = 4$ and $s_2 = \frac{a-\sqrt{a^2+4b}}{2} = 3$.
   The solution is $x_n = A4^n + B3^n + \hat{x}_n$.
   Since c is a constant, then $\hat{x}_n = C$ for some constant C where: $C = 7C - 12C + 18$. Solving that equation gives us $C = 3$. Hence, $x_n = A4^n + B3^n + 3$.
   To find A and B, we use the initial conditions to set up a system of two linear equations:

$$\begin{cases} A + B & = x_0 - 3 = 1 \\ 4A + 3B & = x_1 - 3 = 7 \end{cases}.$$

   Solving the system gives $A = 4$ and $B = -3$. So

$$x_n = 4^{n+1} - 3^{n+1} + 3.$$

2. (a) The solution is of the form $x_n = A6^n + \hat{x}_n$ where $\hat{x}_n = dn + e$ for some constants d and e. Then,

   $\hat{x}_n - 6\hat{x}_{n-1} - 10n - 2 = 0$ and thus $dn + e - 6(d(n-1)+e) - 10n - 2 = 0$.

   Cleaning up, we get $(-5d - 10)n - 5e + 6d - 2 = 0$. Therefore, $-5c - 10 = 0$ and $-5e + 6d - 2 = 0$. Solving those two equations yields: $d = -2,\ e = -\frac{14}{5}$. We now have,

$$x_n = A6^n - 2n - \frac{14}{5}.$$

Using the initial condition, $-3 = x_0 = A6^0 - 2(0) - \frac{14}{5}$, we get $A = -\frac{1}{5}$, and thus, the final solution is:

$$x_n = -\frac{1}{5}6^n - 2n - \frac{15}{4}.$$

(b) This problem is like problem 1(c) except the "c" part is not a constant but is instead $18n + 3$. Therefore, the solution is of the form:

$$x_n = A4^n + B3^n + \hat{x}_n$$

where $\hat{x}_n = dn + e$ for some constants d and e.
Since $\hat{x}_n$ is a solution to the actual recurrence relation, we have

$$\hat{x}_n = 7\hat{x}_{n-1} - 12\hat{x}_{n-2} + 18n + 3$$

Which yields: $dn + e = 7(dn - d + e) - 12(dn - 2d + e) + 18n + 3$. Cleaning that, we obtain,

$$dn + e = (18 - 5d)n + 17d - 5e + 3$$

yielding $d = 18-5d$ and $e = 17d-5e+3$. Solving those two equations, we get $d = 3$ and $e = 9$. This results in:

$$x_n = A4^n + B3^n + 3n + 9$$

To get A and B, use the initial conditions:

$$\begin{cases} A + B & = x_0 - 9 & = 0 \\ 4A + 3B & = x_1 - 3 - 9 & = 3 \end{cases}.$$

Solving this linear system gives $A = 3$ and $B = -3$. That is,

$$x_n = 3 \cdot 4^n - 3^{n+1} + 3n + 9$$

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3. (a) $7!$.
   (b) $\binom{10}{7}$.
   (c) $\binom{30}{5}\binom{25}{4}\binom{21}{3}\binom{18}{2}16!$.
   (d) $26^2 \cdot (36^4 - 26^4)$.
   (e) $(n+1)!$.

4. (a) i. $\binom{10}{5} \cdot 5^5$.
   ii. $\sum_{n=0}^{5} \binom{10}{2n} 5^{10-2n}$.

iii. 3^{10}.

(b) i. $\binom{10}{4}, \sum_{n=0}^{4} \binom{10}{n}, \sum_{n=4}^{10} \binom{10}{n}$.

ii. $\sum_{n=0}^{4} \binom{10}{2n+1}$.

iii. $\binom{10}{5}$.

(c) Assume that an outcome is defined as a *set* of 10 balls. Then the count is $\binom{8}{5}\binom{5}{2}\binom{12}{3}$.

5. (a) $10 \cdot 6 \cdot 8$.

(b) The number of positive integers less than 300 and divisible by 8 is $\lfloor 299/8 \rfloor = 37$. The number of positive integers less than 300 and divisible by 12 is $\lfloor 299/12 \rfloor = 24$. An integer x is divisible by 8 and 12 *if and only if* x is divisible by the least common multiple $\text{lcm}(8, 12)$ of 8 and 12, which is 24. So the number of positive integers less than 300 and divisible by 8 and 12 is $\lfloor 299/24 \rfloor = 12$. Using the fact that if A and B are finite sets, $|A \cup B| = |A| + |B| - |A \cap B|$, the number of positive integers less than 300 that are divisible by 8 or 12 is $37 + 24 - 12 = 49$. The number of positive integers less than 300 that are divisible by neither 8 nor 12 is $299 - 49 = 250$.

6. (a) i. For each i from 1 to n, exactly one mod operation is done to compute m. So the number of mod operations is n.

ii. For each i from 1 to n, exactly $\lfloor n/3 \rfloor$ subtractions are done if and only if $i \bmod 3 = 1$. So the number of subtractions is $\lfloor n/3 \rfloor$ times the number of integers from 1 to n that are congruent to 1 modulo 3. Let B be the set of such integers and let $x \in B$. That is, $1 \leq x \leq n$ and $x = 3q + 1$ for some integer q. It follows that $1 \leq 3q + 1 \leq n$, i.e. $0 \leq q \leq \lfloor (n-1)/3 \rfloor$. Conversely, if $0 \leq q \leq \lfloor (n-1)/3 \rfloor$, then $3q + 1 \in B$. Therefore, the size of B is $\lfloor (n-1)/3 \rfloor + 1$. So the number of subtractions is $\lfloor n/3 \rfloor (\lfloor (n-1)/3 \rfloor + 1)$. If we assume that n is a positive multiple of 3, the expression simplifies to $(n/3)^2$.

iii. For the number of additions, first, let's ignore the computation of loop controls. For each i from 1 to n, if $i \bmod 3 = 0$, $\lfloor n/3 \rfloor$ additions are done, and if $i \bmod 3 = 2$, $(n - 2\lfloor n/3 \rfloor)$ additions are done. Let $A = \{i \in \mathbb{N} \mid 1 \leq i \leq n, i \bmod 3 = 0\}$ and $C = \{i \in \mathbb{N} \mid 1 \leq i \leq n, i \bmod 3 = 2\}$. Then the number of additions is $\lfloor n/3 \rfloor |A| + (n - 2\lfloor n/3 \rfloor)|C|$. Using similar reasoning as that for $|B|$, we get $|A| = \lfloor n/3 \rfloor$ and $|C| = \max(0, \lfloor (n-2)/3 \rfloor + 1)$. If we count the number of additions involved for loop counters, we get additional

$$n + \lfloor n/3 \rfloor |A| + (\lfloor n/3 \rfloor + 1)|B| + (n - 2\lfloor n/3 \rfloor + 1)|C|$$

additions. If we assume n is a positive multiple of 3, the number of additions simplifies to, without considering the loop control,

$$(n/3)(n/3) + (n - 2n/3)(n/3) = 2(n/3)^2,$$

and to, with loop control,

$$\begin{aligned} 2(n/3)^2 + n + (n/3)^2 + (n/3+1)(n/3) + (n/3+1)(n/3) \\ = 5(n/3)^2 + 5n/3 \end{aligned}$$

(b) $T(0) = 0$ because no operation is done for input 0. Else, for each iteration of the for-loop, 4 operations are performed inside the loop, and there are n^2 such iterations. So the number of operations inside the for-loop is $4n^2$. If we take into account the number of increments for the loop counter i, which is n^2, and the computation of $n-1$ and m, then $T(n) = T(n-1) + 5n^2 + 2$. Therefore,

$$\begin{aligned} T(n) &= \sum_{i=1}^{n}(5i^2 + 2) \\ &= 5\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} 2 \\ &= 5(\sum_{i=1}^{n} i^2) + 2n. \end{aligned}$$

It remains to find a closed-form expression for $\sum_{i=1}^{n} i^2$. Using the identity $(i-1)^3 = i^3 - 3i^2 + 3i - 1$, we have

$$\begin{aligned} \sum_{i=0}^{n-1} i^3 = \sum_{i=1}^{n}(i-1)^3 &= \sum_{i=1}^{n} i^3 - 3i^2 + 3i - 1 \\ &= \sum_{i=1}^{n} i^3 - 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1. \end{aligned}$$

Solving for $3\sum_{i=1}^{n} i^2$, we get

$$\begin{aligned} 3\sum_{i=1}^{n} i^2 &= \Big(\sum_{i=1}^{n} i^3 - \sum_{i=0}^{n-1} i^3\Big) + 3\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 \\ &= n^3 + 3n(n+1)/2 - n \\ &= (2n^3 + 3n^2 + n)/2 \end{aligned}$$

That is, $\sum_{i=1}^{n} i^2 = (2n^3 + 3n^2 + n)/6$. Therefore,

$$T(n) = \frac{5(2n^3 + 3n^2 + n)}{6} + 2n.$$

Bonus. (a) Exactly one head: $\sum_{k=1}^{6} \binom{k}{1}$.

(b) Exactly two heads: $\sum_{k=2}^{6} \binom{k}{2}$.

(c) As many heads as tails: $\binom{2}{1} + \binom{4}{2} + \binom{6}{3}$.