Homework 2 Solution

Problem 1

p	q	r	$(p \vee q) \wedge (\neg q \vee \neg r)$	$(\neg p \wedge \neg q) \vee \neg(q \wedge r)$	$p \wedge (q \vee \neg r)$	$(\neg p \vee \neg q) \wedge \neg r$
T	T	T	F	F	T	F
T	T	F	T	T	T	F
T	F	T	T	T	F	F
T	F	F	T	T	T	T
F	T	T	F	F	F	F
F	T	F	T	T	F	T
F	F	T	F	T	F	F
F	F	F	F	T	F	T

Problem 2

a) True.
$A - (B \cap C) = A \cap \overline{B \cap C} = A \cap (\overline{B} \cup \overline{C}) = (A \cap \overline{B}) \cup (A \cap \overline{C}) = (A - B) \cup (A - C)$

b) True.
$(A \cap B) - (A \cap C) = (A \cap B) \cap \overline{A \cap C} = (A \cap B) \cap (\overline{A} \cup \overline{C}) = ((A \cap B) \cap \overline{A}) \cup ((A \cap B) \cap \overline{C})$
$= \emptyset \cup ((A \cap B) \cap \overline{C}) = A \cap (B \cap \overline{C}) = A \cap (B - C)$

c) False.
Prove by counter-example. Let $A = \{1\}, B = \{2\}, C = \{3\}$, then $A \cup (B + C) = \{1, 2, 3\}$, $(A \cup B) + (A \cup C) = \{1, 2\} + \{1, 3\} = \{2, 3\}$. Therefore $A \cup (B + C) \neq (A \cup B) + (A \cup C)$, the statement is false.

d) False.
Prove by counter-example. Let $A = \{1, 3\}, B = \{2\}, C = \{3\}$, then $A + (B - C) = \{1, 2, 3\}$, $(A + B) - C = \{1, 2\}$. Therefore $A + (B - C) \neq (A + B) - C$, the statement is false.

e) True. $A - (B - C) = A - (B \cap \overline{C}) = A \cap \overline{(B \cap \overline{C})} = A \cap (\overline{B} \cup C) = (A \cap \overline{B}) \cup (A \cap C) = (A - B) \cup (A \cap C)$

f) True.

$\forall (x, y) \in A \times (B + C) \Leftrightarrow x \in A \wedge y \in (B + C) \Leftrightarrow x \in A \wedge ((y \in B \wedge y \notin C) \vee (y \in C \wedge y \notin B)) \Leftrightarrow$

$(\mathrm{x} \in A \wedge y \in B \wedge y \notin C) \vee (x \in A \wedge y \in C \wedge y \notin B) \Leftrightarrow (x \in A \wedge y \in B \wedge x \in A \wedge y \notin C) \vee (x \in A \wedge y \in C \wedge x \in A \wedge y \notin B) \Leftrightarrow ((x, y) \in A \times B \wedge (x, y) \notin A \times C) \vee ((x, y) \in A \times C \wedge (x, y) \notin A \times B) \Leftrightarrow (x, y) \in (A \times B + A \times C)$

g) False.
Prove by counter-example. Let $A = \{1, 2\}, B = \{2\}$, then $2^{A-B} = \{\emptyset, \{1\}\}$, $2^A - 2^B = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} - \{\emptyset, \{2\}\} = \{\{1\}, \{1, 2\}\}$. Therefore $2^{A-B} \neq 2^A - 2^B$ the statement is false.

Problem 3

a) Let $x = 3$, then $\lceil \frac{x}{2} \rceil = 2$, $\frac{\lceil x \rceil}{2} = 1.5$, therefore $\lceil \frac{x}{2} \rceil \neq \frac{\lceil x \rceil}{2}$.

b) Let $x = 2, y = 2.6$, then $\lfloor x \times y \rfloor = \lfloor 2 \times 2.6 \rfloor = \lfloor 5.2 \rfloor = 5$, $\lfloor x \rfloor \times \lfloor y \rfloor = \lfloor 2 \rfloor \times \lfloor 2.6 \rfloor = 2 \times 2 = 4$, therefore $\lfloor x \times y \rfloor \neq \lfloor x \rfloor \times \lfloor y \rfloor$.

c) Let $x = 2.5$, then $\lceil 3^{2.5} \rceil = \lceil 15.59 \rceil = 16$, $3^{\lceil 2.5 \rceil} = 3^3 = 27$, therefore $\lceil 3^x \rceil \neq 3^{\lceil x \rceil}$.

d) Let $n = 5$, $2^n + 3 = 2^5 + 3 = 32 + 3 = 35$, 35 is not a prime number $(35 = 5 \times 7)$, therefore the statement is false.

Problem 4

a) Prove that f is one-to-one and onto, and find f^{-1}.

① Prove f is one-to-one.
$f(x) = f(x') \Rightarrow 5x + 12 = 5x' + 12 \Rightarrow x = x'$. Therefore, f is one-to-one.

② Prove f is onto.
Take an arbitrary element $y \in \mathbb{R}$, find $x \in \mathbb{R}$ such that $y = f(x) = 5x + 12$.
$y = 5x + 12 \Leftrightarrow y - 12 = 5x \Leftrightarrow x = \frac{y-12}{5} \in \mathbb{R}$. Therefore, f is onto, and $f^{-1}(y) = \frac{y-12}{5}$

b) ① g is not one-to-one. Let $x_1 = 2, x_2 = -2$, $g(x_1) = g(x_2) = \sqrt{5}$ while $x_1 \neq x_2$; therefore g is not one-to-one.

② g is not onto. To prove it, take $y = 0.5$. $\forall x \in \mathbb{R}, g(x) = \sqrt{x^2 + 1} \geq 1 > 0.5$. Therefore, $\forall x \in \mathbb{R}, g(x) \neq y$. Hence, g is not onto.

c) ① h is not one-to-one. Let $x_1 = 1$, $x_2 = 2$, $h(x_1) = \lfloor \frac{1+1}{4} \rfloor = 0$, $h(x_2) = \lfloor \frac{2+1}{4} \rfloor = 0$; thus, $h(x_1) = h(x_2) = 0$ but $x_1 \neq x_2$; therefore, h is not one-to-one.

② h is onto. Let y be an arbitrary element in $\mathbb{Z}$, find an element $x \in \mathbb{R}$ such that $h(x) = y$. Take $x = 4y - 1 \in \mathbb{R}$; $h(x) = \lfloor \frac{x+1}{4} \rfloor = \lfloor \frac{4y-1+1}{4} \rfloor = \lfloor \frac{4y}{4} \rfloor = \lfloor y \rfloor = y$ because $y \in \mathbb{Z}$. Therefore, h is onto.

d) ① $h\, o\, g(x) = h(g(x)) = \lfloor \frac{g(x)+1}{4} \rfloor = \lfloor \frac{\sqrt{x^2+1}+1}{4} \rfloor$

② $g\, o\, h(x) = g(h(x)) = \sqrt{h(x)^2 + 1} = \sqrt{\lfloor \frac{x+1}{4} \rfloor^2 + 1}$

③ $(f\, o\, h)\, o\, g(x) = f\, o\, (h\, o\, g)(x) = f(h(g(x))) = 5h(g(x)) + 12 = 5\lfloor \frac{g(x)+1}{4} \rfloor + 12$
$= 5\lfloor \frac{\sqrt{x^2+1}+1}{4} \rfloor + 12$

e) ① $f^{\leftarrow}(1) = \{x \in E \mid f(x) = 1\}$, $f(x) = 5x + 12 = 1 \Rightarrow x = \frac{-11}{5} \Rightarrow f^{\leftarrow}(1) \equiv \{\frac{-11}{5}\}$

② $g^{\leftarrow}(3) = \{x \in E \mid g(x) = 3\}$, $g(x) = \sqrt{x^2+1} = 3 \Rightarrow x = \pm\sqrt{8} \Rightarrow g^{\leftarrow}(3) \equiv \{-\sqrt{8}, \sqrt{8}\}$

③ $g^{\leftarrow}(0) = \{x \in E \mid g(x) = 0\}$, $g(x) = \sqrt{x^2+1} = 0 \Rightarrow x^2 = -1$, there does not exist any $x \in \mathbb{R}$ such that $x^2 = -1$, therefore $g^{\leftarrow}(0) \equiv \emptyset$

④ $h^{\leftarrow}(2) = \{x \in E \mid h(x) = 2\}$, $h(x) = \lfloor \frac{x+1}{4} \rfloor = 2 \Rightarrow 2 \le \frac{x+1}{4} < 3 \Rightarrow 7 \le x < 11 \Rightarrow h^{\leftarrow}(2) \equiv [7, 11)$

Problem 5

a) Basic step: $n = 0$, $f(0) = 3$ by definition, and $5^{0+1} - 2 = 5 - 2 = 3$. Therefore, $f(0) = 5^{0+1} - 2$.
Induction step: Assume it is true for $n-1$, $f(n-1) = 5^{n-1+1} - 2 = 5^n - 2$, then for n, with recurrence function $f(n) = 5f(n-1) + 8 = 5(5^n - 2) + 8 = 5^{n+1} - -10 + 8 = 5^{n+1} - 2$.
Therefore $f(n) = 5^{n+1} - 2$, $\forall n \ge 0$.

b) Basic step: $n = 0$, $f(1) = 0$ by definition, $0^2 = 0 = f(0)$, i.e., the result is true for $n = 0$.
Induction step: Assume it is true for $n-1$, $f(n-1) = (n-1)^2$, then for n, with recurrence function $f(n) = f(n-1) + (2n-1) = (n-1)^2 + (2n-1) = n^2 - 2n + 1 + (2n-1) = n^2$.
Therefore $f(n) = n^2$, $\forall n \ge 0$.

c) Basic step: $n = 0$, $f(0) = 0$ by definition, $\frac{0\cdot(2\cdot 0-1)\cdot(2\cdot 0+1)}{3} = 0 = f(0)$, i.e., the result is true for $n = 0$.
Induction step: Assume it is true for $n-1$, $f(n-1) = \frac{(n-1)(2(n-1)-1)(2(n-1)+1)}{3} = \frac{(n-1)(2n-3)(2n-1)}{3}$, then for n, with recurrence function $f(n) = f(n-1) + (2n-1)^2 = \frac{(n-1)(2n-3)(2n-1)+3(2n-1)^2}{3} = \frac{(2n-1)((n-1)(2n-3)+3(2n-1))}{3} = \frac{(2n-1)(2n^2-5n+3+6n-3)}{3} = \frac{(2n-1)(2n^2+n)}{3} = \frac{n(2n-1)(2n+1)}{3}$.
Therefore $f(n) = \frac{n(2n-1)(2n+1)}{3}$, $\forall n \ge 0$.

Problem 6

a) Basic step: $n = 0$, $T(0) = 0$ by definition, $\frac{0\cdot(0+3)}{2} = 0 = T(0)$, i.e., the result is true for $n = 0$.
Induction step: Assume it is true for $n-1$, $T(n-1) = \frac{(n-1)(n+2)}{2}$, then for n, with recurrence function $T(n) = T(n-1) + n + 1 = \frac{(n-1)(n+2)}{2} + n + 1 = \frac{(n^2+n-2)+(2n+2)}{2} = \frac{n^2+3n}{2} = \frac{n(n+3)}{2}$.
Therefore $T(n) = \frac{n(n+3)}{2}$, $\forall n \ge 0$.

b) Let $f(x) = a[0] + a[1]x + a[2]x^2 + \cdots + a[n]x^n$, $T(n)$ is the time to calculate $f(x)$. x^1 takes 0 steps, $x^2 = x \times x$ takes one (multiplication) step, $x^3 = x^2 \times x$ takes one additional (multiplication) step, $x^4 = x^3 \times x$ takes one additional step, and so on, until $x^x = x^{n-1} \times x$ takes one additional step. Therefore, those powers of x take a total of $n-1$ steps.
Next, each $a[i]x^i$ takes one multiplication step, implying that the products $a[1]x$, $a[2]x^2$, $a[3]x^3, \cdots, a[n]x^n$ take n multiplication steps.
Finally, the sum of those products, namely, $a[0] + a[1]x + a[2]x^2 + a[3]x^3 + \cdots + a[n]x^n$ takes n (addition) steps.
Therefore, the total number of steps is $(n-1) + n + n = 3n - 1$.

Bonus Problem

Let A_n represents A with n elements: $A_n \equiv \{a_1, a_2, \cdots, a_n\}$.

Basic step:

$n = 0, \quad A_0 \equiv \emptyset, \quad P(A_0) = \{\emptyset\}, \quad |P(A_0)| = 1, \quad 2^0 = 1 = |P(A_0)|$, i.e., the result is true for $n = 0$.

To make sure $n = 0$ is not a special case, we can also verify it is true for $n = 1, \quad A_1 \equiv \{a_1\}, \quad P(A_1) = \{\emptyset, \{a_1\}\}, \quad |P(A_1)| = 2, \quad 2^1 = 2 = |P(A_0)|$, i.e., the result is true for $n = 1$.

Induction step:

Assume it is true for $n-1$, $|P(A_{n-1})| = 2^{n-1}$.

Let $P(A_{n-1}) = \{Q_1, Q_2, \cdots, Q_{2^{n-1}}\}$, where $Q_i, i = 1, 2, \cdots, 2^{n-1}$, is a subset of $A_{n-1} \equiv \{a_1, a_2, \cdots, a_{n-1}\}$, then $P(A_n)$ can be formed by the union of two non-overlapping sets, $P(A_{n-1})$, and another new set by adding a_n into each element set in $P(A_{n-1})$; let us call this set $P(A_{n-1}, a_n)$.

$P(A_{n-1}, a_n) = \{\{a_n\} \cup Q_1, \{a_n\} \cup Q_2, \cdots, \{a_n\} \cup Q_{2^{n-1}}\}$, which obviously has the same number of elements as $P(A_{n-1})$, and $P(A_{n-1}) \cap P(A_{n-1}, a_n) \equiv \emptyset$.

$P(A_n) \equiv (P(A_{n-1}) \cup P(A_{n-1}, a_n)) \Rightarrow$

$|P(A_n)| = |(P(A_{n-1})| + |P(A_{n-1}, a_n))| - |P(A_{n-1}) \cap P(A_{n-1}, a_n)| = 2^{n-1} + 2^{n-1} - 0 = 2^n$.

Therefore $|P(A_n)| = 2^n, \forall n \geq 0$.