CS 1311 Spring 2019
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Midterm Solution

Problem 1:

a) One-to-one: Prove that $f(x) = f(y) \Rightarrow x = y$.
$f(x) = f(y) \Rightarrow 10x + 35 = 10y + 35 \Rightarrow 10x = 10y \Rightarrow x = y$.

Onto: Prove that $\forall\, y \in \mathbb{R}$, there exists an $\boldsymbol{x} \in \mathbb{R}$ where $f(x) = y$.
Let $y \in \mathbb{R}$. Take $x = \frac{y-35}{10}$, which is in $\mathbb{R}$.
$f(x) = 10x + 35 = 10\left(\frac{y-35}{10}\right) + 35 = y - 35 + 35 = y$. Therefore, $f(x) = y$.
Inverse function: $f^{-1}(y) = \frac{y-35}{10}$, or $f^{-1}(x) = \frac{x-35}{10}$.

b) g is one-to-one. Prove that $f(x) = f(y) \Rightarrow x = y$.
$g(x) = g(y) \Rightarrow 4\sqrt{x} - 7 = 4\sqrt{y} - 7 \Rightarrow 4\sqrt{x} = 4\sqrt{y} \Rightarrow \sqrt{x} = \sqrt{y} \Rightarrow (\sqrt{x})^2 = (\sqrt{y})^2 \Rightarrow x = y$.
g is NOT onto. To prove so, it is enough to find a $y \in \mathbb{R}$ such that for no $x \in \mathbb{N}$, $g(x) = y$. Take $y = -10$. The proof is by contradiction. If there were an $x \in \mathbb{N}$ where $g(x) = -10$, then $4\sqrt{x} - 7 = -10$, that is, $4\sqrt{x} = -3$, but that is impossible because $4\sqrt{x} \geq 0$ while $-3 < 0$.

Problem 2:

a) Basis step: $n = 1$. Need to show that $(a + b)^1 \geq a^1 + b^1$.
Well, $(a + b)^1 = a + b$ and $a^1 + b^1 = a + b$, thus $(a + b)^1 = a^1 + b^1$ and hence $(a + b)^1 \geq a^1 + b^1$.
Induction step: Assume $(a + b)^{n-1} \geq a^{n-1} + b^{n-1}$. Prove $(a + b)^n \geq a^n + b^n$.

$(a + b)^n = (a + b)^{n-1}(a + b)$
$\geq (a^{n-1} + b^{n-1})(a + b)$ By the induction hyposthesis
$= a^{n-1}a + b^{n-1}b + a^{n-1}b + b^{n-1}a$
$= a^n + b^n + (a^{n-1}b + b^{n-1}a)$
$\geq a^n + b^n$ because $(a^{n-1}b + b^{n-1}a) \geq 0$ since a and b are positive.

Therefore, $(a + b)^n \geq a^n + b^n$.

b) Basis step: $n = 1$. Need to show that $1 \times 3 = \frac{1(1+1)(2\times1+7)}{6}$.
$1 \times 3 = 3$, and $\frac{1(1+1)(2\times1+7)}{6} = \frac{1\times2\times9}{6} = \frac{18}{6} = 3$, showing that both are equal (to 3).
Induction step: Assume $1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + (n-1)(n-1+2) = \frac{(n-1)(n)(2(n-1)+7)}{6}$, that is, $1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + (n-1)(n+1) = \frac{(n-1)(n)(2n+5)}{6}$. Prove that

$$1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}.$$

$1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + n(n+2)$
$= 1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + (n-1)(n+1) + n(n+2)$
$= \frac{(n-1)(n)(2n+5)}{6} + n(n+2)$

$=n\left[\frac{(n-1)(2n+5)}{6}+n+2\right]$

$= n\left[\frac{2n^2+3n-5}{6}+\frac{6n+12}{6}\right]$

$= n\left[\frac{2n^2+9n+7}{6}\right]$

$= n\left[\frac{(n+1)(2n+7)}{6}\right]$

$= \frac{n(n+1)(2n+7)}{6}$. Q.E.D.

c) Using part (b), and multiplying both sides by 6, we get:
$n(n+1)(2n+7) = 6[1\times 3 + 2\times 4 + 3\times 5 + \cdots + n(n+2)]$, which is an integer multiple of 6 because the number in green, i.e., $1\times 3 + 2\times 4 + 3\times 5 + \cdots + n(n+2)$, is an integer.

Problem 3: (25 points)

a)

Let $\hat{x}_n = cn + d$

$cn + d = 2c(n-1) + 2d + n$

$cn + d = (2c+1)n + 2d - 2c$

$\begin{cases} c = 2c+1 \\ d = 2d - 2c \end{cases} \Rightarrow \begin{cases} c = -1 \\ d = -2 \end{cases}$

$\therefore \hat{x}_n = -n - 2$

$x_n = A2^n - n - 2$

Since $x_0 = 0, 0 = A2^0 - 0 - 2$. Thus, $A = 2$

$\therefore x_n = 2^{n+1} - n - 2$

b)

Since $a^2 + 4b = 2^2 + 4 = 8 > 0$

$x_n = A\left(\frac{a+\sqrt{a^2+4b}}{2}\right)^n + B\left(\frac{a-\sqrt{a^2+4b}}{2}\right)^n$

$x_n = A\left(1+\sqrt{2}\right)^n + B\left(1-\sqrt{2}\right)^n$

$\begin{cases} x_0 = 0 \\ x_1 = 4 \end{cases} \Rightarrow \begin{cases} 0 = A + B \\ 4 = A\left(1+\sqrt{2}\right) + B\left(1-\sqrt{2}\right) \end{cases} \Rightarrow \begin{cases} A = \sqrt{2} \\ B = -\sqrt{2} \end{cases}$

$\therefore x_n = \sqrt{2}\left(1+\sqrt{2}\right)^n - \sqrt{2}\left(1-\sqrt{2}\right)^n$

Problem 4: (25 points)

a) 8!

b) $5! \times 3!$

Explanation: 5! is the number of arrangements of the 5 women, and 3! is the number of arrangements of the 3 men.

c) $\binom{6}{3} \times 3! \times 5!$

Explanation:

The arrangements must look like $: W : W : W : W : W :$ where each W is a woman, each $:$ is a man or is empty, and three of the $:\,'s$ are chosen to be filled with men (while the other three remain empty). There are $\binom{6}{3}$ selections of $:\,'s$ where to put the men, and for each selection, the three men can be in any of 3! arrangements/permutations. The 5 women (in the W positions) can be in any of 5! arrangements/permutations. The answer is the product of those 3 quantities.

d) $\frac{7!}{3! \times 2! \times 2!}$ (Other forms such as $\binom{7}{3}\binom{4}{2}$ is also accepted).

Explanation 1: The problem is counting the number of permutations of $n = 7$ objects that are "a", "a", "a", "b", "b", "c" and "c", where $n_1 = 3$ objects are identical (all "a"s), $n_2 = 2$ objects are identical (all "b"s), and another $n_2 = 2$ objects are identical (all "c"s). We have a formula for that: $\frac{n!}{n_1! \times n_2! \times n_3!} = \frac{7!}{3! \times 2! \times 2!}$.

Explanation 2:

Number the positions of the letters in the string from 1 to 7. The problem can be viewed as follows: How many selections of three locations (out 7) where to place the three "a"s, and for each such selection how many selections of 2 locations (out of the remaining 4) where to place the two "b"s; note the remaining two locations will necessarily be filled with "c"s. The first number is $\binom{7}{3}$, and the second number is $\binom{4}{2}$, and the final answer is the product.

e) $\sum_{i=5}^{8} \binom{8}{i}$

Explanation: The desired outcomes have 5 heads or 6 heads or 7 heads or 8 heads. The number of outcomes with 5 heads (out of 8) is $\binom{8}{5}$ because in each such outcome the problem is identical to selecting which 5 tosses out of the 8 tosses turn heads. The number of the outcomes that have 6 heads is counted the same way, i.e., $\binom{8}{6}$; also, the number of outcomes with 7 heads is the same way equal to $\binom{8}{7}$; and finally, the number of outcomes with all 8 heads is $\binom{8}{8} = 1$. Because of the "OR", we add those numbers.