CS 1311 March 28, 2019
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Homework 5
Due Date: April 11, 2019

**Problem 1**: (20 points)

a) Show the multiplication table (mod 7), and also the multiplication table (mod 10).
b) What is the inverse (mod 7) of every integer between 1 and 6?
c) Give all the numbers between 1 and 9 that have no inverse mod 10.
d) Calculate $5^{612}$ (mod 7), giving an answer between 0 and 6, and using a small number of steps. Show your steps.

**Problem 2**: (20 points)

Let G = (V,E) be the following <u>undirected</u> graph: V = {1, 2, 3, 4, 5, 6, 7, 8, 9}, and
E = {(1,2) , (3,1) , (2,3) , (1,4) , (5,1) , (4,5) , (6,1) , (1,7) , (7,6) , (7,8), (9,7), (8,9)}.

a) Draw G, and give the adjacency matrix A of G.
b) Do a depth-first search (DFS) and a breadth-first search (BFS) on G, starting from node 1, and show the DFS tree and BFS tree. Tie-breaking is by choosing the smallest node.
c) Is G connected? How can you tell?
d) An *articulation point* (or *single point of failure*) of a connected graph is a node whose removal disconnects the graph. List the articulation points of G, if any.
e) Does G have an Eulerian cycle? If yes, show such a cycle; if no, prove your answer.
f) Does G have a Hamiltonian cycle? If yes, show such a cycle; if no, prove your answer.

**Problem 3**: (20 points)

Take the undirected graph G of Problem 2.

a) The *distance* *d*(x,y) between two nodes x and y is the length of the shortest path between x and y. Recall that the length of a path is the number of the edges in the path. Compute the *distance matrix* D of G where D[i,j]=*d*(i,j), for all i=1,2,…,9 and all j=1,2,…,9. (Hint: you can use BFSs to compute the distances)
b) The *diameter* of a graph is the largest distance in that graph. Give the diameter of G?
c) The *radius* of the graph from a node *x* is the distance from *x* to the farthest node. For each node *x* in G, compute the radius of G from *x*.
d) The *center* of a graph is a node *u* where the radius of G from *u* is the smallest. Note that there can be ties, in which case the graph has multiple centers. Find the center or centers of G.
e) The *average radius* of the graph from a node *x* is the average distance from *x* to all the nodes of the graph, that is, $\frac{d(x,1)+d(x,2)+\cdots+d(x,9)}{9}$. For each node *x* in G, compute the average radius of G from *x*. Which node(s) has/have the smallest average radius?

**Problem 4**: (10 points)

Let G be the same graph given in Problem 2 except that it is now <u>directed</u>.

a) Draw G and give its adjacency matrix A.
b) Compute the indegree and outdegree of each node.
c) For every node, list all the nodes reachable from it. Is G strongly connected?

**Problem 5:** (20 points)
Let $T$ be a binary tree. Denote by $L_T$ the number of leaves in $T$, and by $N_T$ the number of nodes in $T$. Let $P$ be a perfect binary tree, and let $L_P$ and $N_P$ be defined similarly. Assume that $T$ and $P$ have the same height $h$.

a) Prove that $N_T \leq N_P$ and $L_T \leq L_P$.
b) Prove that $L_T \leq 2^h$ and $N_T \leq 2^{h+1} - 1$. (Hint: Use part (a) and what you know about perfect trees)
c) Prove $h \geq \log_2 L_T$ and $h \geq \log_2 \frac{N_T+1}{2}$. (Hint: Use part (b) and the fact that $\log_2 2^k = k$, and that $\log_2 x$ is an increasing function of $x$)

**Problem 6**: (10 points)
A *binary search tree* (BST) is a binary tree where (1) every node holds a numerical value (called *key*), and (2) for every node X in the tree, the keys of the nodes of the left subtree of X are all $\leq$ the key of X, and the keys of the nodes of the right subtree of X are all $>$ the key of X. Prove by induction on the number of nodes in a BST that the minimum key of every BST is the "leftmost" node, i.e., the farthest node reachable from the root of that tree by always going to the left child.

**Bonus problem**: (5 points)
Prove that if you remove a single edge (without removing its end nodes) from an arbitrary tree, the tree becomes disconnected.