CS 1311 February 28, 2019
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Homework 4
Due Date: March 28, 2019

Problem 1: (20 points)

a) Let P be the set of (logical) propositions. For each of the following relations, state without proof if the relation is reflexive, symmetric, antisymmetric, transitive, an equivalence relation, and/or a partial order.
   i. p R q if $p \Rightarrow q$
   ii. p R q if $p \Leftrightarrow q$
   iii. p R q if $p \Leftrightarrow \bar{q}$
   iv. p R q if $p \not\Leftrightarrow \bar{q}$

b) Let S be the set of students in this class. For each of the following relations, state without proof if the relation is reflexive, symmetric, antisymmetric, transitive, an equivalence relation, and/or a partial order. In the case of equivalence relations, explain what every equivalence class signifies. In the case of partial orders, indicate if the set S has a minimum, a maximum, minimals, and/or maximals, and explain what minimum, maximum, minimal and maximal mean in that context.
   i. x R y if x and y have the same hobbies
   ii. x R y if x=y or x is younger than y
   iii. x R y if x and y have declared major and both have the same primary major
   iv. x R y x=y or x views himself/herself as a friend of y

Problem 2: (20 points)

Let E be the set of 4-bit binary strings: E={0000, 0001, 0010, 0011, … , 1111}.

a) Let R be the following relation in E: x R y if x and y have the same number of 1's. Prove that R is an equivalence relation, and give the equivalence classes of R.
b) The *Hamming distance* between two strings x and y in E, denoted H(x,y), is the number of bit-wise differences between x and y, that is, the minimum number of bits that have to be flipped in the first string to become identical to the second string. Compute H(0000, 0101), H(1101,0110), and H(1001,0110).
c) Let R be the following relation in E: x R y if H(x,y) is even. Prove that R is an equivalence relation, and give the equivalence classes of R.

Problem 3: (20 points)

Let A be the set of 3-bit binary strings: A={000, 001, 010, 011, 100, 101, 110, 111}. Treat "0" as equivalent to "False" (F), and "1" as equivalent to 'True" (T). Let $\oplus$ be the "bit-wise logical **or**" operation between the elements of A. For example, $100 \oplus 001 = 101$.

Define the following relation R in A:

x R y if $x \oplus y = y$

a) Prove that R is a partial order.
b) Is R a total order?
c) Draw the graph of R and the Hess diagram of R.
d) Does A have a maximum? If so, what is it?
e) Does A have a minimum? If so, what is it?

f) Let B= {001, 010, 011, 100, 101}. If B has a maximum, give its maximum; otherwise, give the maximals of B. Also, if B has a minimum, give its minimum; otherwise, give the minimals of B.

Problem 4: (16 points)
Let R be a relation in a set A, and derive from R another relation S in A as follows:
x S y if (x R y **or** y R x).

a) Prove that if R is reflexive, then S is reflexive.
b) Prove that S is symmetric.
c) Prove that if R is transitive, S is not necessarily transitive (by a counterexample).
d) If R is antisymmetric, is S antisymmetric? Prove your answer.
e) If R is an equivalence relation, is S an equivalence relation? Prove your answer.
f) If R is a partial order, is S a partial order? Prove your answer.

Problem 5: (16 points)
Let R be a relation in a set A, and derive from R another relation S in A as follows:
x S y if (x R y **and** y R x).

a) Prove that if R is reflexive, then S is reflexive.
b) Prove that S is symmetric.
c) Prove that if R is transitive, S is transitive.
d) If R is antisymmetric, is S antisymmetric? Prove your answer.
e) If R is an equivalence relation, is S an equivalence relation? Prove your answer.
f) If R is a partial order, is S a partial order? Prove your answer.

Problem 6: (8 points)
Let R be a relation in a set A, and derive from R another relation S in A as follows:
x S y if (x R y **xor** y R x).
Recall that **xor**, exclusive or, is defined as: p **xor** q is true if (p is true and q is false, or p is false and q is true).

g) Prove that S is irreflexive.
a) Prove that S is symmetric.
b) Prove that if R is transitive, S is not necessarily transitive (by a counterexample).

Bonus Problem: (5 points)
Let $\mathbb{Z}$ be the set of integers, and $\mathbb{Q}$ the set of rational numbers, that is, $\mathbb{Q}$={$\frac{p}{q}$ | p and q are in $\mathbb{Z}$ and $q \neq 0$}. Let $\mathbb{Z}^* = \mathbb{Z} - \{\mathbf{0}\}$. Finally, let R be the following relation in $\mathbb{Z}$x$\mathbb{Z}^*$:
(a, b) R (c, d) if $ad = bc$ (that is, a times $d = b$ times c).

a) Prove that R is an equivalence relation.
b) Denote by $[a, b]$ the equivalence class of element (a, b), and let E be the set of all equivalence classes of R. Let $f: E \rightarrow \mathbb{Q}$ be a function where for all $[a, b]$, $f([a, b]) = \frac{a}{b}$. Prove that f is one-to-one and onto.