CS 1311 January 31, 2019

Youssef

Homework 2

Due Date: February 14, 2019

Problem 1: (20 points)

Let p, q, and r be 3 propositions. Give the truth tables of the following propositions:

a) $(p \vee q) \wedge (\neg q \vee \neg r)$

b) $(\neg p \wedge \neg q) \vee \neg(q \wedge r)$

c) $p \wedge (q \vee \neg r)$

d) $(\neg p \vee \neg q) \wedge \neg r$

Problem 2: (21 points)

Let A, B and C be three arbitrary sets. For each of the following statements, indicate if the statement is true or false, and prove your answer.

a) $A - (B \cap C) = (A - B) \cup (A - C)$

b) $A \cap (B - C) = (A \cap B) - (A \cap C)$

c) $A \cup (B+C) = (A \cup B) + (A \cup C)$

d) $A + (B - C) = (A + B) - C$

e) $A - (B - C) = (A - B) \cup (A \cap C)$

f) $A \times (B+C) = (A \times B) + (A \times C)$

g) $2^{A-B} = 2^A - 2^B$

Problem 3: (16 points)

Find counterexamples to the following statements (where *x* and *y* are arbitrary <u>real</u> numbers):

a) $\left\lceil \frac{x}{2} \right\rceil = \frac{\lceil x \rceil}{2}$ for all *x*, where $\lceil x \rceil$ is the *ceiling* of x, i.e., the smallest integer $\geq x$.

b) $\lfloor x \times y \rfloor = \lfloor x \rfloor \times \lfloor y \rfloor$ for all *x* and *y*, where $\lfloor x \rfloor$ is the *floor* of *x*, i.e., the largest integer $\leq x$.

c) $\lceil 3^x \rceil = 3^{\lceil x \rceil}$ for all *x*.

d) $2^n + 3$ is prime for every integer $n \geq 1$

Problem 4: (20 points)

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^+$ the set of non-negative real numbers, and $\mathbb{Z}$ the set of integers. Let also $f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}^+$, and $h: \mathbb{R} \to \mathbb{Z}$ be 3 functions defined as follows:

$$f(x) = 5x+12, \quad g(x) = \sqrt{x^2+1}, \text{ and } h(x) = \left\lfloor \frac{x+1}{4} \right\rfloor.$$

a) Prove that f is one-to-one and onto, and find f^{-1}.

b) Is *g* one-to-one? Onto? Prove your answer.

c) Is *h* one-to-one? Onto? Prove your answer.

d) Calculate $h \circ g(x)$, $g \circ h(x)$, $(f \circ h) \circ g(x)$, *and* $f \circ (h \circ g)(x)$.

e) Given two sets E and F, a function v: E→F, and an element y ∈ F, define $v^{\leftarrow}(y)$ to be the following set: $v^{\leftarrow}(y) = \{x \in E \mid v(x) = y\}$. Determine $f^{\leftarrow}(1)$, $g^{\leftarrow}(3)$, $g^{\leftarrow}(0)$, $h^{\leftarrow}(2)$.

Problem 5: (15 points)

Let $\mathbb{N}$ be the set of natural numbers, and $\mathbb{R}$ the set of real numbers. Let $f: \mathbb{N} \to \mathbb{R}$ be a function.

a) If $f(0) = 3$ and $f(n) = 5f(n-1) + 8$ $\forall n \geq 1$, prove by induction on *n* that $f(n) = 5^{n+1} - 2$.

b) Let $f(n) = 1 + 3 + 5 + \cdots + (2n-1)$ $\forall n \geq 1$, and $f(0) = 0$. Prove by induction on *n* that $f(n) = n^2$ $\forall n \geq 0$.

c) Let $f(n) = 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2$ $\forall n \geq 1$, and $f(0) = 0$. Prove by induction on *n* that $f(n) = \frac{n(2n-1)(2n+1)}{3}$ $\forall n \geq 0$.

d)

Problem 6: (8 points)
Designers of algorithms are often interested to compute the number of computation steps $T(n)$ of their algorithm, where *n* is an indicator of the input size. T(*n*) is called the time of the algorithm.

a) Consider an algorithm that computes *polynomials*, that is an algorithm that takes as input an array of numbers $a[0], a[1], a[2], \dots, a[n]$ and a number x, and computes the value:
$$a[0] + a[1]x^1 + a[2]x^2 + \cdots + a[n]x^n$$
as output. Let $T(n)$ be the time of the algorithm. One naïve way to compute the output is to first compute "recursively" the value $a[0] + a[1]x^1 + a[2]x^2 + \cdots + a[n-1]x^{n-1}$ (in time $T(n-1)$), and then compute $a[n]x^n$ (using *n* steps), and finally add the two values in one step. The total time is then:
$$T(n) = T(n-1) + n + 1 \quad \forall n \geq 1.$$
Note that when $n = 0$, the value to be returned is simply $a[0]$ which does not need any computation; therefore, $T(0) = 0$.
Prove by induction on n that $T(n) = \frac{n(n+3)}{2}$.

b) A more clever algorithm will first compute the sequence $x^1,\ x^2, \dots, x^n$ by computing each x^k as $x^{k-1} \times x$, making use of the value x^{k-1} computed earlier. Afterwards, it computes the products $a[1]x^1, a[2]x^2, \dots, a[n]x^n$ in n steps, and finally computes the sum $a[0] + a[1]x^1 + a[2]x^2 + \cdots + a[n]x^n$ in n steps. What is the time of this algorithm?

Bonus Problem: (5 points)
Let A be a set of n element, and let $P(A)$ be the power set of A. Prove by induction on n that $|P(A)| = 2^n$. (Do not look up the solution anywhere. It has to be your work.)