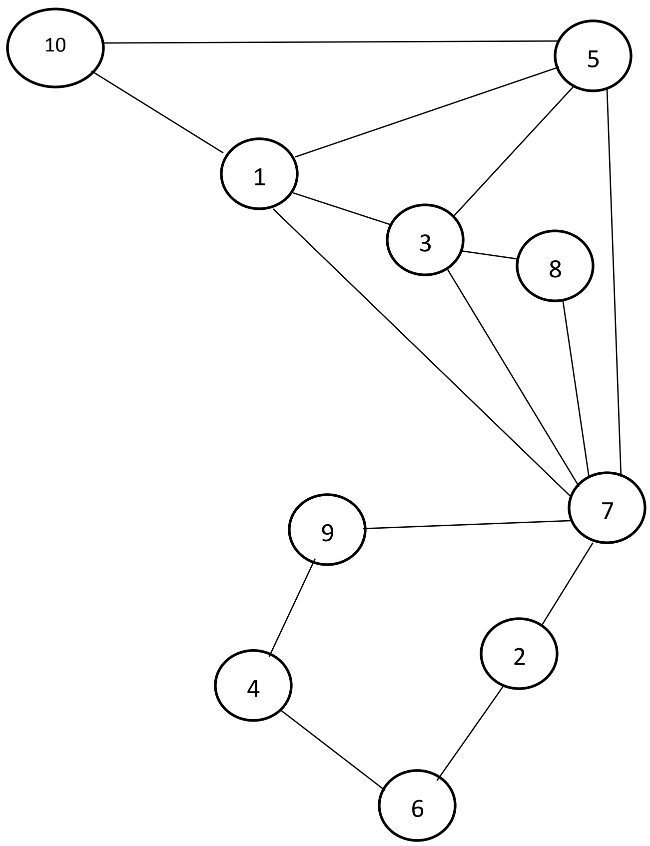
CS1311 **7^{th} Apr 2016**

Homework 5 Solutions

Problem 1:

a)

Graph G: (undirected)

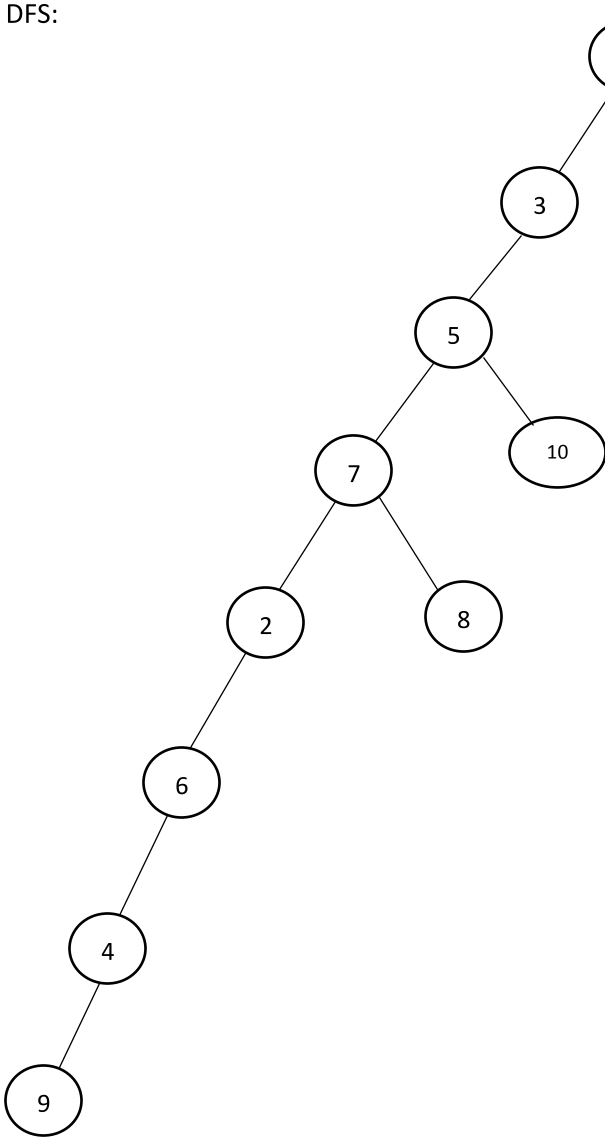


Adjacency Matrix:

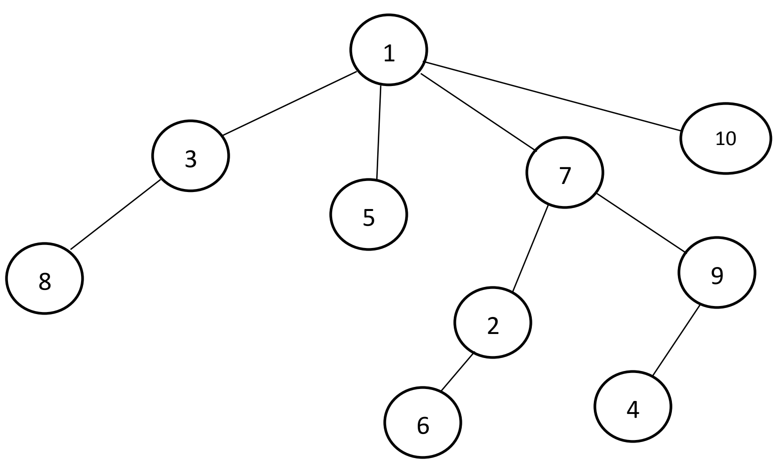
	1	2	3	4	5	6	7	8	9	10
1	0	0	1	0	1	0	1	0	0	1
2	0	0	0	0	0	1	1	0	0	0
3	1	0	0	0	1	0	1	1	0	0
4	0	0	0	0	0	1	0	0	1	0
5	1	0	1	0	0	0	1	0	0	1
6	0	1	0	1	0	0	0	0	0	0
7	1	1	1	0	1	0	0	1	1	0
8	0	0	1	0	0	0	1	0	0	0
9	0	0	0	1	0	0	1	0	0	0
10	1	0	0	0	1	0	0	0	0	0

b)

DFS:



BFS:



c) $Yes. One\ DFS\ traversal\ could\ traverse\ all\ the\ nodes\ of\ the\ Graph\ into\ a\ single\ tree.$

d) $Yes. Node/Vertex\ 7.$

e) $Yes. Eulerian\ cycle\ (as\ a\ sequence\ of\ nodes)$: 1, 10, 5, 1, 3, 5, 7, 8, 3, 7, 2, 6, 4, 9, 7, 1. It traverses all the edges, once per edge.

f)
$No. Because\ the\ graph\ G\ above\ has\ an\ articulation\ point\ and\ thus\ cannot\ have\ a\ Hamiltonian\ cycle.$

Problem 2:

a)

u	1	2	3	4	5	6	7	8	9	10
d(1,u)	0	2	1	3	1	3	1	2	2	1

b) $Diameter = 4$, because d(4,10)=4 and no pair of nodes are more than 4 edges apart.

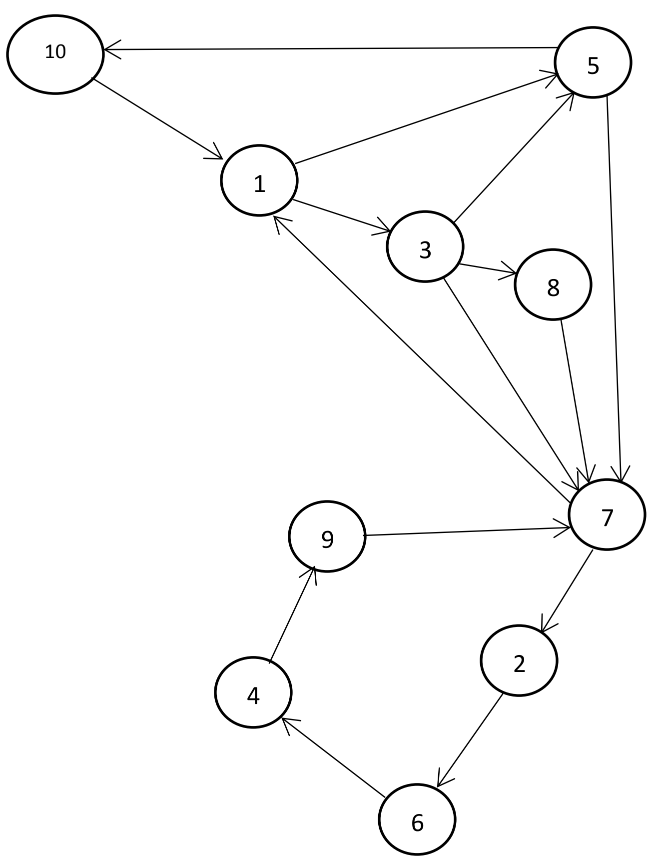
c)

u	1	2	3	4	5	6	7	8	9	10
r(u)	3	3	3	4	3	4	2	3	3	4

d) $Center\ of\ G = node\ 7$, because the radius of G relative to node 7 is 2, and is the smallest radius.

Problem 3:

a) Graph G: (directed)



Adjacency Matrix:

	1	2	3	4	5	6	7	8	9	10
1	0	0	1	0	1	0	0	0	0	0
2	0	0	0	0	0	1	0	0	0	0
3	0	0	0	0	1	0	1	1	0	0
4	0	0	0	0	0	0	0	0	1	0
5	0	0	0	0	0	0	1	0	0	1
6	0	0	0	1	0	0	0	0	0	0
7	1	1	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	1	0	0	0
9	0	0	0	0	0	0	1	0	0	0
10	1	0	0	0	0	0	0	0	0	0

b)

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
In-degree	2	1	1	1	2	1	4	1	1	1
Out-degree	2	1	3	1	2	1	2	1	1	1

c)

$1: all\ (2,3,4,5,6,7,8,9,10)$

$2: all\ (1,3,4,5,6,7,8,9,10)$

$3: all\ (1,2,4,5,6,7,8,9,10)$

$4: all\ (1,2,3,5,6,7,8,9,10)$

$5: all\ (1,2,3,4,6,7,8,9,10)$

$6: all\ (1,2,3,4,5,7,8,9,10)$

$7: all\ (1,2,3,4,5,6,8,9,10)$

$8: all\ (1,2,3,4,5,6,7,9,10)$

$9: all\ (1,2,3,4,5,6,7,8,10)$

$10: all\ (1,2,3,4,5,6,7,8,9)$

$Yes, G\ is\ Strongly\ connected\ because\ every\ node\ can\ reach\ every\ other\ n$

d) No, because otherwise every node in G would have to be of out-degree 5, but G has a node of out-degree $1 \neq 5$.

e)

$No. G$ is $strongly$ $connected$ $\Longrightarrow$ Any $digraph$ $Isomorphic$ to G $must$ $also$ be $Strongly$ $connected.$

Problem 4:

a)

$Claim$: $Number\ of\ nodes\ in\ a\ perfect\ binary\ tree\ at\ level\ i = 2^i. Proof\ by\ induction$:

$Basis\ step$: $i = 0,$

$At\ level\ 0, only\ node\ is\ Root.\quad \therefore\ number\ of\ nodes\ at\ level\ 0\ is\ 1 = 2^0$

$\therefore\ Number\ of\ nodes\ at\ level\ is\ 2^i for\ i = 0$

$Induction\ step$: $Assume\ that\ Number\ of\ nodes\ at\ level\ i - 1 = 2^{i-1}(the\ induction\ hypothesis)$

$Prove\ that\ the\ number\ of\ nodes\ at\ level\ i = 2^i$

$Number\ of\ nodes\ at\ level\ i =$
$(number\ of\ nodes\ at\ level\ i - 1) * 2\quad because\ every\ node\ at\ level\ i - 1$ has 2 children

$$= 2^{i-1} * 2 = 2^i$$

$\therefore\ Number\ of\ nodes\ in\ a\ perfect\ binary\ tree\ at\ level\ i = 2^i, \forall\ i \geq 0$

$Now, Total\ number\ of\ nodes\ in\ the\ perfect\ binary\ tree\ (n)$ is:

$$n = \sum_{i=0}^{k-1} number\ of\ nodes\ at\ i = \sum_{i=0}^{k-1} 2^i$$

$$= \frac{2^{(k-1)+1} - 1}{2 - 1}$$

$$= 2^k - 1$$

$\therefore\ The\ number\ of\ nodes\ in\ a\ perfect\ binary\ tree\ of\ k\ levels\ is\ 2^k - 1.$

b)

$Claim$: $Number\ of\ nodes\ in\ a$ 2-3 $tree\ at\ level\ i\ is\ between\ 2^i and\ 3^i inclusive, i.e.,$

$2^i \leq Number\ of\ nodes\ at\ level\ i \leq 3^i. Proof\ by\ induction$:

$Basis\ step$: $i = 0, 2^i = 2^0 = 1\ and\ 3^i = 3^0 = 1$

$At\ level\ 0, the\ only\ node\ is\ Root, and\ 2^0 \leq 1 \leq 3^0$

$\therefore\ 2^i \leq Number\ of\ nodes\ at\ level\ i \leq 3^i, for\ i = 0$

$Induction\ step$: $Assume\ that,\ 2^{i-1} \leq Number\ of\ nodes\ at\ level\ (i-1)$
$\leq 3^{i-1}\ (the\ induction\ hypothesis).$

$Prove\ that$: $2^i \leq Number\ of\ nodes\ at\ level\ i \leq 3^i$

$We\ have:$

- $total\ number\ of\ nodes\ at\ level\ i-1\ is\ \geq\ 2^{i-1} and\ each\ node\ has\ \geq\ 2\ child\ nodes.$
  $Therefore, the\ number\ of\ nodes\ at\ level\ i\ is\ \geq\ 2 * 2^{i-1} = 2^i.$
- $total\ number\ of\ nodes\ at\ level\ i-1\ is\ \leq\ 3^{i-1} and\ each\ node\ has\ \leq\ 3\ child\ nodes.$
  $Therefore, the\ number\ of\ nodes\ at\ level\ i\ is\ \leq\ 3 * 3^{i-1} = 3^i.$

$\Longrightarrow 2^i \leq Number\ of\ nodes\ at\ level\ i \leq 3^i$

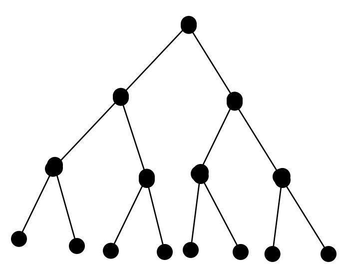
$Now, Total\ number\ of\ nodes\ n\ in$ a 2-3 $tree\ with\ k\ levels\ satisfies \sum_{i=0}^{k-1} 2^i \leq n \leq \sum_{i=0}^{k-1} 3^i$

$$\Longrightarrow \frac{2^{(k-1)+1} - 1}{2-1} \leq n \leq \frac{3^{(k-1)+1} - 1}{3-1} \Longrightarrow 2^k - 1 \leq n \leq \frac{3^k - 1}{2}$$

$\therefore\ In\ a\ 2-3\ tree\ of\ n\ nodes\ and\ k\ levels, we\ must\ have\ 2^k - 1 \leq n \leq \frac{3^k - 1}{2}$

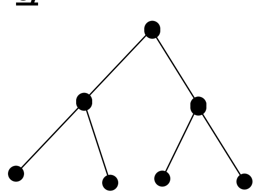
<u>Problem 5:</u>

<u>a)</u>



<u>b)</u> $A\ perfect\ binary\ tree\ with\ k\ levels\ has\ 2^k - 1\ nodes.\ Therefore, the\ number\ of\ nodes\ of$ *perfect binary trees can only be*: 1, 3, 7, 15, 31, etc., *in particular, it cannot be 10.*

<u>c)</u>



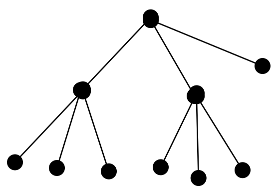
d) $Using\ 4b, 2-3\ tree\ of\ 3\ levels\ will\ have\ nodes\ between\ 2^3-1\ and\ \frac{3^3-1}{2} \Longrightarrow 7 \leq n \leq 13.$

$But\ 13 < 15, so, it\ is\ not\ possible\ to\ have\ a\ 2-3\ of\ 3\ levels\ with\ 15\ nodes.$

e) $Number\ of\ edges\ in\ a\ tree = number\ of\ nodes - 1.$

$Therefore\ it\ is\ not\ possible\ to\ have\ a\ tree\ with\ 10\ nodes\ and\ 12\ edges\ because\ 12 \neq 10-1.$

f)



g) There can't be a tree with 10 nodes where there are at least 3 internal nodes and each internal node has 4 children. The reason is that the 3 (or more) internal nodes have 4 children each (and can't share children),implying that those three nodes have a total of 12 children, exceeding the 10 allowed nodes.

Bonus Problem:

$Basis\ step: k = 0.$ one-node tree $(root\ node\ only) \Longrightarrow No. of\ leaves = 1 = 0 + 1 = k + 1$

$Induction\ step: Assume\ for\ any\ full\ binary\ tree\ of\ k-1\ internal\ nodes, number\ of\ leaves = (k-1)+1 = k.\ \ (the\ induction\ hypothesis)$

$Prove\ that\ for\ full\ binary\ tree\ of\ k\ internal\ nodes, the\ number\ of\ leaves = k+1.$

$Let\ T\ be\ a\ full\ binary\ tree\ of\ k\ internal\ nodes. Pick\ any\ internal\ node\ whose\ 2\ children\ are\ leaves.$

$Delete\ those\ 2\ leaves\ from\ T, resulting\ in\ a\ tree\ T'.$

a. $Number\ of\ leaves\ in\ T' = Number\ of\ leaves\ in\ T - 2 + 1 = Number\ of\ leaves\ in\ T - 1$
b. $Number\ of\ internal\ nodes\ in\ T' = Number\ of\ internals\ in\ T - 1 = k - 1$

$By\ induction\ hypothesis\ on\ T', Number\ of\ leaves\ in\ T' = k.$

$By\ item\ (a) above, Number\ of\ leaves\ in\ T - 1 = Number\ of\ leaves\ in\ T' = k \Longrightarrow$
$Number\ of\ leaves\ in\ T = k + 1.$
$\therefore In\ a\ full\ binary\ tree\ with\ k\ internal\ nodes, the\ number\ of\ leaves = k + 1$

$Now, total\ number\ of\ nodes\ in\ a\ full\ binary\ tree\ with\ k\ internal\ nodes\ and\ k+1\ leaves = k + (k+1) = 2k+1. Q.E.D.$