CS1311 **March 24, 2016**

Homework 4 Solutions

Problem 1:

a) $R\ is\ reflexive, symmetric, transive\ and\ therefore\ an\ equivalence\ relation.$
$R\ is\ not\ antisymmetric\ and\ therefore\ not\ a\ partial\ order.$

$Equivalance\ class: [a] = \{\ x\ \in Y \mid a\ R\ x\ \}$
$= All\ YouTube\ videos\ available\ Ja. 1\ 2016\ and\ having\ the\ same\ number\ of\ views\ as\ video\ a.$

b) $R\ is\ reflexive, symmetric, transive\ and\ therefore\ an\ equivalence\ relation.$
$R\ is\ not\ antisymmetric\ and\ therefore\ not\ a\ partial\ order.$

$Equivalance\ class: [a] = \{\ x\ \in Y \mid a\ R\ x\ \}$
$= All\ YouTube\ videos\ available\ Ja. 1\ 2016\ and\ having\ the\ same\ number\ of\ likes\ as\ video\ a..$

c) $R\ is\ reflexive, antisymmetric, transitive\ and\ therefore\ a\ partial\ order.$
$R\ is\ not\ symmetric\ and\ therefore\ not\ an\ equivalence\ relation.$

$Y\ may\ or\ may\ not\ have\ a\ minimum\ and/or\ maximum.$
$Y\ will\ have\ minimals\ and\ maximals.$

$minimum = the\ YouTube\ video\ in\ Y\ with\ the\ least\ number\ of\ likes,$
$if\ it\ is\ the\ only\ video\ with\ as\ many\ likes\ .$
$maximum = the\ YouTube\ video\ in\ Y\ with\ the\ largest\ number\ of\ likes,$
$if\ it\ is\ the\ only\ video\ with\ as\ many\ likes\ .$
.

$minimals = youtube\ videos\ in\ Y\ which\ have\ the\ least\ number\ of\ likes.$
$maximals =\ youtube\ videos\ in\ Y\ which\ have\ the\ highest\ number\ of\ likes.$

d) $R\ is\ neither\ reflexive\ nor\ symmetric\ and\ therfore\ neither\ an\ equivalance\ relation$
$nor\ a\ partial\ order.$
$But\ R\ is\ antisymmetric\ and\ transitive.$

Problem 2:

a)

Note: In this part, when we say that two letters are the same, that also means they have the same case.

i) $\alpha\ R\ \alpha?\ \alpha\ contains\ the\ same\ first\ letter\ as\ itself\ \Longrightarrow\ \alpha\ R\ \alpha. \therefore R\ is\ reflexive$

$\alpha\ R\ \beta \implies \alpha\ and\ \beta\ have\ the\ same\ first\ letter\ (equal\ and\ same\ case)$
$\implies \beta\ and\ \alpha\ have\ the\ same\ first\ letter \implies \beta\ R\ \alpha$
$\therefore R\ is\ symmetric.$

$\alpha\ R\ \beta\ and\ \beta\ R\ \gamma$
$\implies \alpha\ and\ \beta\ have\ the\ same\ first\ letter\ and$
$also\ \beta\ and\ \gamma\ have\ the\ same\ first\ letter.$
$\implies \alpha\ and\ \gamma\ have\ the\ same\ first\ letter \implies \alpha\ R\ \gamma \therefore R\ is\ transitive.$

$Since\ R\ is\ reflexive, symmetric\ and\ transitive, \quad R\ is\ an\ equivalence\ relation.$

ii) $\#\ Equivalence\ classes =$
$26\ (lower\ case\ english\ letters\ ) + 26\ (upper\ case\ english\ letters) = 52$

iii) $Set\ of\ all\ strings\ which\ have\ the\ same\ first\ letter\ as\ in\ the\ string\ "apple"\ which\ is\ "a"$

b)

i) $\alpha\ R\ \alpha?\ \alpha\ correspond\ to\ a\ lowercase\ string\ as\ itself. \implies \alpha\ R\ \alpha. \therefore R\ is\ reflexive$

$\alpha\ R\ \beta \implies \alpha\ and\ \beta\ correspond\ to\ the\ same\ lowercase\ string$
$\implies \beta\ and\ \alpha\ correspond\ to\ that\ same\ lowercase\ string \implies \beta\ R\ \alpha$
$\therefore R\ is\ symmetric.$

$\alpha\ R\ \beta\ and\ \beta\ R\ \gamma$
$\implies \alpha\ and\ \beta\ correspond\ to\ the\ same\ lowercase\ string\ and\ so\ do\ \beta\ and\ \gamma$
$\implies \alpha\ and\ \gamma\ correspond\ to\ that\ same\ lowercase\ string \implies \alpha\ R\ \gamma$
$\therefore R\ is\ transitive.$

$R\ is\ reflexive, symmetric\ and\ transitive \implies R\ is\ an\ equivalence\ relation.$

ii) $[app] = \{\ app, apP, aPP, aPp, APP, ApP, APp, App\ \}$

iii) $|[apple]| = 2^5 = 32$

Problem 3:

a)

i) $\alpha\ R\ \alpha?\ \ \alpha\ is\ lexographically\ at\ itself \implies \alpha\ R\ \alpha\ \therefore R\ is\ reflexive.$

For Antisymmetry, we need to show that $\alpha\ R\ \beta\ and\ \alpha \neq \beta \implies not(\beta\ R\ \alpha)$.
$\alpha\ R\ \beta\ and\ \ \alpha \neq \beta \implies \alpha\ is\ lexographically\ before\ \beta$
$\implies \beta\ cannot\ be\ lexographically\ before\ \alpha.$
$\therefore R\ is\ antisymmetric.$

For transitivity, let $\alpha\ R\ \beta\ and\ \beta\ R\ \gamma\ . We\ need\ to\ show\ that\ \alpha\ R\ \gamma.$

$If\ \alpha = \beta, then\ \alpha\ R\ \gamma\ because\ \beta\ R\ \gamma. Also\ if\ \beta = \gamma, then\ \alpha\ R\ \gamma\ because\ \alpha\ R\ \beta.$
$Now\ if\ \alpha \neq \beta\ and\ \beta \neq \gamma,$
$then\ \alpha\ R\ \beta\ and\ \beta\ R\ \gamma\ implies\ that\ \alpha\ is\ lexographically\ before\ \beta\ and$
$\beta\ is\ lexographically\ before\ \gamma$
$\Longrightarrow\ \alpha\ is\ lexographically\ before\ \gamma \Longrightarrow \alpha\ R\ \gamma\ \therefore R\ is\ transitive.$

$\therefore R\ is\ a\ partial\ order\ relation.$

ii) $Yes, L\ has\ a\ minimum\ w.r.t\ R =$ "a"

iii) $No, L\ does\ not\ have\ a\ maximum\ w.r.t\ R.$

iv) $Yes, E\ has\ a\ maximum\ w.r.t\ R =$ "$zzzzz$".

$No, E\ has\ one\ maximal: zzzzz.$

b)

i) $\alpha\ R\ \alpha?\ \ \alpha\ is\ a\ prefix\ of\ itself \Longrightarrow \alpha\ R\ \alpha\ \therefore R\ is\ reflexive.$

For Antisymmetry, we need to show that $\alpha\ R\ \beta\ and\ \alpha \neq \beta \Longrightarrow not(\beta\ R\ \alpha).$
$\alpha\ R\ \beta\ and\ \alpha \neq \beta \Longrightarrow \alpha\ is\ a\ prefix\ of\ \beta\ and\ \alpha \neq \beta$
$\Longrightarrow \beta\ contains\ all\ letters\ in\ \alpha\ and\ at\ least\ one\ letter\ more\ than\ \alpha$
$\Longrightarrow \beta\ cannot\ be\ a\ prefix\ of\ \alpha \Longrightarrow not(\beta\ R\ \alpha).$
$\therefore R\ is\ antisymmetric.$

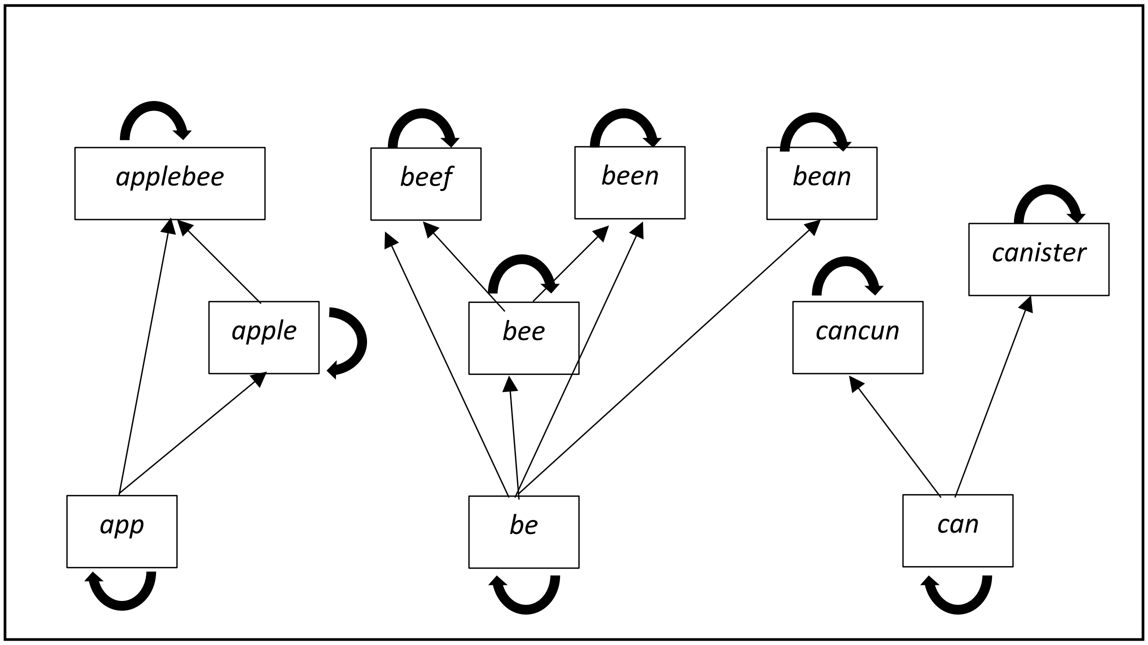
$\alpha\ R\ \beta\ and\ \beta\ R\ \gamma \Longrightarrow \alpha\ is\ a\ prefix\ of\ \beta\ and\ \beta\ is\ a\ prefix\ of\ \gamma$
$\Longrightarrow \alpha\ is\ a\ prefix\ of\ \gamma \Longrightarrow \alpha\ R\ \gamma\ \therefore R\ is\ transitive.$

$\therefore R\ is\ a\ partial\ order\ relation.$

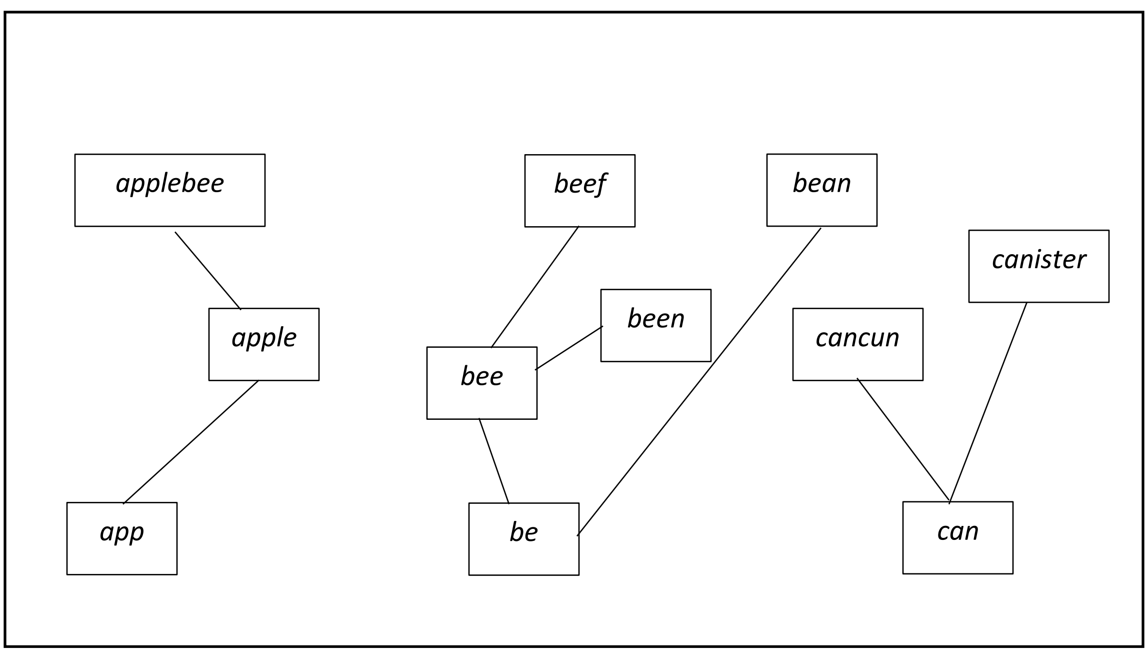
ii) $L\ cannot\ have\ a\ minimum\ w.r.t.\ R\ because\ no\ nonempty\ word\ is\ a\ prefix\ of\ all\ other\ words.$

iii) $Yes, L\ has\ minimals\ w.r.t\ R, namely, \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$

c) *Graph for E*:



Hasse Diagram of R for E:



d) *No, E does not have a minimum or maximum w.r.t R.*

$maximals: applebee, beef, been, bean, cancun, canister$
$minimals: app, be, can$

Problem 4:

a)
$Reflexive: (\forall x \in A)\ (x\ R \cap S\ x)?$
$R\ is\ reflexive \implies x\ R\ x$
$S\ is\ reflexive \implies x\ S\ x$

$\implies x\ R\ x\ and\ x\ S\ x \implies x\ R \cap S\ x$

b)
$Symmetric: (\forall x, y \in A)\ (x\ R \cap S\ y) \implies (y\ R \cap S\ x)?$
$x\ R \cap S\ y \implies x\ R\ y\ and\ x\ S\ y \implies y\ R\ x\ and\ y\ S\ x\ (\ because\ R\ and\ S\ are\ symmetric)$
$\implies y\ R \cap S\ x\ by\ definition\ of\ R\ \cap S$

c)
$Antisymmetric: (\forall x, y \in A)\ (x\ R \cap S\ y\ and\ y\ R \cap S\ x) \implies x = y?$
$x\ R \cap S\ y\ and\ y\ R \cap S\ x \implies x\ R\ y\ and\ y\ R\ x \implies x = y\ because\ Ris\ antisymmetric.$

d)
$Transitive: (\forall x, y, z \in A)(x\ R \cap S\ y\ and\ y\ R \cap S\ z) \implies (x\ R \cap S\ z)?$
$x\ R \cap S\ y\ and\ y\ R \cap S\ z \implies x\ R\ y\ and\ x\ S\ y\ and\ y\ R\ z\ and\ y\ S\ z$
$\implies x\ R\ z\ and\ x\ S\ z\ (because\ R\ and\ S\ are\ symmetric) \implies x\ R \cap S\ z$

Problem 5:

a) $\forall x\ and\ y \in Z, x\ R\ y\ if\ (x - y)\ is\ a\ multiple\ of\ 4$
$(x - y)\ is\ a\ multiple\ of\ 4 \implies 4 \mid (x - y) \implies x - y = 4k\ for\ some\ k \in Z.$

$Reflexive: x\ R\ x?$
$x - x = 0 = 4 * 0, 0 \in Z$
$\therefore x\ R\ x$

$Symmetric: x\ R\ y \implies y\ R\ x?$
$x - y = 4k\ for\ some\ k \in Z \implies (multiply\ by - 1\ on\ both\ sides) \implies -(x - y) = -4k$
$\implies (y - x) = 4(-k)$
$we\ know\ that\ \forall k \in Z, -k \in Z. Therefore, \quad y - x\ is\ a\ multiple\ and\ thus\ y\ R\ x.$
$\therefore x\ R\ y \implies y\ R\ x$

$Transitive: x\ R\ y\ and\ y\ R\ z \implies x\ R\ z?$
$x\ R\ y\ and\ y\ R\ z \implies x - y = 4k\ and\ y - z = 4l\ for\ k\ and\ l\ in\ Z$
$\implies (adding\ these\ two\ equations, we\ get)\ x - y + y - z = 4k + 4l$
$\implies x - z = 4(k + l)$
$We\ know\ that, \forall\ k, l \in Z, k + l \in Z. Therefore, x - z\ is\ a\ multiple\ of\ 4\ and\ thus\ x\ R\ z.$

$\therefore x\ R\ y\ and\ y\ R\ z\ \Longrightarrow x\ R\ z$

$\therefore R\ is\ an\ equivalance\ relation.$

b) $[x] = x\ mod\ 4\ \Longrightarrow x - [x] = 4k\ for\ k\ \in Z?$
$x = 4k + r\ for\ k\ \in Z\ and\ 0 \le r \le 3, [x] = r.$
$\Longrightarrow x = 4k + [x] \Longrightarrow x - [x] = 4k\ \Longrightarrow x - [x]\ is\ a\ multiple\ of\ 4 \Longrightarrow x\ R\ [x]$

c) The equivalence classes are:

$C(0) = \{ x \in Z\ | x\ R\ 0\} = \{x = 4k\ | k \in Z\}$
$C(1) = \{ x \in Z\ | x\ R\ 1\} = \{x = 4k + 1\ | k \in Z\}$
$C(2) = \{ x \in Z\ | x\ R\ 2\} = \{x = 4k + 2\ | k \in Z\}$
$C(3) = \{ x \in Z\ | x\ R\ 3\} = \{x = 4k + 3\ | k \in Z\}$

Bonus Problem:

a)
$Reflexive: (m, n)R\ (m, n)?$
$(m, n)R(m, n)\ because\ m + n = m + n\ which\ is\ true.$
$\therefore (m, n)\ R\ (m, n)$

$Symmetric: (m_1, n_1)R\ (m_2, n_2) \Longrightarrow (m_2, n_2)R\ (m_1, n_1)?$
$(m_1, n_1)R\ (m_2, n_2) \Longrightarrow m_1 + n_2 = m_2 + n_1 \Longrightarrow m_2 + n_1 = m_1 + n_2$
$\therefore (m_1, n_1)R\ (m_2, n_2) \Longrightarrow (m_2, n_2)R\ (m_1, n_1)$

$Transitive: (m_1, n_1)R\ (m_2, n_2)\ and\ (m_2, n_2)R\ (m_3, n_3)\ \Longrightarrow (m_1, n_1)R\ (m_3, n_3)?$
$(m_1, n_1)R\ (m_2, n_2)\ and\ (m_2, n_2)R\ (m_3, n_3)\ \Longrightarrow m_1 + n_2 = m_2 + n_1 and\ m_2 + n_3 = m_3 + n_2$
$\Longrightarrow (adding\ equations) \Longrightarrow (m_1 + n_2) + (m_2 + n_3)$
$= (m_2 + n_1) + (m_3 + n_2)\ \Longrightarrow m_1 + n_3 = m_3 + n_1$
$\therefore\ (m_1, n_1)R\ (m_2, n_2)\ and\ (m_2, n_2)R\ (m_3, n_3)\ \Longrightarrow (m_1, n_1)R\ (m_3, n_3)$

b) $[m, n] = \{ (m', n') \in N * N\ |\ m' - n' = m - n\}$

$one - to - one: f([m_1, n_1]) = f([m_2, n_2]) \Longrightarrow [m_1, n_1] = [m_2, n_2]?$

$f([m_1, n_1]) = f([m_2, n_2]) \Longrightarrow m_1 - n_1 = m_2 - n_2 \Longrightarrow \{(m', n')\ |\ m' - n' = m_1 - n_1\}$
$= \{(m', n')|\ m' - n' = m_2 - n_2\}\ \Longrightarrow [m_1, n_1] = [m_2, n_2]$
$\therefore f\ is\ one - to - one$

$onto: (\forall k \in Z)(\exists [m, n] \in E)(f([m, n]) = k)?$
$2\ cases: k \ge 0\ and\ k < 0$
$case\ 1: k \ge 0: Take\ (m, n) = (k, 0) \in N * N$
$f([m, n]) = f([k, 0]) = k - 0 = k\ and\ [k, 0] \in E$
$case\ 2: k < 0: Take\ (m, n) = (0, -k) \in N * N$

$$f([m,n]) = f([0,-k]) = 0 - \; -k = k \;\; and \; [0,-k] \in E$$

$\therefore f \; is \; onto.$