CS1311 Spring 2016

Homework 3 Solution

Problem 1 (5 points / question)

a) $x_n^{(h)} = \alpha_1 5^n$

$F(n) = 12 \Longrightarrow x_n^{(p)} = p_0$

$p_0 = 5p_0 + 12 \Longrightarrow p_0 = -3$

$x_n^{(p)} = -3$

$x_n = x_n^{(h)} + x_n^{(p)} = \alpha_1 5^n - 3$

$x_0 = 2 = \alpha_1 - 3 \Longrightarrow \alpha_1 = 5$

$x_n = 5 \cdot 5^n - 3 = 5^{n+1} - 3$

b) $r^2 - 8r + 15 = 0 \Longrightarrow (r-3)(r-5) = 0 \Longrightarrow r_1 = 3, r_2 = 5$

$x_n = \alpha_1 3^n + \alpha_2 5^n$

$$\begin{cases} x_0 = 1 = \alpha_1 + \alpha_2 \\ x_1 = 2 = 3\alpha_1 + 5\alpha_2 \end{cases} \Longrightarrow \alpha_1 = \frac{3}{2}, \alpha_2 = -\frac{1}{2}$$

$x_n = \frac{3}{2} \cdot 3^n - \frac{1}{2} \cdot 5^n$

c) $r^2 - 2r - 8 = 0 \Longrightarrow (r-4)(r+2) = 0 \Longrightarrow r_1 = 4, r_2 = -2$

$x_n^{(h)} = \alpha_1 4^n + \alpha_2 (-2)^n$

$F(n) = 12 \Longrightarrow x_n^{(p)} = p_0$

$p_0 = 2p_0 + 8p_0 + 12 \Longrightarrow p_0 = -\frac{4}{3}$

$x_n^{(p)} = -\frac{4}{3}$

$x_n = x_n^{(h)} + x_n^{(p)} = \alpha_1 4^n + \alpha_2 (-2)^n - \frac{4}{3}$

$$\begin{cases} x_0 = 1 = \alpha_1 + \alpha_2 - \frac{4}{3} \\ x_1 = 1 = 4\alpha_1 - 2\alpha_2 - \frac{4}{3} \end{cases} \Longrightarrow \alpha_1 = \frac{7}{6}, \alpha_2 = \frac{7}{6}$$

$x_n = \frac{7}{6} \cdot 4^n + \frac{7}{6}(-2)^n - \frac{4}{3}$

Problem 2 (10 points / question)

a) $x_n^{(h)} = \alpha_1 4^n$

$F(n) = 9n + 3 \Longrightarrow x_n^{(p)} = p_1 n + p_0$

$p_1 n + p_0 = 4[p_1(n-1) + p_0] + 9n + 3 \Longrightarrow (3p_1 + 9)n - 4p_1 + 3p_0 + 3 = 0$

$\begin{cases} 3p_1 + 9 = 0 \\ -4p_1 + 3p_0 + 3 = 0 \end{cases} \Longrightarrow p_1 = -3, p_0 = -5$

$x_n^{(p)} = -3n - 5$

$x_n = x_n^{(h)} + x_n^{(p)} = \alpha_1 4^n - 3n - 5$

$x_0 = 7 = \alpha_1 - 5 \Longrightarrow \alpha_1 = 12$

$x_n = 12 \cdot 4^n - 3n - 5$

b) $r^2 - 8r + 15 = 0 \Longrightarrow (r-3)(r-5) = 0 \Longrightarrow r_1 = 3, r_2 = 5$

$x_n^{(h)} = \alpha_1 3^n + \alpha_2 5^n$

$F(n) = 16n + 4 \Longrightarrow x_n^{(p)} = p_1 n + p_0$

$p_1 n + p_0 = 8[p_1(n-1) + p_0] - 15[p_1(n-2) + p_0] + 16n + 4$

$\Longrightarrow (-8p_1 + 16)n + 22p_1 - 8p_0 + 4 = 0$

$\begin{cases} -8p_1 + 16 = 0 \\ 22p_1 - 8p_0 + 4 = 0 \end{cases} \Longrightarrow p_1 = 2, p_0 = 6$

$x_n^{(p)} = 2n + 6$

$x_n = x_n^{(h)} + x_n^{(p)} = \alpha_1 3^n + \alpha_2 5^n + 2n + 6$

$\begin{cases} x_0 = 1 = \alpha_1 + \alpha_2 + 6 \\ x_1 = 3 = 3\alpha_1 + 5\alpha_2 + 8 \end{cases} \Longrightarrow \alpha_1 = -10, \alpha_2 = 5$

$x_n = -10 \cdot 3^n + 5 \cdot 5^n + 2n + 6 = -10 \cdot 3^n + 5^{n+1} + 2n + 6$

Problem 3 (2 points / question)

a) $C(10,4) = 210$

b) $P(9,9) = 362880$

c) $C(9,2)C(7,3)P(4,4) = 30240$

d) $C(8,3) = 56$

e) i. $26^3 \times 10^4 = 1.7576 \times 10^8$

ii. $10^3 \times 5 = 5 \times 10^3$

iii. $P(26,3)P(10,4) = 7.8624 \times 10^7$

f) i. $20^3 = 8000$
ii. $P(20,3) = 6840$

g) $C(5,2)C(7,3)C(10,4) = 73500$

Problem 4 (5 points / question)

a) $\{R, G, B, W, Y, T\} \times \{F, R\} \times \{S, M, L, XL\}$
$Cardinality = 6 \times 2 \times 4 = 48$

b) $\lfloor 199 \div 6 \rfloor = 33$, $\lfloor 199 \div 9 \rfloor = 22$, $lcm(6,9) = 18 \Longrightarrow \lfloor 199 \div 18 \rfloor = 11$, $33 + 22 - 11 = 44$, $199 - 44 = 155$

c) The problem is the same as proving $2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$ for all positive integers n
Basis step:
When $n = 1$, $2^1 = 2 = 2^{1+1} - 2$
Inductive step:
Assume that $2 + 2^2 + 2^3 + \cdots + 2^k = 2^{k+1} - 2$ is true for an arbitrary positive integer k;
We must show that $2 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} = 2^{k+2} - 2$

$$\begin{aligned} 2 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} &= 2^{k+1} - 2 + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 2 \\ &= 2^{k+2} - 2 \end{aligned}$$

d) m^n
Suppose without loss of generality that $E = \{1,2,3,\cdots,n\}$. Every function from E to F can be viewed as a sequence$\big(f(1), f(2), f(3), \cdots f(n)\big)$, where in that sequence order matters and repetition is allowed. Therefore, the number of functions from E to F is m^n functions.

Problem 5 (10 points / question)

a) The problem is the same as proving $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$ for all positive integers n
Basis step:
When $n = 1$, $1 = \frac{1}{2} \cdot 1 \cdot (1 + 1)$
Inductive step:
Assume that $1 + 2 + 3 + \cdots + k = \frac{1}{2}k(k+1)$ is true for an arbitrary positive integer k;
We must show that $1 + 2 + 3 + \cdots + k + (k+1) = \frac{1}{2}(k+1)(k+2)$

$$1+2+3+\cdots+k+(k+1)=\frac{1}{2}k(k+1)+(k+1)$$
$$=\left(\frac{1}{2}k+1\right)(k+1)$$
$$=\frac{1}{2}(k+1)(k+2)$$

b) The problem is the same as proving $1+3+6+\cdots+\frac{1}{2}n(n+1)=\frac{1}{6}n(n+1)(n+2)$ for all positive integers n

Basis step:

When $n=1$, $\frac{1}{2}\cdot 1\cdot(1+1)=1=\frac{1}{6}\cdot 1\cdot(1+1)(1+2)$

Inductive step:

Assume that $1+3+6+\cdots+\frac{1}{2}k(k+1)=\frac{1}{6}k(k+1)(k+2)$ is true for an arbitrary positive integer k;

We must show that $1+3+6+\cdots+\frac{1}{2}k(k+1)+\frac{1}{2}(k+1)(k+2)=\frac{1}{6}(k+1)(k+2)(k+3)$

$$1+3+6+\cdots+\frac{1}{2}k(k+1)+\frac{1}{2}(k+1)(k+2)$$
$$=\frac{1}{6}k(k+1)(k+2)+\frac{1}{2}(k+1)(k+2)$$
$$=\left(\frac{1}{6}k+\frac{1}{2}\right)(k+1)(k+2)$$
$$=\frac{1}{6}(k+1)(k+2)(k+3)$$

Bonus Problem (2.5 points / question)

a) $x_0=1,\ x_1=1,\ x_n=x_{n-1}+x_{n-2}+1$

b) $r^2-r-1=0\Longrightarrow r=\frac{1\pm\sqrt{1+4}}{2}\Longrightarrow r_1=\frac{1+\sqrt{5}}{2}, r_2=\frac{1-\sqrt{5}}{2}$

$$x_n^{(h)}=\alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^n+\alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$F(n)=1\Longrightarrow x_n^{(p)}=p_0$

$p_0=p_0+p_0+1\Longrightarrow p_0=-1$

$x_n^{(p)}=-1$

$$x_n=x_n^{(h)}+x_n^{(p)}=\alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^n+\alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^n-1$$

$$\begin{cases} 1 = \alpha_1 + \alpha_2 - 1 \\ 1 = \frac{1+\sqrt{5}}{2}\alpha_1 + \frac{1-\sqrt{5}}{2}\alpha_2 - 1 \end{cases} \Longrightarrow \alpha_1 = 1 + \frac{1}{\sqrt{5}}, \alpha_2 = 1 - \frac{1}{\sqrt{5}}$$

$$x_n = \left(1 + \frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(1 - \frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n - 1$$