CS1311 **Spring 2016**

Homework 2 Solutions

Problem 1:

							a)			b)
$\boldsymbol{p}$	q	r	$\neg p$	$\neg q$	$\neg r$	$\neg q \vee \neg r$	$p \wedge (\neg q \vee \neg r)$	$q \wedge r$	$\neg(q \wedge r)$	$p \vee \neg(q \wedge r)$
T	T	T	F	F	F	F	F	T	F	T
T	T	F	F	F	T	T	T	F	T	T
T	F	T	F	T	F	T	T	F	T	T
T	F	F	F	T	T	T	T	F	T	T
F	T	T	T	F	F	F	F	T	F	F
F	T	F	T	F	T	T	F	F	T	T
F	F	T	T	T	F	T	F	F	T	T
F	F	F	T	T	T	T	F	F	T	T

								c)		b)
$\boldsymbol{p}$	q	r	$\neg p$	$\neg q$	$\neg r$	$q \wedge \neg r$	$\neg(q \wedge \neg r)$	$p \vee \neg(q \wedge \neg r)$	$p \wedge \neg q$	$(p \wedge \neg q) \wedge r$
T	T	T	F	F	F	F	T	T	F	F
T	T	F	F	F	T	T	F	T	F	F
T	F	T	F	T	F	F	T	T	T	T
T	F	F	F	T	T	F	T	T	T	F
F	T	T	T	F	F	F	T	T	F	F
F	T	F	T	F	T	T	F	F	F	F
F	F	T	T	T	F	F	T	T	F	F
F	F	F	T	T	T	F	T	T	F	F

Problem 2:

a) The statement is True.

Proof:

$A-(B \cup C) = A \cap (B \cup C)^c$ by Set difference law

$= A \cap A \cap (B^c \cap C^c)$ by Idempotent law and De Morgan's laws

$= A \cap (A \cap B^c) \cap C^c$ by Associative law

$= A \cap (B^c \cap A) \cap C^c$ by Commutative law

$= (A \cap B^c) \cap (A \cap C^c)$ by Associative law

$= (A-B) \cap (A-C)$ by Set difference law

b) The statement is True.

Proof:

$(A \cap B) - C = (A \cap B) \cap C^c$ by Set difference law

$= A \cap (B \cap C^c)$ by Associative law

$= A \cap (B - C)$ by Set difference law

c) The statement is False.

Proof by counter example:

Let A = { a, b, c }, B= { a, d, e } and C = { a, f, g }

Then we have, B + C = {d, e, f, g } and A – B = { b, c }

Now, A – (B + C) = { a, b, c } and (A – B) – C = { b, c }

$\therefore A - (B + C) \neq (A - B) - C$

d) The statement is False.

Proof by counter example:

Let A = { a, b, c }, B= { a, d, e } and C = { a, f, g }

Then we have, B – C = { d, e } and A – B = { b, c }

Now, A – (B – C) = { a, b, c } and (A – B) + C = { a, b, c, f, g }

$\therefore A - (B - C) \neq (A - B) + C$

e) The statement is False.

Proof by counter example:

Let A = { a, b, c }, B= { a, d, e } and C = { a, f, g }

Then we have, B – C = { d, e } and C – B = { f, g }

Now, A – (B – C) = { a, b, c } and $\mathrm{A} \cap (\mathrm{C} - \mathrm{B}) = \emptyset$

$\therefore A - (B - C) \neq A \cap (C - B)$

f) This statement is True.

Proof:

$X \in 2^{A \cap B} \Leftrightarrow X \subseteq A \cap B \Leftrightarrow X \subseteq A \wedge X \subseteq B \Leftrightarrow X \in 2^A \wedge X \in 2^B \Leftrightarrow X \in 2^A \cap 2^B$

$\therefore 2^{A \cap B} = 2^A \cap 2^B$

Problem 3:

a) Counter example, for n = 1,

$3^n + 1 = 3^1 + 1 = 3 + 1 = 4$, 4 is not a prime number.

b) Counter example, Let A = { 1, a } and B = { a, b }, then

we have, $2^A = \{ \emptyset, \{1\}, \{a\}, \{1, a\}\}$ and $2^B = \{ \emptyset, \{a\}, \{b\}, \{a, b\}\}$

and A + B = { 1, b } => $2^{A+B} = \{ \emptyset, \{1\}, \{b\}, \{1, b\}\}$

But $2^A + 2^B = \{ \{1\}, \{b\}, \{1, a\}, \{a, b\}\}$

$\therefore 2^{A+B} \neq 2^A + 2^B$

c) Counter example, Let A = { 1, 2, 3 }, B = { 3, 4, 5 } and C = { 1, 5, 6 }

then we have, A + B = { 1, 2, 4, 5 } and B – C = { 3, 4 }

Now, A + (B – C) = { 1, 2, 4 } and (A + B) – C = { 2, 4 }

$\therefore A + (B - C) \neq (A + B) - C$

Problem 4:

a)

Proof :

(*f* is one-to-one):

$f(x_1) = f(x_2)$ => $5x_1 + 9 = 5x_2 + 9$ => $5x_1 = 5x_2$ => $x_1 = x_2$.

Therefore, f is one-to-one.

(*f* is onto):

$Let\ y \in R. [We\ must\ show\ that\ \exists x\ in\ R\ such\ that\ f(x) = y.]\ Let\ x = \frac{y-9}{5}.$

$Then\ x\ is\ a\ real\ number\ since\ sums\ and\ quotients\ (other\ than\ by\ 0) of\ real\ numbers\ are\ real\ numbers.$

$It\ follows\ that,$

$$f(x) = f\left(\frac{y-9}{5}\right) \quad \textit{by substitution}$$

$$= 5.\left(\frac{y-9}{5}\right) + 9 \quad \textit{by the definition of } f$$

$$= (y-9) + 9 \quad \textit{by basic algebra}$$

$$= y$$

Therefore, f is onto.

Finding Inverse function (f^{-1}):

$For\ any\ [particular\ but\ arbitrarily\ chosen]\ y\ in\ R, by\ the\ definition\ of\ f^{-1},$

$$f^{-1}(y) = that\ unique\ real\ number\ x\ such\ that\ f(x) = y.$$

But, $f(x) = y$

$$\Leftrightarrow 5x + 9 = y \quad \textit{by definition of } f$$

$$\Leftrightarrow x = \frac{y-9}{5} \quad \textit{by algebra}$$

$Hence,$ $f^{-1}(y) = \dfrac{y-9}{5}.$

b) g is not one-to-one but it is onto.

$g\ is\ not\ one-to-one.\ proof\ (by\ counter\ example):$

$Let, x_1 = 0\ and\ x_2 = 2\ be\ 2\ real\ numbers. Then,$

$we\ have: g(x_1) = (0-1)^2 = (-1)^2 = 1$

$and\ g(x_2) = (2-1)^2 = (1)^2 = 1.$

$That\ is, g(x_1) = g(x_2) but\ x_1 \neq x_2.$

$\therefore\ g\ is\ not\ one-to-one.$

$g\ is\ onto.\ proof: Let\ y\ \in R^+ [we\ must\ show\ that\ \exists x \in R\ such\ that\ g(x) = y.]\ Let\ x = \sqrt{y} + 1.$

$Then\ x\ is\ a\ real\ number. It\ follows\ that,$

$$g(x) = g(\sqrt{y} + 1)$$

$= \left((\sqrt{y}+1)-1\right)^2$ *by definition of g*

$= \left(\sqrt{y}\right)^2$

$= y$

$\therefore\ g\ is\ onto.$

c) h is one-to-one but not onto.

Proof :

(*h* is one-to-one):

We must $h(x_1) = h(x_2) \Rightarrow x_1 = x_2$.

$h(x_1) = h(x_2) \Rightarrow 2x_1 - 5 = 2x_2 - 5 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$.

Therefore, h is one-to-one.

(*h* is not onto):

We must show that for some $y \in R, there\ is\ no\ x\ in\ R^+\ such\ that\ h(x) = y$. Take $y = -11 \in R$. Assume by contradiction that $\exists\ x \in R^+\ such\ that\ h(x) = y$. It follows that

$-11 = 2x - 5$ *by the definition of y and h*

$-11 + 5 = 2x \Longrightarrow 2x = -6$

$\Longrightarrow x = -3 \notin R^+,\ contradicting\ that\ x \in R^+.$

Therefore, h is not onto.

d)

$h\ o\ g\ (x) = h(g(x))$

$= 2\ g(x) - 5$

$= 2\ (x-1)^2 - 5$

$= 2\ (x^2 - 2x + 1) - 5$

$= (2x^2 - 4x + 2) - 5$

$\therefore h\ o\ g\ (x) = 2x^2 - 4x - 3$

$g\ o\ h\ (x) = g\big(h(x)\big)$

$= (h(x) - 1)^2$

$= \left((2x - 5) - 1\right)^2$

$= (2x - 6)^2$

$= 2^2\ (x - 3)^2$

$\therefore g\ o\ h\ (x) = 4\ (x^2 - 6x + 9)$

$(f\ o\ h)\ o\ g(x) = (f\ o\ h)\big(g(x)\big) = (f\ o\ h)(\ (x-1)^2) = f(h((x-1)^2)))$

$= f(2(x-1)^2 - 5) = 5[2(x-1)^2 - 5] + 9$

$= 10(x-1)^2 - 25 + 9$

$\therefore (f\ o\ h)\ o\ g(x) = 10(x-1)^2 - 16$

$f\ o\ (h\ o\ g)(x) = f\big(h\ o\ g\ (x)\big) = f\left(h\big(g(x)\big)\right) = f\big(h\ ((x-1)^2)\big) = f(2(x-1)^2 - 5)$

$= 5[2(x-1)^2 - 5] + 9 = 10(x-1)^2 - 25 + 9$

$\therefore f\ o\ (h\ o\ g)(x) = 10(x-1)^2 - 16$

e)

$f^{\leftarrow}(1) = \{\ x\ \in R \mid f(x) = 1\}$

$f(x) = 1\ \Leftrightarrow 5x + 9 = 1 \Leftrightarrow 5x = 1 - 9 = -8 \Leftrightarrow x = -\dfrac{8}{5}$

$\therefore f^{\leftarrow}(1) = \left\{-\dfrac{8}{5}\right\}$

$g^{\leftarrow}(9) = \{\ x\ \in R \mid g(x) = 9\}$

$g(x) = 9 \Leftrightarrow (x-1)^2 = 9 \Leftrightarrow x - 1 = 3\ or\ x - 1 = -3 \Leftrightarrow x = 4\ or\ x = -2$

$\therefore g^{\leftarrow}(9) = \{\ -2, 4\ \}$

$g^{\leftarrow}(0) = \{\ x\ \in R \mid g(x) = 0\}$

$g(x) = 0 \Leftrightarrow (x-1)^2 = 0 \Leftrightarrow x - 1 = 0 \Leftrightarrow x = 1$

$\therefore g^{\leftarrow}(0) = \{\ 1\ \}$

$h^{\leftarrow}(9) = \{ x \in R \mid h(x) = 9\}$

$h(x) = 9 \Leftrightarrow 2x - 5 = 9 \Leftrightarrow 2x = 9 + 5 = 14 \Leftrightarrow x = \frac{14}{2} = 7$

$\therefore h^{\leftarrow}(9) = \{ 7 \}$

Problem 5:

a)

$Basis\ step$: $n = 0$,

$f(0) = 1$ by definition, and $6 * 2^0 - 5 = 6 * 1 - 5 = 6 - 5 = 1$.

Therefore, $f(n) = 6 * 2^n - 5$ for $n = 0$.

$Induction\ step$: Assume $f(n-1) = 6 * 2^{n-1} - 5$ (the induction hypothesis).

Prove that $f(n) = 6 * 2^n - 5$.

$f(n) = 2\, f(n-1) + 5$, by definition of f

$\Longrightarrow f(n) = 2\,(6 * 2^{n-1} - 5) + 5$ using the Induction Hypothesis.

$\Longrightarrow f(n) = 6 * 2 * 2^{n-1} - 10 + 5$

$\Longrightarrow f(n) = 6 * 2^{(n-1)+1} - 5$

$\Longrightarrow f(n) = 6 * 2^n - 5$

$\therefore f(n) = 6 * 2^n - 5\ for\ all\ n \geq 0.$

b)

$Basis\ step$: $n = 0$,

$f(0) = 0\ and\ \frac{0(0+1)}{2} = 0$. Therefore, $f(n) = \frac{n(n+1)}{2}$ for $n = 0$.

$Induction\ step: Assume\ f(n-1) = \frac{(n-1)n}{2}\ \ (the\ induction\ hypothesis)\ .\ Prove\ that\ f(n) = \frac{n(n+1)}{2}$

$f(n) = 1 + 2 + 3 + \cdots + n$, by definition

$f(n) = 1 + \underbrace{2 + 3 + \cdots + (n-1)}_{} + n$

$\Longrightarrow f(n) = \quad f(n-1) \quad + n$

$\Longrightarrow f(n) = \frac{(n-1)n}{2} + n$ Using the Induction Hypothesis.

$$\Longrightarrow f(n) = \frac{(n-1)n + 2n}{2} = \frac{n^2 - n + 2n}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}$$

$$\therefore f(n) = \frac{n(n+1)}{2} \; for\ any\ n \geq 0.$$

c)

$Basis\ step$: $n = 0$,

$f(0) = 0$ by definition, and $\frac{0(0+1)(2*0+1)}{6} = \frac{0}{6} = 0$.

Therefore, $f(n) = \frac{n\,(n+1)(2n+1)}{6}$ for $n = 0$.

$Induction\ Step$: $Assume f(n-1) = \frac{(n-1)((n-1)+1)(2(n-1)+1)}{6} = \frac{(n-1)n(2n-1)}{6}$ (Induction hypothesis).

$Prove\ that\ f(n) = \frac{n(n+1)(2n+1)}{6}$.

$f(n) = 1^2 + 2^2 + 3^3 + \cdots + n^2$ by definition of f

$$\Longrightarrow f(n) = \underbrace{1^2 + 2^2 + 3^2 + \cdots + (n-1)^2}_{} + n^2$$

$$\Longrightarrow f(n) = \qquad f(n-1) \qquad + n^2$$

$\Longrightarrow f(n) = \frac{(n-1)n(2n-1)}{6} + n^2$ Using the Induction Hypothesis

$$\Longrightarrow f(n) = \frac{\big(n(2n^2 - 3n + 1)\big) + 6n^2}{6} = \frac{2n^3 - 3n^2 + n + 6n^2}{6} = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(2n^2 + 3n + 1)}{6}$$
$$= \frac{n(2n^2 + 2n + n + 1)}{6} = \frac{n\big(2n(n+1) + 1(n+1)\big)}{6} = \frac{n(n+1)(2n+1)}{6}$$

$$\therefore f(n) = \frac{n(n+1)(2n+1)}{6} \; for\ all\ n \; \geq 0.$$

d)

$Basis\ step$: $n = 0$,

$f(0) = 0$ by definition, and $\left(\frac{0(0+1)}{2}\right)^2 = \left(\frac{0}{2}\right)^2 = 0$. Therefore, $f(n) = \left(\frac{n\,(n+1)}{2}\right)^2$ for $n = 0$.

$Induction\ Step$: $Assume\ f(n-1) = \left(\frac{(n-1)n}{2}\right)^2$, the Induction hypothesis.

$Prove\ that\ f(n) = \left(\frac{n(n+1)}{2}\right)^2$.

$f(n) = 1^3 + 2^3 + 3^3 + \cdots + n^3$ by definition of f

$\Longrightarrow f(n) = \underbrace{1^3 + 2^3 + 3^3 + \cdots + (n-1)^3} + n^3$

$\Longrightarrow f(n) = \quad f(n-1) \quad + n^3$

$\Longrightarrow f(n) = \frac{n^2(n-1)^2}{4} + n^3$ Using the Induction Hypothesis

$$\Longrightarrow f(n) = \frac{((n-1)^2 n^2) + 4n^3}{4} = \frac{n^2(n^2 - 2n + 1) + 4n^3}{4} = \frac{n^4 - 2n^3 + n^2 + 4n^3}{4} = \frac{n^4 + 2n^3 + n^2}{4}$$
$$= \frac{n^2(n^2 + 2n + 1)}{4} = \frac{n^2(n+1)^2}{4} = \left(\frac{n(n+1)}{2}\right)^2$$

$\therefore f(n) = \left(\frac{n(n+1)}{2}\right)^2\ for\ all\ n\ \geq 0.$

e)

$Basis\ step$: $n = 0$,

$f(0) = 1$ by definition, and $\frac{a^{0+1}-1}{a-1} = \frac{a^1-1}{a-1} = \frac{a-1}{a-1} = 1$. Therefore, $f(n) = \frac{a^{n+1}-1}{a-1}\ for\ n = 0$.

$Induction\ Step$: $Assume\ f(n-1) = \frac{a^n-1}{a-1}, (Induction\ Hypothesis)$. Prove $f(n) = \frac{a^{n+1}-1}{a-1}$.

$f(n) = 1 + a + a^2 + \cdots + a^n$ by definition

$\Longrightarrow f(n) = \underbrace{1 + a + a^2 + \cdots + a^{(n-1)}} + a^n$

$\Longrightarrow f(n) = \quad f(n-1) \quad + a^n$

$\Longrightarrow f(n) = \frac{a^n-1}{a-1} + a^n$ *Using Induction Hypothesis.*

$$\Longrightarrow f(n) = \frac{(a^n - 1) + a^n(a-1)}{a-1} = \frac{a^n - 1 + a^{n+1} - a^n}{a-1} = \frac{a^{n+1} - 1}{a-1}$$

$$\therefore f(n) = \frac{a^{n+1} - 1}{a-1} \; for \; all \; n \geq 0.$$

f)

$Basis\ step: n = 0,$

$f(0) = 0$ by definition, and $\frac{0(0+1)(0+2)}{3} = \frac{0}{3} = 0$. Therefore, $f(n) = \frac{n\,(n+1)(n+2)}{3}\ for\ n = 0.$

$Induction\ Step: Assume\ f(n-1) = \frac{(n-1)((n-1)+1)((n-1)+2)}{3} = \frac{(n-1)n(n+1)}{3}$, (the Induction Hypothesis).

$Prove\ that\ f(n) = \frac{n(n+1)(n+2)}{3}.$

$f(n) = 1 * 2 + 2 * 3 + 3 * 4 + \cdots + n * (n+1)$ by definition

$$\Longrightarrow f(n) = \underbrace{1 * 2 + 2 * 3 + 3 * 4 + \cdots + (n-1) * n}_{} + n * (n+1)$$

$$\Longrightarrow f(n) = \qquad f(n-1) \qquad + n * (n+1)$$

$\Longrightarrow f(n) = \frac{(n-1)n(n+1)}{3} + \big(n(n+1)\big)$ Using the Induction Hypothesis.

$$\Longrightarrow f(n) = \frac{\big(n(n^2-1)\big) + 3(n^2+n)}{3} = \frac{n^3 - n + 3n^2 + 3n}{3} = \frac{n^3 + 3n^2 + 2n}{3} = \frac{n(n^2 + 3n + 2)}{3}$$
$$= \frac{\big(n(n^2 + 1n + 2n + 2)\big)}{3} = \frac{n\big(n(n+1) + 2(n+1)\big)}{3} = \frac{n(n+1)(n+2)}{3}$$

$$\therefore f(n) = \frac{n(n+1)(n+2)}{3}\ for\ all\ n\ \geq 0.$$

g)

$Basis\ step: n = 0,$ and $n = 1$

$f(0) = 0$ by definition, and $0(0+1) = 0$. Also, $f(1) = 2$ by definition, and $1(1+1) = 1 * 2 = 2$. Therefore, $f(n) = n(n+1) for\ n = 0$ and $n = 1$.

$Induction\ Step: Assume\ f(m) = m(m+1) for\ all\ m \leq n - 1, (the\ induction\ hypoethsis)$. Prove that $f(n) = n(n+1)$.

$f(n-1) = (n-1)\big((n-1)+1\big)$, by applying the induction hypothesis at m=n-1. Therefore, $f(n-1) = (n-1)n = n^2 - n$.

Similarly, by applying the induction hypothesis at m=n-2, we get

$f(n-2) = (n-2)\big((n-2)+1\big) = (n-2)(n-1) = n^2 - 3n + 2$.

$f(n) = \frac{f(n-1)+f(n-2)}{2} + (3n-1)$ by definition of f

$$\Rightarrow f(n) = \frac{(n^2-n)+(n^2-3n+2)}{2} + (3n-1) = \frac{2n^2-4n+2}{2} + 3n - 1$$
$$= (n^2-2n+1) + 3n - 1 = n^2 + n = n(n+1)$$

$\therefore f(n) = n(n+1)\ for\ all\ n \geq 0.$

Bonus Problem:

$Basis\ step: for\ n = 0, A = \emptyset.\ |A| = 0 \Rightarrow 2^{|A|} = 2^0 = 1.\ And, 2^A = \{\emptyset\} \Rightarrow |2^A| = 1.$

$\therefore\ |2^A| = 2^{|A|}\ for\ n = 0.$

$Induction\ Step: Assume\ |2^A| = 2^{|A|} = 2^{n-1}\ for\ all\ sets\ A\ of\ cardinality\ |A| = n-1.$

$Prove\ that, |2^A| = 2^{|A|} for\ a\ set\ A\ where\ |A| = n.$

$Let\ A\ be\ an\ arbitrary\ set\ where\ |A| = n. We\ can\ assume, A = \{1, 2, 3, \ldots n\}$, without loss of generality.

$$2^A = \{X \mid X \subseteq A\} = \underbrace{\{X \mid X \subseteq A\ \&\ n \notin X\}}_{B} \cup \underbrace{\{X \mid X \subseteq A\ \&\ n \in X\}}_{C}$$

$Let\ B = \{X \mid X \subseteq A\ \&\ n \notin X\}\ and\ C = \{X \mid X \subseteq A\ \&\ n \in X\}.$

$Observe\ that\ B\ and\ C\ are\ disjoint (B \cap C = \emptyset)$. Therefore, $|2^A| = |B| + |C|$.

$B = \{X \mid X \subseteq A\ \&\ n \notin X\} = \{X \mid X \subseteq \{1, 2, 3, \ldots, n-1\}\} = 2^{\{1,2,3,\ldots,n-1\}}$

$|B| = \left|2^{\{1,2,3,\ldots,n-1\}}\right| = 2^{|\{1,2,3,\ldots,n-1\}|} = 2^{n-1}, by\ the\ Induction\ Hypothesis.$

$C = \{X \mid X \subseteq A\ \&\ n \in X\}$

$Let, C' = \{X - \{n\} \mid X \in C\}.$

The elements of C are in one-to-one correspondence with the elements of C'.

$\therefore |C| = |C'|$

$C' = \{ X - \{n\} | X \in C\} = \{ X - \{n\} | X \subseteq A \,\&\, n \in X\}$

$= \{ X - \{n\} | X - \{n\} \subseteq A\}$

$= \{ X - \{n\} | X - \{n\} \subseteq \{1, 2, 3, \ldots, n-1\}\}$

$= 2^{\{1,2,3,\ldots,n-1\}} = B$

$\Longrightarrow |C'| = 2^{n-1} \Longrightarrow |C| = 2^{n-1}$

$Since, |2^A| = |B| + |C| = 2^{n-1} + 2^{n-1} = 2 * 2^{n-1} = 2^n.$

$\therefore |2^A| = 2^{|A|}$ $for\ any\ finite\ set\ A.$