CS1311 **March 21, 2016**

Midterm Solutions

Problem 1:

a)

f is one-to-one: Need to prove that $f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$.

$10\, x_1 - 30 = 10\, x_2 - 30 \Longrightarrow 10\, x_1 = 10\, x_2 \Longrightarrow x_1 = x_2$

$(adding\ 30\ on\ both\ sides\ and\ then\ dividing\ by\ 10\ on\ both\ sides)$

$\therefore f\ is\ one - to - one$

$f\ is\ onto$: need tp prove that $\forall\, y \in R, \exists\, x \in R, such\ that\ f(x) = y$.

Let $y \in R$. Tane $x = \frac{y+30}{10}$, which is a real number.

$f(x) = 10x - 30 = 10\frac{y+30}{10} - 30 = y + 30 - 30 = y$.

$\therefore\ f\ is\ onto$.

$f^{-1}(y) = \frac{y+30}{10}$.

b)

g is not one-to-one because $g(-1)$=$g(1)$=2, but $-1 \neq 1$.

g is not onto because for all $x \in R$, $g(x) > 0$, and thus for $y = -1$, there is no $x \in R$ where $g(x) = y$ because $-1 < 0$.

Problem 2:

a) $No, its\ not. Consider\ an\ example\ where\ n = 5$.

$3^n + 2 \Longrightarrow 3^5 + 2 = 245\ which\ is\ also\ divisible\ by\ 5, and\ thus\ not\ a\ prime\ number$

b)

$Basis: for\ n = 0$. Prove that $f(0) = 3^{0+1} - 1$. $f(0) = 2$ by definition. Also, $3^{0+1} - 1 = 3 - 1 = 2$. Therefore, $f(0) = 3^{0+1} - 1$.

$Induction\ step$: $Assume\ that\ the\ statement\ is\ true\ for\ n-1, i.e., we\ have\ f(n-1) = 3^n - 1.$

$We\ need\ to\ prove\ that\ \ f(n+1) = 3^{n+1} - 1.$

$f(n) = 3\,f(n-1) + 2$ by definition of f

$f(n) = 3\,(3^n - 1) + 2 \qquad (substituting\ for\ f(n-1) from\ I.H.)$

$\quad = 3 * 3^n - 3 + 2$

$\quad = 3^{n+1} - 1 \qquad Q.E.D$

$\therefore f(n) = 3^{n+1} - 1\ is\ true\ for\ all\ non-negative\ integers\ n.$

Problem 3:

a)

$General\ solution\ for\ equation\ of\ the\ form\ x_n = ax_{n-1} + f(n)\ is$

$$x_n = Aa^n + \widehat{x_n}$$

$Step\ 1$: $a = 6\ and\ \widehat{x_n} = kn + l. Plug\ kn + l\ into\ the\ recurrence\ relation\ to\ find\ k\ and\ l.$

$kn + l = 6(k(n-1) + l) + 2n - 3$

$kn + l = 6kn - 6k + 6l + 2n - 3$

$kn + l = n\,(6k + 2) - 6k + 6l - 3$

Therefore, $k = 6k + 2$ and $l = -6k + 6l - 3$.

$k = 6k + 2 \implies k = -\frac{2}{5}$

$l = -6k + 6l - 3 \implies -5l = \frac{12}{5} - 3 \implies l = \frac{3}{25}$

$\widehat{x_n} = -\frac{2}{5}n + \frac{3}{25}$

$Step\ 2$: $x_0 = A * 6^0 - \frac{2}{5} * 0 + \frac{3}{25}$

$1 = A + \frac{3}{25} \implies A = \frac{22}{25}$

$$\therefore x_n = \frac{22}{25}6^n - \frac{2}{5}n + \frac{3}{25}$$

b)

$General\ solution\ is\ of\ the\ form$: $x_n = As_1^n + Bs_2^n + \widehat{x_n}$

$Step\ 1: a = 7, b = -10, c = 4$

$Step\ 2: Characteristic\ equation: x^2 - 7x + 10 = 0, solving\ for\ roots\ of\ this\ equation:$

$x^2 - 2x - 5x + 10 = 0 \implies (x-2)(x-5) = 0.$

$\therefore roots: s_1 = 2, s = 5$

$$Step\ 3: \widehat{x_n} = \frac{c}{1 - b - a}, b - a \neq 1$$

$$\widehat{x_n} = \frac{4}{1—10} - 7 = \frac{4}{4}$$

$$\widehat{x_n} = 1$$

$Step\ 4: x_0 = A * 2^0 + B * 5^0 + \widehat{x_0} \implies A + B + 1 = 0 \implies A = -B - 1$

$x_1 = A * 2^1 + B * 5^1 + \widehat{x_1} \implies 2A + 5B + 1 = 0 \implies 2A + 5B = 0$

$Solving\ for\ A\ and\ B\ in\ the\ above\ equations:$

$\implies 2(-B - 1) + 5B = 0$

$\implies 3B - 2 = 0 \implies B = \frac{2}{3}$

$A = -B - 1 = -\frac{2}{3} - 1 = -\frac{5}{3}$

$$\therefore x_n = \left(-\frac{5}{3}\right)2^n + \left(\frac{2}{3}\right)5^n + 1$$

<u>Problem 4:</u>

a) $Order\ does\ not\ matter\ and\ repetitions\ are\ not\ allowed.$

$$12C_5 = \frac{12!}{(12-5)!\,5!} = \frac{12!}{7!\,5!}$$

b) $For\ the\ first\ place\ there\ are\ 8C_2$

$For\ the\ second\ place\ there\ are\ 6C_3$

$For\ the\ rest, there\ are\ 3!$

$$\therefore no.of\ ways = 8C_2 * 6C_3 * 3! = \frac{8!}{(8-2)!\,(2)!}\left(\frac{(6)!}{(6-3)!\,(3)!}\right)3! = \frac{8!}{2!\,3!}$$

c)

- $Let\ A\ be\ Positive\ integers\ \leq 400\ divisible\ by\ 4, A = \{\ 4k \mid 1 \leq k \leq 400/4\} \Longrightarrow |A| = 100$
- $Let\ B\ be\ positive\ integers \leq 400\ divisible\ by\ 10, B = \{\ 10l \mid 1 \leq l \leq 400/10\} \Longrightarrow |B| = 40$
- $Let\ C\ be\ the\ positive\ integers \leq 400\ divisible\ by\ 4\ and\ 10, Since\ LCM\ (4,10) = 20, we\ get,$

$$C = A \cap B = \{20m \mid 1 \leq m \leq \frac{400}{20}\} \Longrightarrow |C| = 20$$

- $Positive\ integers \leq 400\ divisible\ by\ 4\ or\ 10\ is\ |A \cup B| = |A| + |B| - |A \cap B|$

$$= 100 + 40 - 20 = 120$$

- $Positive\ integers \leq 400\ which\ are\ neither\ divisible\ by\ 4\ nor\ 10{:}\ 400 - |A \cup B|$

$$= 400 - 120 = 280$$