CS 1311 March 22, 2016

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Homework 5

Due Date: April 7, 2016

Problem 1: (20 points)

Let G = (V,E) be the following undirected graph: V = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, and E = {(1,5) , (1,3) , (7,1) , (5,7) , (3,7) , (3,8) , (8,7) , (9,7) , (7,2) , (4,9), (3,5) , (2,6) , (6,4), (5,10), (10,1)}.

a) Draw G, and give the adjacency matrix A of G.
b) Do a depth-first search (DFS) and a breadth-first search (BFS) on G, starting from node 1, and show the DFS tree and BFS tree. Tie-breaking is by choosing the smallest node.
c) Is G connected? How can you tell?
d) An *articulation point* (or *single point of failure*) of a connected graph is a node such that the removal of that nodes makes the graph disconnected. Does G have an articulation point? If so, which one(s)?
e) Does G have an Eulerian cycle? If yes, show such a cycle; if no, prove your answer.
f) Does G have a Hamiltonian cycle? If yes, show such a cycle; if no, prove your answer.

Problem 2: (20 points)

Take the undirected graph G of Problem 1.

a) The distance d(x,y) between two nodes x and y is the length of the shortest path between x and y. Recall that the length of a path is the number of the edges in the path. Find the distance between node 1 and each node in the graph. You can use the BFS tree to help you compute those distances.
b) The *diameter* of a graph is the largest distance in that graph. Give the diameter of G?
c) The *radius* of the graph relative to a node *u* is: *maximum*{distance(*u*,*y*) | for all nodes *y* in G}. For each node *u* in G, compute the radius of G with respect to *u*.
d) The *center* of a graph is a node *u* where the radius of G relative to *u* is the smallest. Note that there can be ties, in which case the graph has multiple centers. Find the center or centers of G.

Problem 3: (20 points)

Let G be the same graph given in Problem 1 except that it is now directed.

a) Draw G and give its adjacency matrix A.
b) Compute the indegree and outdegree of each node.
c) For every node, list all the nodes reachable from it. Is G strongly connected?
d) Can G be isomorphic to a graph G' where the outdegree of one of the nodes (in G') is 5? Why or why not?
e) Can G be isomorphic to a graph G' that is not strongly connected? Why or why not?

Problem 4: (20 points)

Let T be a general tree of n nodes and e edges. Recall that T is a connected acyclic graph (T has no cycles).

a) A perfect binary tree is a binary tree where every non-leaf has two children and all the leaves are at the bottom level. Prove that the number of nodes in a perfect tree of k levels is 2^k-1.
b) A *2-3 tree* is a rooted where all the leaves are at the same level, and every internal node has 2 or 3 children. Prove that in a 2-3 tree of *n* nodes and *k* levels, we must have

$$2^k - 1 \leq n \leq \frac{3^k - 1}{2}.$$

Problem 5: (20 points)
In each of the following cases, draw a tree with the given specifications or explain why no such tree exists:

a) A perfect binary tree of height 3
b) A perfect binary tree of 10 nodes
c) A 2-3 tree of 7 nodes
d) A 2-3 tree of 3 levels and 15 nodes
e) A tree of 10 nodes and 12 edges
f) A rooted tree of 10 nodes where there are at least 3 internal nodes and every internal node has at least 3 children.
g) A rooted tree of 10 nodes where there are at least 3 internal nodes and every internal node has at least 4 children.

Bonus problem: (5 points)
A full binary tree is a binary tree where every node is either a leaf or has two children. Prove by induction on *k* that in a full binary tree with *k* internal nodes, the number of leaves is *k*+1 and the total number of nodes is 2*k*+1.