CS 1311 January 28, 2016

Youssef

Homework 2

Due Date: February 16, 2016

Problem 1: (20 points)

Let p, q, and r be 3 propositions. Give the truth tables of the following propositions:

a) $p \wedge (\neg q \vee \neg r)$
b) $p \vee \neg(q \wedge r)$
c) $p \vee \neg(q \wedge \neg r)$
d) $(p \wedge \neg q) \wedge r$

Problem 2: (25 points)

Let A, B and C be three arbitrary sets. For each of the following statements, indicate if the statement is true or false, and prove your answer.

a) $A - (B \cup C) = (A - B) \cap (A - C)$
b) $(A \cap B) - C = A \cap (B - C)$
c) $A - (B+C) = (A - B) - C$
d) $A - (B - C) = (A - B) + C$
e) $A - (B - C) = A \cap (C - B)$
f) $2^{A \cap B} = 2^A \cap 2^B$

Problem 3: (15 points)

Find counterexamples to the following statements:

a) $3^n + 1$ is prime for every integer $n \geq 0$
b) For every two sets A and B, $2^{A+B} = 2^A + 2^B$
c) For every three sets A, B and C, $A+(B - C)=(A+B) - C$

Problem 4: (20 points)

Let **R** be the set of real numbers, $\mathbf{R}^+$ the set of non-negative real numbers, and **Z** the set of integers. Let also $f : \mathbf{R} \rightarrow \mathbf{R}$, $g: \mathbf{R} \rightarrow \mathbf{R}^+$ and $h: \mathbf{R}^+ \rightarrow \mathbf{R}$ be 3 functions defined as follows:

$$f(x)= 5x+9,\ g(x)= (x - 1)^2,\ h(x)= 2x - 5.$$

a) Prove that f one-to-one and onto, and find f^{-1}.
b) Is g one-to-one? Onto? Prove your answer.
c) Is h one-to-one? Onto? Prove your answer.
d) Calculate $h \circ g(x)$, $g \circ h(x)$, $(f \circ h) \circ g(x)$, *and* $f \circ (h \circ g)(x)$.
e) Given two sets E and F, a function $u : E \rightarrow F$, and an element $y \in F$, define $u^{\leftarrow}(y)$ to be the following set: $u^{\leftarrow}(y) = \{x \in E \mid u(x) = y\}$. Determine $f^{\leftarrow}(1)$, $g^{\leftarrow}(9)$, $g^{\leftarrow}(0)$, $h^{\leftarrow}(9)$.

Problem 5: (20 points)

Let **N** be the set of natural numbers, and **R** the set of real numbers. Let $f: \mathbf{N} \rightarrow \mathbf{R}$ be a function.

a) If $f(0) = 1$ and $f(n) = 2f(n - 1) + 5$ for all $n \geq 1$, prove by induction on n that
$$f(n) = 6 \times 2^n - 5$$
b) If $f(0) = 0$ and $f(n) = 1 + 2 + 3 ... + n$ for all $n \geq 1$, prove by induction on n that
$$f(n) = \frac{n(n+1)}{2} \text{ for all } n \geq 0.$$
c) If $f(0) = 0$ and $f(n) = 1^2 + 2^2 + 3^2 ... + n^2$ for all $n \geq 1$, prove by induction on n that
$$f(n) = \frac{n(n+1)(2n+1)}{6} \text{ for all } n \geq 0.$$
d) If $f(0) = 0$ and $f(n) = 1^3 + 2^3 + 3^3 ... + n^3$ for all $n \geq 1$, prove by induction on n that $f(n) = [\frac{n(n+1)}{2}]^2$ for all $n \geq 0$.

e) Let a be a real number $\neq 1$, and let $f(n) = 1 + a + a^2 + a^3 + \cdots + a^n$ for all $n \geq 0$. Prove by induction on n that

$$f(n) = \frac{a^{n+1}-1}{a-1} \text{ for all } n \geq 0.$$

f) If $f(0) = 0$ and $f(n) = 1 \times 2 + 2 \times 3 + 3 \times 4 \ldots + n \times (n+1)$ for all $n \geq 1$, prove by induction on n that

$$f(n) = \frac{n(n+1)(n+2)}{3} \text{ for all } n \geq 0.$$

g) If $f(0) = 0$, $f(1) = 2$ and $f(n) = \frac{f(n-1)+f(n-2)}{2} + 3n - 1$ for all $n \geq 2$, prove by induction on n that

$$f(n) = n(n+1) \text{ for all } n \geq 0.$$

Bonus Problem: (5 points)

Let A be a finite set. Prove by induction on the cardinality of A that $|2^A| = 2^{|A|}$.