A Lie Group Variational Integrator for the Attitude dynamics of a Rigid body

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Motivation

- Dynamics and Control of a System
  - The motion of a system is described by a *Differential Equation*.
  - Geometric feature is crucial for a dynamics and control problem.
  - The analytic solution of a differential equation is hard to find.

- Numerical Integration
  - The numerical solution is used to study dynamics of the system, to design/verify a controller.
  - There has been little attention given to computational approaches.
  - Generally, numerical integration methods DO NOT preserve the characteristics of the system.

Geometric Numerical Integration
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**Geometric Numerical Integration**
Motivation

Numerical Example I: Two Body Problem

- Exact flow of the Two Body Problem
  - Normalized Equation of motion
    \[
    \ddot{q}_1 = -\frac{q_1}{(q_1^2 + q_2^2)^{3/2}}, \quad \ddot{q}_2 = -\frac{q_2}{(q_1^2 + q_2^2)^{3/2}}.
    \]
    \[
    \left( q_{10} = 1 - e, \; q_{20} = 0, \; \dot{q}_{10} = 0, \; \dot{q}_{20} = \sqrt{\frac{1 + e}{1 - e}} \right)
    \]
  - The trajectory is a conic section.
  - The total energy and the angular momentum are conserved.
    \[
    H = \frac{1}{2} (\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{(q_1^2 + q_2^2)^{1/2}}
    \]
    \[
    h = q_1\dot{q}_2 - q_2\dot{q}_1
    \]
Motivation
Numerical Example I: Two Body Problem

- Numerical integration result
  - Variational step-size Runge-Kutta method (Matlab `ode45.m`)

The trajectory is not closed orbit. ($e = 0.6$)
- The total energy and the angular momentum are not conserved.
Motivation
Numerical Example II: Pendulum

- Exact flow of a Pendulum
  - Equation of motion
    \[ \ddot{q} = - \sin q \]
  - The total energy is conserved.
    \[ H = \frac{1}{2} \dot{q}^2 - \cos q \]
  - The flow is symplectic.
    (Area conservation in phase plane)
Motivation
Numerical Example II: Pendulum

- Numerical integration result
  - Variational step-size Runge-Kutta method (Matlab `ode45.m`)

The position and velocity errors increase as $t$.
The total energy is not conserved.
Motivation
Numerical Example II: Pendulum

- Numerical integration result
  - The numerical flow is not symplectic.
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Numerical Example II: Pendulum

- Numerical integration result
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Geometric Numerical Integration

- **Energy Momentum method** [LaBudde 1976], [Simo 1992]
  - The parameters of numerical integration is chosen such that energy and momentum are conserved.
  - The states are projected to the constant energy momentum surface.

- **Variational Integrator** [Moser 1991], [Marsden 2001]
  - A systematic method of constructing structure-preserving integrator
  - It approximates the continuous dynamics by discretizing Hamilton’s principle.
  - It inherits the properties of the continuous dynamics.

- **Lie group method** [Iserles 2000]
  - Many practical differential equations evolve on a Lie group.
  - It preserves the geometry of the configuration space.
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Problem Definition

Attitude Dynamics of a Rigid body under a Potential depends on the Attitude

- **Assumption**
  - The $O - e_1 e_2 e_3$ is an inertial frame.
  - $V : SO(3) \hookrightarrow \mathbb{R}$

- **Geometric Feature**
  - The configuration space is $SO(3)$.
  - The Hamiltonian is conserved.
  - Other first integral may exist according to the symmetries of $V$.

- **Goal**
  - A numerical integrator that respects the geometric features.
  - The integrator updates $\omega, R$. 

![Diagram of attitude dynamics](image-url)
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Variational Principle
The basic idea of the variational integrator

**Continuous time**

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<th>((q, \dot{q}) \in TQ)</th>
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Two equivalent forms of Lagrangian

\[ L(R, \omega) = \frac{1}{2} \int_{\text{Body}} \|\omega \times \tilde{\rho}\|^2 \, dm - V(R). \]

Using \( \|\omega \times \tilde{\rho}\|^2 = \|S(\tilde{\rho})\omega\|^2 \),

\[ L(R, \omega) = \frac{1}{2} \omega^T \left( \int_{\text{Body}} S(\tilde{\rho})^T S(\tilde{\rho}) \, dm \right) \omega - V(R), \]

\[ \overset{\Delta}{=} \frac{1}{2} \omega^T J \omega - V(R). \quad (1) \]

Using \( \|\omega \times \tilde{\rho}\|^2 = \|S(\omega)\tilde{\rho}\|^2 \),

\[ L(R, \omega) = \frac{1}{2} \int_{\text{Body}} \text{tr} [S(\omega)\tilde{\rho}\tilde{\rho}^T S(\omega)^T] \, dm - V(R), \]

\[ = \frac{1}{2} \text{tr} \left[ S(\omega) \left( \int_{\text{Body}} \tilde{\rho}\tilde{\rho}^T \, dm \right) S(\omega)^T \right] - V(R), \]

\[ \overset{\Delta}{=} \frac{1}{2} \text{tr} [S(\omega)J_d S(\omega)^T] - V(R), \quad (2) \]
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\[ \triangleq \frac{1}{2} \text{tr} \left[ S(\omega) J_d S(\omega)^T \right] - V(R), \] (2)
Lagrangian

- **Relationship between $J$ and $J_d$**

  Using the relationship, $S(\tilde{\rho})^T S(\tilde{\rho}) = \tilde{\rho}^T \tilde{\rho} I_{3 \times 3} - \tilde{\rho} \tilde{\rho}^T$, $J$ and $J_d$ are related as

  $$J = \left( \int_{\text{Body}} \tilde{\rho}^T \tilde{\rho} dm \right) I_{3 \times 3} - J_d$$

  $$= \text{tr}[J_d] I_{3 \times 3} - J_d.$$

  If we express $\tilde{\rho}$ in coordinates $(x, y, z)$, $J$ and $J_d$ are expressed as

  $$J = \begin{bmatrix} \int y^2 + z^2 dm & -\int xy dm & -\int xz dm \\ -\int xy dm & \int z^2 + x^2 dm & -\int yz dm \\ -\int xz dm & -\int yz dm & \int x^2 + y^2 dm \end{bmatrix}, \quad J_d = \begin{bmatrix} \int x^2 dm & \int xy dm & \int xz dm \\ \int xy dm & \int y^2 dm & \int yz dm \\ \int xz dm & \int yz dm & \int z^2 dm \end{bmatrix}.$$

  Furthermore, the following equation is satisfied for $\omega \in \mathbb{R}^3$.

  $$S(J\omega) = S(\omega)J_d + J_d S(\omega). \quad (3)$$
Variation

- **Action Integral**
  \[
  \mathcal{S} = \int_{t_0}^{t_f} L(R, \omega) dt = \int_{t_0}^{t_f} \frac{1}{2} \text{tr} \left[ S(\omega) J_d S(\omega)^T \right] - V(R) dt.
  \]  

- **Variation of** \(R, \omega\)

  The variation should respect the geometry of the configuration space.

  The varied rotation matrix \(R^\epsilon \in SO(3)\) can be expressed as

  \[
  R^\epsilon = R e^{\epsilon \eta},
  \]  

  where \(\epsilon \in \mathbb{R}, \eta \in so(3)\)

  Using the kinematic relationship \(\dot{R} = RS(\omega)\), the variation of \(\omega\) is induced from \(R^\epsilon\).

  \[
  S(\omega^\epsilon) = R^{\epsilon T} \dot{R}^\epsilon = e^{-\epsilon \eta} S(\omega) e^{\epsilon \eta} + \epsilon \dot{\eta},
  \]

  \[
  = S(\omega) + \epsilon \left[ \dot{\eta} + S(\omega) \eta - \eta S(\omega) \right] + \mathcal{O}(\epsilon^2).
  \]
Equation of Motion

- Variation of Action Integral

\[ \delta \mathcal{G} = \frac{d}{d \epsilon} \mathcal{G}^\epsilon \bigg|_{\epsilon = 0} = \int_{t_0}^{t_f} \frac{1}{2} \text{tr} \left[ \eta \left\{ S(\dot{J} \omega + \omega \times J \omega) + 2R^T \frac{\partial V}{\partial R} \right\} \right] dt. \]

- Continuous Equation of motion

From Hamilton’s principle, \( \delta \mathcal{G} = 0 \). Then, we obtain

\[ S(\dot{J} \omega + \omega \times J \omega) = \frac{\partial V^T}{\partial R} R - R^T \frac{\partial V}{\partial R}, \]

or equivalently,

\[ \dot{J} \omega + \omega \times J \omega = M, \]

where \( M \in \mathbb{R}^3 \) is determined by

\[ S(M) = \frac{\partial V^T}{\partial R} R - R^T \frac{\partial V}{\partial R}. \]

More explicitly, it can be shown that

\[ M = r_1 \times v_{r_1} + r_2 \times v_{r_2} + r_3 \times v_{r_3}, \]

where \( r_i, v_{r_i} \in \mathbb{R}^{1 \times 3} \) are the \( i \)th row vectors of \( R \) and \( \frac{\partial V}{\partial R} \), respectively.
### Variational Principle

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\(\mathcal{G} = \int_0^f L(q, \dot{q}) \, dt\)

\(\delta \mathcal{G} = \frac{d}{de} \mathcal{G}^e = 0\)

\(\dot{q} = f(q, \dot{q})\)
Discrete Lagrangian

- **Discrete Lagrangian**
  Define new variable $F_k \in SO(3)$ such that $R_{k+1} = R_k F_k$.
  Using the kinematic relationship $\dot{R} = RS(\omega)$, $S(\omega_k)$ can be approximated as

  $$S(\omega_k) = R_k^T \dot{R}_k \approx R_k^T \left( \frac{R_{k+1} - R_k}{h} \right) = \frac{1}{h} (F_k - I_{3 \times 3}).$$  

- **Action Sum**
  Since the discrete Lagrangian $L_d$ approximates $\int_{tk}^{tk+1} L(R, \omega) \, dt$,

  $$\mathcal{G}_d = \sum_{k=0}^{N-1} L_d(R_k, F_k) = \sum_{k=0}^{N-1} \frac{1}{h} \text{tr}[(I_{3 \times 3} - F_k) J_d] - \frac{h}{2} V(R_k) - \frac{h}{2} V(R_{k+1}).$$
Variation of $R_k, F_k$

The varied rotation matrix $R_k^\epsilon \in \text{SO}(3)$ can be expressed as

$$R_k^\epsilon = R_k e^{\epsilon \eta_k},$$

(12)

where $\epsilon \in \mathbb{R}$, $\eta_k \in \mathfrak{so}(3)$.

Using the definition of $F_k$, the variation of $F_k$ is induced as

$$F_k^\epsilon = R_k^\epsilon T R_{k+1}^\epsilon = e^{-\epsilon \eta_k} R_k^T R_{k+1} e^{\epsilon \eta_{k+1}} = e^{-\epsilon \eta_k} F_k e^{\epsilon \eta_{k+1}}.$$

(13)

Variation of Action Sum

$$\frac{d}{d\epsilon} \mathcal{G}_d^\epsilon \bigg|_{\epsilon=0} = \sum_{k=0}^{N-1} \frac{1}{h} \text{tr}[(\eta_k F_k - F_k \eta_{k+1}) J_d] + \frac{h}{2} \text{tr} \left[ \eta_k R_k^T \frac{\partial V}{\partial R_k} + \eta_{k+1} R_{k+1}^T \frac{\partial V}{\partial R_{k+1}} \right],$$

$$= \sum_{k=1}^{N-1} \text{tr} \left[ \eta_k \left\{ \frac{1}{h} (F_k J_d - J_d F_{k-1}) + h R_k^T \frac{\partial V}{\partial R_k} \right\} \right].$$

(14)
Discrete Equation of Motion

- Discrete Equation of Motion in $R_k, F_k$
  From Hamilton’s principle, $\delta \mathcal{G}_d = 0$. Then, we obtain
  \[
  \frac{1}{h} \left( F_{k+1} J_d - J_d F_k - J_d F_{k+1}^T + F_k^T J_d \right) = h S(M_{k+1}),
  \]
  \[
  R_{k+1} = R_k F_k. 
  \]
  This yields a discrete map $(R_0, F_0) \mapsto (R_1, F_1)$ and the process can be repeated.

- Discrete Equation of Motion in $R_k, \omega_k$
  Inertial and body-fixed angular momentum are related as
  \[
  S(J\omega) = S(\omega) J_d + J_d S(\omega),
  \]
  \[
  S(RJ\omega) = RS(J\omega) R^T = \dot{R}J_d R^T - RJ_d \dot{R}^T.
  \]
  If we approximate $\dot{R}_k = \frac{R_{k+1} - R_k}{h}$, we obtain
  \[
  S(R_k J\omega_k) = \frac{1}{h} \left( R_{k+1} J_d R_k^T - R_k J_d R_{k+1}^T \right),
  \]
  \[
  S(J\omega_k) = R_k^T S(R_k J\omega_k) R_k = \frac{1}{h} \left( F_k J_d - J_d F_k^T \right). 
  \]
Discrete Equation of Motion
Variational Integrator for the Attitude dynamics of a Rigid Body

- Discrete Equation of Motion in $R_k, \omega_k$

Using the above equations, the *discrete equations of motion* can be expressed as

$$J\omega_{k+1} = F_k^T J\omega_k + hM_{k+1}, \quad (18)$$

$$S(J\omega_k) = \frac{1}{h} \left( F_k J_d - J_d F_k^T \right), \quad (19)$$

$$R_{k+1} = R_k F_k, \quad (20)$$

where $M_k \in \mathbb{R}^3$ is determined by $S(M_k) = \frac{\partial V}{\partial R_k}^T R_k - R_k^T \frac{\partial V}{\partial R_k}$. More explicitly,

$$M_k = r_1 \times v_{r_1} + r_2 \times v_{r_2} + r_3 \times v_{r_3}, \quad (21)$$

where $r_i, v_{r_i} \in \mathbb{R}^{1 \times 3}$ are the $i$th row vectors of $R_k$ and $\frac{\partial V}{\partial R_k}$, respectively.

Given $R_0$, and $\omega_0$, we can obtain $F_0$ implicitly by solving (19), $R_1$ is updated by (20), and the angular velocity $\omega_1$ is updated by (18). This yields a map $(R_0, \omega_0) \mapsto (R_1, \omega_1)$ and the process is repeated.
Discrete Equation of Motion
Properties of the Variational Integrator

- Preservation of conserved quantities
  - *Noether’s theorem*
    Symmetries in the Lagrangian result in the conservation of the associated momentum map.
  - Variational integrators exhibit a discrete analogue of *Noether’s theorem*.
    All the conserved momenta in the continuous dynamics are preserved in the discrete dynamics.

- Preservation of the configuration space
  - The state is automatically evolves on the Rotation group.
    Since $\text{SO}(3)$ is closed under multiplication, the attitude matrix $R_k$ remains on $\text{SO}(3)$. ($R_{k+1} = R_k F_k$)
  - The attitude is globally defined without any projection or constrains.
Discrete Equation of Motion

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Solution of \( hS(J\omega) = FJ_d - J_dF^T \).

Using the Rodrigues’ formula,

\[
F = e^{S(f)} = I_{3 \times 3} + \frac{\sin \|f\|}{\|f\|} S(f) + \frac{1 - \cos \|f\|}{\|f\|^2} S(f)^2,
\]

where \( f \) is a vector in \( \mathbb{R}^3 \). Then the matrix equation can be transformed into

\[
hJ\omega = \frac{\sin \|f\|}{\|f\|} Jf + \frac{1 - \cos \|f\|}{\|f\|^2} f \times Jf \triangleq G(f).
\]  \( 22 \)

Newton iteration method gives

\[
f_{i+1} = f_i + \nabla G(f_i)^{-1} (hJ\omega - G(f_i)),
\]

where \( \nabla G(f) \) can be expressed as

\[
\nabla G(f) = \frac{\cos \|f\| \|f\| - \sin \|f\|}{\|f\|^3} J f f^T + \frac{\sin \|f\| \|f\| - 2(1 - \cos \|f\|)}{\|f\|^4} (f \times Jf) f^T
\]

\[
+ \frac{\sin \|f\|}{\|f\|} J + \frac{1 - \cos \|f\|}{\|f\|^2} \{ -S(Jf) + S(f)J \}.
\]
3D Pendulum Model

- Uniform gravity potential
  \[ V = -mge_3^TR\rho, \quad \frac{\partial V}{\partial R} = -mge_3\rho^T. \]

- Equation of motion
  Since \( M = r_3 \times v_{r3} = -e_3^TR \times mg\rho^T, \)
  \[ J\omega_{k+1} = F_k^TJ\omega_k + hmg\rho \times R_{k+1}^Te_3. \]

- Conserved quantities
  Since the Lagrangian is independent of time, and
  is invariant under the inertial rotation about \( e_3 \).
  \[ H = \frac{1}{2}\omega^TJ\omega - mge_3^TR\rho, \quad \pi_3 = e_3^TRJ\omega. \]
  are conserved. The equation of motion can be
  written as \( R_{k+1}J\omega_{k+1} - R_kJ\omega_k = hmgR_{k+1}\rho \times e_3, \)
  which shows the conservation of \( \pi_3 \) directly.
Simulation Parameters

- Mass properties
  \[ J = \text{diag} [1, 2.8, 2] \text{kg m}^2, \ m = 1 \text{kg}, \ \rho = [0, 0, 1] \text{m}. \]

- Initial conditions
  1. Small perturbation from the hanging equilibrium
     \[ \omega_0 = [0.5, -0.5, 0.4] \text{rad/s}, \ R_0 = I_{3\times3}. \]
  2. Small perturbation from the inverted equilibrium
     \[ \omega_0 = [0.5, -0.5, 0.4] \text{rad/s}, \ R_0 = \text{diag} [-1, 1, -1]. \]
  3. Perturbation from an initially horizontal position
     \[ \omega_0 = [2.5, 2.5, 0] \text{rad/s}, \ R_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}. \]
Simulation Result
(i) Small perturbation from the hanging equilibrium

Angular velocity
\( h = 10^{-3} \text{(sec)}, \) Variational integrator \( \text{, Runge-Kutta } \)

Conserved Quantities

\[
\begin{align*}
H(t) &\approx -9.174 \\
\pi_3 &\approx 0.8005 \\
\|I-RTR\| &\approx 4 \times 10^{-4}
\end{align*}
\]
Simulation Result
(ii) Small perturbation from the inverted equilibrium

Angular velocity
\( (h = 10^{-5} \text{ (sec)}, \text{ Variational integrator } \quad \text{, Runge-Kutta } \quad ) \)
Simulation Result

(iii) Perturbation from an initially horizontal position

Angular velocity
\( (h = 10^{-5} \text{ (sec)}, \text{ Variational integrator } \quad \text{, Runge-Kutta }) \)

Conserved Quantities
Simulation Result
Chaotic Motion: (iii) Perturbation from an initially horizontal position
Variational Integrator for the Attitude dynamics of a Rigid body with the Potential depends on the Attitude

- The integrator is obtained from a discrete variational principle.
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- It adopts the approach of Lie group integrators.
- The discrete state evolves on a rotation group automatically without any reprojection and constraints.
- The general idea can be applied to other Lie groups.

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- The Lie group variational integrator is used to study the dynamics of the 3D Pendulum.
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