Abstract—This paper studies formation control of an arbitrary number of spacecraft based on a serial network structure. The leader controls its absolute position and absolute attitude with respect to an inertial frame, and the followers control its relative position and attitude with respect to another spacecraft assigned by the serial network. The unique feature is that both the absolute attitude and the relative attitude control systems are developed directly in terms of the line-of-sight observations between spacecraft, without need for estimating the full absolute and relative attitudes, to improve accuracy and efficiency. Control systems are developed on the nonlinear configuration manifold, guaranteeing exponential stability. Numerical examples are presented to illustrate the desirable properties of the proposed control system.

I. INTRODUCTION

Spacecraft formation flight has been intensively studied as distributing tasks over a group of low-cost spacecraft is more efficient and robust than operating a single large and powerful spacecraft [1]. For cooperative spacecraft missions, precise control of relative configurations among spacecraft is critical for success. For interferometer missions like Darwin, spacecraft in formation should maintain specific relative position and relative attitude configurations precisely. Precise relative position control and estimation have been addressed successfully, for example, by utilizing carrier-phase differential GPS [2], [3].

For relative attitude control, there have been various approaches, including leader-follower strategy [4], [5], behavior-based controls [6], [7] and virtual structures [8], [9]. These approaches have distinct features, but there is a common framework: the absolute attitude of each spacecraft in formation is determined independently and individually by using an on-board sensor, such as inertial measurement units and star trackers, and they are transmitted to other spacecraft to determine relative attitude between them. As the relative attitudes are determined indirectly by comparing absolute attitudes, there is a fundamental limitation in accuracies. More explicitly, measurement and estimation errors of multiple sensors are accumulated in determination of the relative attitudes.

Vision-based sensors have been widely applied for navigation of autonomous vehicles, and recently, they are proposed for determination of relative attitudes. It is shown that the line-of-sight (LOS) measurements between two spacecraft determine the relative attitude between them completely, and based on it, an extended Kalman filter is developed [10], [11]. Recently, these are also utilized in stabilization of relative attitude between two spacecraft [12], and tracking control of relative attitude formation between multiple spacecraft [13], [14], where control inputs are directly expressed in terms of line-of-sight measurements, without need for constructing the full, absolute attitude or the relative attitudes.

However, these prior results are restrictive in the sense that the relative positions among spacecraft, and therefore the lines-of-sight with respect to the inertial frame, are assumed to be fixed during the whole attitude maneuvers. Therefore, they cannot be applied to the cases where both the relative positions and the relative attitudes should be controlled concurrently at the similar time scale.

The objective of this paper is to eliminate such restrictions. In this paper, the translational dynamics and the rotational dynamics of each spacecraft are considered, and a serial network structure is defined. The first spacecraft at the network, namely the leader controls its absolute position and absolute attitude with respect to an inertial frame, and the remaining spacecraft, namely followers control its relative position and relative attitude with respect to another spacecraft ahead in the serial chain of network. The main contribution is that both the absolute attitude controller of the leader, and the relative attitude controller of the followers are defined directly in terms of the line-of-sight measurements, and exponential stability is guaranteed without the restrictive assumption that the relative positions are fixed.

Therefore, the control system proposed in this paper inherits the desirable features of the relative attitude controls based on the lines-of-sight [12], [13], [14], namely low cost and long-term stability, while requiring no corrections in measurements as opposed to gyros, or eliminating needs for computationally expensive star tracking algorithms. But, it can be applied to more realistic cases where the relative positions are controlled simultaneously. Another distinct feature of this paper is that the absolute attitude of the leader is also controlled, whereas the preliminary works are only focused on relative attitude formation control [12]. All of these are constructed on the special orthogonal group to avoid singularities, complexities associated with local parameterizations or ambiguities of quaternions in representing attitudes.

II. PROBLEM FORMULATION

Consider \( n \) spacecraft, where each spacecraft is modeled as a rigid body. Define an inertial frame, and the body-fixed frame for each spacecraft. The configuration of the
i-th spacecraft is defined by \((R_i, x_i) \in SE(3)\), where the special Euclidean group \(SE(3)\) is the semi-direct product of the special orthogonal group \(SO(3) = \{ R \in \mathbb{R}^{3 \times 3} \mid R^T R = I, \ \det(R) = 1 \}\), and \(\mathbb{R}^3\). The rotation matrix \(R_i \in SO(3)\) represents the linear transformation of the representation of a vector from the \(i\)-th body fixed frame to the inertial frame, and the vector \(x_i \in \mathbb{R}^3\) denote the location of the mass center of the \(i\)-th spacecraft with respect to the inertial frame.

A. Dynamic Model

Let \(m_i \in \mathbb{R}\) and \(J_i \in \mathbb{R}^{3 \times 3}\) be the mass and the inertia matrix of the \(i\)-th spacecraft. The equations of motion are given by

\[
\begin{align*}
    m_i \ddot{x}_i &= f_i, \\
    \dot{x}_i &= v_i, \\
    J_i \dot{\Omega}_i + \Omega_i \times J_i \dot{\Omega}_i &= u_i, \\
    \dot{R}_i &= R_i \dot{\Omega}_i,
\end{align*}
\]

where \(v_i, \Omega_i \in \mathbb{R}^3\) are the translational velocity and the rotational angular velocity of the \(i\)-th spacecraft, respectively. The control force acting on the \(i\)-th spacecraft is denoted by \(f_i \in \mathbb{R}^3\), and the \(i\)-th control moment is denoted by \(M_i \in \mathbb{R}^3\). The vectors \(v_i, \dot{\Omega}_i\) are represented with respect to the inertial frame, and \(\Omega_i, M_i\) are represented with respect to the \(i\)-th body-fixed frame.

The hat map \(\hat{\cdot} : \mathbb{R}^3 \rightarrow so(3)\) transforms a vector in \(\mathbb{R}^3\) to a \(3 \times 3\) skew-symmetric matrix such that \(\hat{x} y = (x) y = x \times y\) for any \(x, y \in \mathbb{R}^3\). The inverse of the hat map is denoted by the vee map \(\vee : so(3) \rightarrow \mathbb{R}^3\). A few properties of the hat map are summarized as follows:

\[
\begin{align*}
    \hat{x} \times y &= \hat{x} y - \hat{y} x = yx^T - xy^T, \\
    \text{tr} [\hat{x} A] &= \frac{1}{2} [\text{tr} [\hat{x} (A - A^T)]] = -x^T (A - A^T)^\vee, \\
    \hat{x} A + A^T \hat{x} &= (\{\text{tr}[A]I_{3 \times 3} - A\} x)^\vee, \\
    R \hat{x} R^T &= (R x)^\vee, \\
    \exp(\hat{R} y) &= R \exp(\hat{y}) R^T,
\end{align*}
\]

for any \(x, y \in \mathbb{R}^3\), \(A \in \mathbb{R}^{3 \times 3}\), and \(R \in SO(3)\).

Suppose that \(n\) spacecraft are serially connected by daisy-chaining. For notational convenience, it is assumed that spacecraft indices are ordered along the serial network. The first spacecraft of the serial chain, namely Spacecraft 1 is selected to be the leader, and its absolute position and its absolute attitude are controlled with respect to the inertial frame. The remaining spacecraft are considered as followers, where each follower controls its relative position and its relative attitude with respect to the spacecraft that is one step ahead in the serial chain. For example, Spacecraft 2 controls its relative configuration with respect to Spacecraft 1.

It can be shown that the controller structure of each of followers are identical. To make the subsequent derivations more concrete and concise, the control systems are defined for the leader, Spacecraft 1, and the first follower, Spacecraft 2, only. Later, it is generalized for control systems of the remaining spacecraft.

Each spacecraft is assumed to measure the line-of-sight toward two assigned objects via vision-based sensors to control its attitude. Line-of-sight measurements for the leader and the followers are described as follows.

B. Line-of-Sight Measurements of Leader

Suppose that there are two distinct objects, namely \(A, B\), such as distant stars, whose locations in the inertial reference frame are available. Let \(s_A, s_B \in S^2 = \{ q \in \mathbb{R}^3 \mid \|q\| = 1 \}\) be the unit-vectors representing the directions from Spacecraft 1 to the object A and to the object B, respectively. The vectors \(s_A, s_B\) are expressed with respect to the inertial frame, and we have \(s_A \times s_B \neq 0\), as they are distinct objects. The time-derivatives of these unit-vectors are given by

\[
\begin{align*}
    \dot{s}_A &= \mu_A \times s_A, \\
    \dot{s}_B &= \mu_B \times s_B, \quad (10)
\end{align*}
\]

where \(\mu_A, \mu_B \in \mathbb{R}^3\) are angular velocities of \(s_A, s_B\), respectively, that can be determined explicitly by the relative translational motion of the leader with respect to the objects \(A, B\).

Assume the leader is equipped with a vision-based sensor that can measure the directions toward the objects \(A\) and \(B\). These two line-of-sight measurements are expressed with respect to the first body-fixed frame, and they are defined as \(b_A, b_B \in S^2\). Note that \(s_A, s_B\) and \(\hat{b}_A, \hat{b}_B\) are related by the rotation matrix, i.e.,

\[
\begin{align*}
    s_j &= R_1 b_j \text{ or equivalently, } b_j = R_1^T s_j, \quad (11)
\end{align*}
\]

for \(j \in \{A, B\}\). From (10) and (11), we can obtain the kinematic equations for \(b_A, b_B\) as

\[
\begin{align*}
    \dot{b}_j &= \hat{R}_1^T s_j + R_1^T \dot{s}_j = -\dot{\Omega}_1 R_1^T s_j + R_1^T \dot{\mu}_j s_j \\
    &= -\Omega_1 b_j + R_1^T \dot{\mu}_j R_1 R_1^T s_j = b_j \times (\Omega_1 - R_1^T \mu_j), \quad (12)
\end{align*}
\]

for \(j \in \{A, B\}\).
C. Line-of-Sight Measurements of Follower

Spacecraft 2 controls its relative attitude and its relative position with respect to Spacecraft 1. The relative configurations are defined as follows. The relative attitude of Spacecraft 2 with respect to Spacecraft 1 is represented by a rotation matrix $Q_{21} \in SO(3)$, given by

$$Q_{21} = R_{12}^T R_2.$$  

(13)

From [4], the time-derivative of $Q_{21}$ is given by

$$\dot{Q}_{21} = -\Omega_{21} R_{12}^T R_2 + R_{12}^T R_2 \dot{\Omega}_2 = Q_{21} \dot{\Omega}_2 - \Omega_2 Q_{21},$$

(14)

where the relative angular velocity vector of Spacecraft 2 with respect to Spacecraft 1 is defined as $\Omega_{21} = \Omega_2 - Q_{21}^T \Omega_1 \in \mathbb{R}^3$. The relative position of Spacecraft 2 with respect to Spacecraft 1 is given by

$$x_{21} = x_2 - x_1.$$  

(15)

To control the relative attitude between the pair of Spacecraft 1 and Spacecraft 2, a third common object, namely Spacecraft 3 is assigned to form a triangular structure. It is assumed that Spacecraft 1 and Spacecraft 2 measure the line-of-sight toward each other, and also toward Spacecraft 3. Furthermore, Spacecraft 3 does not lie on the line joining Spacecraft 1 and 2.

Let $s_{ij} \in S^2$ be the unit-vector from the $i$-th spacecraft toward the $j$-th spacecraft, i.e., $s_{ij} = \frac{x_j - x_i}{\|x_j - x_i\|}$, and let $b_{ij} \in S^2$ be the line-of-sight measurement from the $i$-th spacecraft toward the $j$-th spacecraft, represented with respect to the $i$-th body-fixed frame, i.e., $s_{ij} = R_{ij} b_{ij}$. According to the above assumptions, we have $s_{13} \times s_{23} \neq 0$, and there are four measurements $\{b_{12}, b_{13}, b_{21}, b_{23}\}$ for the pair of Spacecraft 1 and 2.

It has been shown that the relative attitude $Q_{21}$ can be completely determined from the following constraints [12]:

$$b_{12} = -Q_{21} b_{21},$$

(16)

$$b_{23} = -Q_{21} b_{23},$$

(17)

where $b_{123} = \frac{b_{12} \times b_{13}}{\|b_{123}\|} \in S^2$ and $b_{213} = \frac{b_{21} \times b_{23}}{\|b_{213}\|} \in S^2$.

The first constraint (16) states that the unit vector from Spacecraft 1 to Spacecraft 2 is opposite to the unit vector from Spacecraft 2 to Spacecraft 1, i.e., $s_{12} = -s_{21}$. The second constraint (17) implies that the plane spanned by $s_{12}$ and $s_{13}$ should be co-planar with the plane spanned by $s_{21}$ and $s_{23}$. For given LOS measurements $\{b_{12}, b_{13}, b_{21}, b_{23}\}$, the relative attitude $Q_{21}$ is uniquely determined by solving (16) and (17) for $Q_{21}$.

Similar to the definitions in [11]-[12], we have

$$s_{ij}(t) = R_{i}(t) b_{ij}(t), \quad b_{ij}(t) = R_{i}(t)^T s_{ij}(t),$$

(18)

$$\dot{s}_{ij} = \mu_{ij} \times s_{ij},$$

(19)

$$\dot{b}_{ij} = b_{ij} \times (\Omega_{i} - R_{i}^T \mu_{ijk}),$$

(20)

for $(i,j) \in \{(1,2), (1,3), (2,1), (2,3)\}$. Further, for $(i,j,k) \in \{(1,2,3), (2,1,3)\}$, we have

$$\dot{s}_{ijk} = \mu_{ijk} \times s_{ijk},$$

(21)

where $s_{ijk} = \frac{s_{ij} \times s_{ik}}{\|s_{ij} \times s_{ik}\|} \in S^2$ and $\mu_{ijk} \in \mathbb{R}^3$ is the angular velocity of $s_{ijk}$. Similar to [20], we can write

$$b_{ijk} = R_{i}^T s_{ijk} + R_{i}^T \dot{s}_{ijk} = b_{ij} \times (\Omega_{i} - R_{i}^T \mu_{ijk}).$$

(22)

Notice that $\|b_{ij} \times s_{ik}\| = \|R_{i}^T s_{ij} \times R_{i}^T s_{ik}\| = \|s_{ij} \times s_{ik}\|$, and it is non-zero.

D. Spacecraft Formation Control Problem

Next, we define formation control problem. For the leader, Spacecraft 1, the desired absolute attitude trajectory $R_{11}^d(t) \in SO(3)$ is given, and it satisfies the following kinematic equation

$$\dot{R}_{11}^d = R_{11}^d \Omega_{11}^d,$$

(23)

where $\Omega_{11}^d \in \mathbb{R}^3$ is the desired angular velocity of Spacecraft 1. We transform this into the desired line-of-sight and its angular velocity as follows. The corresponding desired line-of-sight measurements are given by

$$\dot{b}_{ij}^d = b_{ij}^d \times (\Omega_{11}^d - R_{i}^d \mu_{ij}^d), \quad i \in \{A, B\}.$$  

(24)

Similar with [12], the kinematic equations for desired line-of-sight measurements can be obtained as

$$\dot{\Omega}_{21}^d = Q_{21}^d \hat{\Omega}_{21}^d,$$  

(26)

where $\hat{\Omega}_{21}^d \in \mathbb{R}^3$ is the desired relative angular velocity vector, satisfying

$$\Omega_{21}^d = \Omega_2 - Q_{21}^T \Omega_1.$$  

(27)

Let the desired relative position and the relative velocity of Spacecraft 2 be $x_{21}^d(t), v_{21}^d(t) \in \mathbb{R}^3$, and they satisfy

$$\dot{x}_{21}^d = v_{21}^d = v_2^d - v_1.$$  

(28)

Assumption 1: The magnitude of each desired angular velocity is bounded by

$$\|\Omega_{ij}^d\| \leq B_{\Omega_{ij}},$$

(29)

where $\Omega_{ij}^d \in \{\Omega_{21}^d, \Omega_{22}^d, \Omega_{11}^d\}$.

Assumption 2: The magnitude of angular velocity of each line-of-sight measurement is bounded by

$$\|\mu_{ij}\| \leq B_{\mu},$$

(30)

where $B_{\mu} \in \mathbb{R}^+$ is a positive constant. The $\mu$ here includes $\{\mu_{1A}, \mu_{1B}, \mu_{12}, \mu_{13}, \mu_{123}, \mu_{213}\}$.

The first assumption is natural in any tracking problem, and the second assumption is satisfied if there is no collision between spacecraft and the velocities of each spacecraft are bounded.

The goal is to design control inputs $(u_1, u_2, f_1, f_2)$ such that the zero equilibrium of the position and attitude tracking errors becomes asymptotically stable, and in particular, it is required that the control moments $u_1, u_2$ are directly expressed in terms of the line-of-sight measurements.
III. Attitude and Position Tracking Error Variables

As a preliminary work before designing control systems, in this section, we present error variables for each of absolute attitude tracking error, relative attitude tracking error, and position tracking error.

A. Absolute Attitude Error Variables

The fundamental idea of constructing attitude control system in terms of LOS measures is utilizing the property that the attitude error becomes zero if the LOS are aligned to their desired values, i.e., we have $R_1 = R_1^d$ if $b_A = b_A^d$ and $b_B = b_B^d$ [14]. This motivates the following definition of the configuration error function for each line-of-sight:

$$
\Psi_j = 1 - b_j \cdot b_j^d = 1 - R_1^{ds} s_j \cdot R_1^{ds} s_j,
$$

for each object $j \in \{A, B\}$. They are combined into

$$
\Psi_1 = k_{b_A} \Psi_A + k_{b_B} \Psi_B,
$$

where $k_{b_A} \neq k_{b_B}$ are positive constants. The error function $\Psi$ refers to the corresponding error of LOS measurements, and $\Psi_1$ represents the combined errors for Spacecraft 1.

By finding the derivatives of the configuration error functions, we obtain the configuration error vectors as follows.

$$
e_{b_A} = b_A \times b_A^d, \quad e_{b_B} = b_B \times b_B^d,
$$

$$
e_b = k_{b_A} e_{b_A} + k_{b_B} e_{b_B}.
$$

Additionally, the angular velocity error vector is defined as

$$
e_{\Omega_1} = \Omega_1 - \Omega_1^d.
$$

It can be shown that the above error variables satisfy the following properties.

**Proposition 1:** (i) The error function $\Psi_1$ and the error vector $e_b$ can be rewritten as

$$
\Psi_1 = \operatorname{tr}[K_1(I_{3 \times 3} - R_1 R_1^{dT})],
$$

$$
e_b = [R_1^{dT} K_1 R_1 - R_1^{dT} K_1 R_1^{dT}]^T,
$$

where $K_1 = k_{b_A} S_A A_T^d + k_{b_B} S_B A_T^d \in \mathbb{R}^{3 \times 3}$ is symmetric and $I_{3 \times 3}$ is the $3 \times 3$ identity matrix.

(ii) $\Psi_1$ is locally quadratic, more explicitly,

$$
\bar{\psi}_b \| e_b \|^2 \leq \Psi_1 \leq \bar{\psi}_b \| e_b \|^2,
$$

where $\bar{\psi}_b = \frac{h_1}{h_2 + h_3}$ and $\bar{\psi}_b = \frac{h_4}{h_5(h_3 - 1)}$ for

$$
h_1 = 2 \min\{k_{b_A}, k_{b_B}\},
$$

$$
h_2 = 4 \max\{k_{b_A}^2 - k_{b_B}^2, k_{b_A}^2, k_{b_B}^2\},
$$

$$
h_3 = 4(k_{b_A} + k_{b_B}) \cdot k_{b_A},
$$

$$
h_4 = 2(k_{b_A} + k_{b_B}),
$$

$$
h_5 = 4 \min\{k_{b_A}^2, k_{b_B}^2\},
$$

and $\psi_1$ is a positive constant satisfying $\psi_1 < \psi_1 < h_1$.

(iii) $\| e_b \| \leq k_{b_A} + k_{b_B}$.

(iv) $\frac{d}{dt} \Psi_1 = e_b \cdot e_{\Omega_1} + \Gamma \| e_b \|$. 

(v) $\| e_b \| \leq \frac{1}{2}(k_{b_A} + k_{b_B}) \| e_{\Omega_1} \| + (B_{1d} + B_1) \| e_b \|$. 

**Proof:** See Appendix, Section B.

B. Relative Attitude Error Variables

Similarly, we define error variables for the relative attitude. It is also based on the fact that we have $Q_2^d = Q_2^d$ provided that $b_12 = -Q_2^d b_{12}$ and $b_123 = -Q_2^d b_{123}$ [13]. Configuration error functions that represent the errors in satisfaction of (16) and (17) are defined as

$$
\Psi_2 = 1 + b_{12} \cdot Q_2^d b_{21},
$$

$$
\Psi_2 = 1 + b_{123} \cdot Q_2^d b_{213}.
$$

For positive constants $k_{21} \neq k_{21}^d$, these are combined into

$$
\Psi_2 = k_{21}^d \Psi_2 + k_{21} \Psi_2^d.
$$

These yield the configuration error vectors as

$$
e_{21} = (Q_2^d b_{12}) \times b_{21}, \quad e_{21} = (Q_2^d b_{123}) \times b_{213},
$$

$$
e_{21} = k_{21}^d \Psi_2 + k_{21} \Psi_2^d.
$$

We have $\| e_{21}^d \|, \| e_{21}^d \| \leq 1$ as $b_{12}, b_{123}$ are unit vectors. And the angular velocity error vector is defined as:

$$
e_{\Omega_2} = \Omega_2 - \Omega_2^d,
$$

where $\Omega_2^d$ can be determined by (27). The properties of relative error variables are summarized as follows:

**Proposition 2:** (i) The error function $\Psi_2$ and the error vector $e_{21}$ can be rewritten as

$$
\Psi_2 = \operatorname{tr}[K_2(I_{3 \times 3} - R_2 R_2^d T)],
$$

$$
e_{21} = [Q_2^d R_1^T K_2 R_2 - R_2^T K_2 R_1 Q_2^d]^T,
$$

where $K_2 = k_{21}^d S_{21} s_{21}^T + k_{21}^d s_{21} s_{213} \in \mathbb{R}^{3 \times 3}$.

(ii) $\Psi_2$ is locally quadratic, more explicitly,

$$
\bar{\psi}_{21} \| e_{21} \|^2 \leq \Psi_2 \leq \bar{\psi}_{21} \| e_{21} \|^2,
$$

where $\bar{\psi}_{21} = \frac{n_1 + n_2 + n_3}{n_4(n_1 - \phi_{21})}$, for

$$
n_1 = 2 \min\{k_{21}^d, k_{21}^d\},
$$

$$
n_2 = 4 \max\{(k_{21}^d - k_{21}^d)^2, (k_{21}^d)^2, (k_{21}^d)^2\},
$$

$$
n_3 = 4(k_{21}^d + k_{21}^d)^2,
$$

$$
n_4 = 2(k_{21}^d + k_{21}^d),
$$

$$
n_5 = 4 \min\{k_{21}^d, k_{21}^d\},
$$

and $\phi_{21}$ is a positive constant satisfying $\Psi_2 < \phi_{21} < n_1$.

(iii) $\frac{d}{dt} \Psi_2 = e_{21} \cdot e_{\Omega_2} + \Gamma \| e_{21} \|^2$.

(iv) $\| \frac{d}{dt} e_{21} \| = \sqrt{k_{21}^d + k_{21}^d} \| e_{\Omega_2} \| + (B_{1d} + B_{21}) \| e_{21} \|$. 

**Proof:** See Appendix, Section B.

C. Position Tracking Error Variables

The position and velocity error vectors of Spacecraft 1 are given as

$$
e_{x_1} = x_1 - x_1^d,
$$

$$
e_{v_1} = v_1 - v_1^d = \dot{e}_{x_1}.
$$
Similarly, the relative position and relative velocity error vectors are given by
\[ e_{x_{21}} = x_{21} - x_{21}^d, \quad e_{v_{21}} = v_{21} - v_{21}^d = \dot{e}_{x_{21}}, \] (46) (47)

IV. FORMATION CONTROL SYSTEM

Based on the error variables defined at the previous section, control systems are designed as follows.

A. Formation Control for Two Spacecraft

Proposition 3: Consider the formation of two spacecraft shown in Fig. [1] For positive constants, \( k_{\Omega i}, k_x, k_v, i \in \{1, 2\} \), the control inputs are chosen as follows
\[ u_1 = -e_b - k_{\Omega 1} e_{\Omega 1} + \Omega_1^d J_1(e_{\Omega 1} + \Omega_1^d) + J_1 \Omega_1^d, \] (48)  
\[ u_2 = -e_{21} - k_{\Omega 2} e_{\Omega 2} + \Omega_2^d J_2(e_{\Omega 2} + \Omega_2^d) + J_2 \Omega_2^d, \] (49)  
\[ f_1 = -k_x e_{x_{21}} - k_v e_{v_{21}} + m_1 \dot{x}_1, \] (50)  
\[ f_2 = -k_x e_{x_{21}} - k_v e_{v_{21}} + m_2 \dot{x}_1 + \dot{x}_2, \] (51)

Then, the zero equilibrium of tracking errors is exponentially stable.

Proof: The error dynamics are as follows. From [1], we have
\[ \dot{e}_{v_i} = \frac{f_1}{m_1} - \dot{x}_1^d, \quad \dot{e}_{v_{21}} = \frac{f_2}{m_2} - \dot{x}_1 - \dot{x}_2, \] (52)

By using (3), (53), (42), we can obtain
\[ J_i \dot{e}_{\Omega_i} = [J(e_{\Omega_i} + \Omega_i^d)]^T e_{\Omega_i} - \Omega_i^d J_i(e_{\Omega_i} + \Omega_i^d) - J_i \Omega_i^d + u_i, \] (53)

for \( i \in \{1, 2\} \).

For positive constants \( c_r \) and \( c_v \), let the Lyapunov function be
\[ V = V_i^1 + V_{21}^1 + V_i^d + V_{21}^d, \] (54)

where
\[ V_1^1 = \frac{1}{2} e_{\Omega 1}^T J_1 e_{\Omega 1} + \Psi_1 + c_r J_1 e_{\Omega 1} \cdot e_b, \] (55)  
\[ V_2^1 = \frac{1}{2} e_{\Omega 2}^T J_2 e_{\Omega 2} + \Psi_2 + c_r J_2 e_{\Omega 2} \cdot e_{21}, \] (56)  
\[ V_i^d = \frac{1}{2} k_x e_{x_i}^T e_{x_i} + \frac{1}{2} m_i e_{v_i}^T e_{v_i} + c_v e_{x_i} \cdot e_{v_i}, \] (57)  
\[ V_{21}^d = \frac{1}{2} k_x e_{x_{21}}^T e_{x_{21}} + \frac{1}{2} m_2 e_{v_{21}}^T e_{v_{21}} + c_v e_{x_{21}} \cdot e_{v_{21}}, \] (58)

From (53), (43), the Lyapunov function is positive-definite about the equilibrium \((e_{\theta_1}^d, e_{\theta_2}, e_{x_{21}}, e_{v_{21}}, e_{v_{21}}) = (0, 0, 0, 0, 0)\) for \( i \in \{1, 2\} \), provided that the constants \( c_r, c_v \) are sufficiently small.

The time-derivative of \( V_i^1, V_{21}^1 \) are given by
\[ \dot{V}_i^1 = (e_{\Omega_i} + c_r e_b)^T J_i \dot{e}_{\Omega_i} + \Psi_1 + c_r J_1 e_{\Omega_i} \cdot \dot{e}_b, \] \[ \dot{V}_{21}^1 = (e_{\Omega_{21}} + c_v e_{21})^T J_2 \dot{e}_{\Omega_{21}} + \dot{\Psi}_{21} + c_r J_2 e_{\Omega_{21}} \cdot \dot{e}_{21}. \]

By using the properties (iv),(v) of Proposition 1 and the properties (iii),(iv) of Proposition 2 and also substituting (53), (48), and (49), the time-derivative of \( V_i^1, V_{21}^1 \) can be rearranged as
\[ \dot{V}_i^1 \leq -[k_{\Omega i} - c_r \lambda M_i(\frac{1}{\sqrt{2}} + 1)k_b] ||e_{\Omega i}||^2 + c_r [\lambda M_i(2B\Omega i + B_1) + k_{\Omega i}] ||e_{\Omega i}|| ||e_b|| - (c_r - \Gamma) ||e_b||^2, \] (59)  
\[ \dot{V}_{21}^1 \leq -[k_{\Omega_{21}} - c_r \lambda M_{21}(\frac{1}{\sqrt{2}} + 1)k_{21}] ||e_{\Omega_{21}}||^2 + c_r [\lambda M_{21}(2B\Omega_{21} + B_{21}) + k_{\Omega_{21}}] ||e_{\Omega_{21}}|| ||e_{21}|| - (c_r - \Gamma_{21}) ||e_{21}||^2, \] (60)

For the translational dynamics, from (45) and (47), we have
\[ \dot{\mathbf{v}}_i = k_x e_{x_{21}} + c_v ||e_{v_i}||^2 + \Omega_1 e_{v_1} + \Omega_2 e_{v_2}, \] (59)  
\[ \dot{v}_{21} = k_x e_{x_{21}} + c_v ||e_{v_{21}}||^2 + \Omega_1 e_{v_1} + \Omega_2 e_{v_2}. \] (60)

Substituting (52) with the control inputs (58), (59), we obtain
\[ \dot{V}_i^1 = -\frac{c_k e_{x_i}}{m_1} ||e_{x_i}||^2 - (k_{v_i} - c_v) ||e_{v_i}||^2 - \frac{c_k e_{v_1} e_{x_{21}}}{m_1}, \] (61)  
\[ \dot{V}_{21}^1 = -\frac{c_k e_{x_{21}}}{m_2} ||e_{x_{21}}||^2 - (k_{v_{21}} - c_v) ||e_{v_{21}}||^2 - \frac{c_k e_{v_{21}} e_{x_{21}}}{m_2}. \] (62)

From (59), (60), (61) and (62), the time-derivative of the complete Lyapunov function \( V \) can be written as
\[ \dot{V} \leq -\langle \xi_1^T M_1 \zeta_1 + \xi_2^T M_2 \zeta_2 + \xi_1^T N_1 \xi_1 + \xi_2^T N_2 \xi_2 \rangle, \] (63)

where \( \xi_1 = [||e_b||, ||e_{\Omega i}||]^T, \quad \xi_2 = [||e_{v_i}||]^T, \quad \zeta_1 = [||e_{v_1}||, ||e_{v_2}||]^T, \quad \zeta_2 = [||e_{v_{21}}||, ||e_{v_{21}}||]^T \in \mathbb{R}^2, \) and the matrices \( M_1, M_2, N_1, N_2 \in \mathbb{R}^{2 \times 2} \) are defined as
\[ M_1 = \frac{1}{2} \begin{bmatrix} 2(c_r - \Gamma) & -c_v \xi_1 \\ -c_v \xi_1 & 2k_{\Omega i} - c_v \lambda M_i k_b(\sqrt{2} + 2) \end{bmatrix}, \] (64)  
\[ M_2 = \frac{1}{2} \begin{bmatrix} 2(c_r - \Gamma) & -c_v \xi_2 \\ -c_v \xi_2 & 2k_{21} - c_v \lambda M_{21} k_{21}(\sqrt{2} + 2) \end{bmatrix}, \] (65)  
\[ N_1 = \frac{1}{2m_1} \begin{bmatrix} 2c_k \xi_{21} & c_k \xi_{21} \\ c_k \xi_{21} & 2m_1(k_{v_1} - \xi_1) \end{bmatrix}, \] (66)  
\[ N_2 = \frac{1}{2m_2} \begin{bmatrix} 2c_k \xi_{21} & c_k \xi_{21} \\ c_k \xi_{21} & 2m_2(k_{v_2} - \xi_2) \end{bmatrix}, \] (67)

where \( \Lambda_1 = \lambda M_i(2B\Omega_i + B_1) + k_{\Omega i} \in \mathbb{R} \) and \( \Lambda_2 = \lambda M_{21}(2B\Omega_{21} + B_{21}) + k_{\Omega_{21}} \in \mathbb{R} \). If the constants \( c_r \) and \( c_v \) are sufficiently small, we can show that all of matrices \( M_1, M_2, N_1 \) and \( N_2 \) are positive definite. This implies that the equilibrium \((e_b, e_{\theta_1}, e_{\theta_2}, e_{x_{21}}, e_{v_{21}}, e_{v_{21}}) = (0, 0, 0, 0, 0, 0)\) for \( i \in \{1, 2\} \), i.e., the desired formation, is exponentially stable.

B. Formation Control for Multiple Spacecraft

The preceding results for two spacecraft are readily generalized for an arbitrary number of spacecraft. Here, we present the controller structures as follows, without stability proof.
that can be obtained by generalizing the proof of Proposition 3.

**Proposition 4:** Consider \( n \) spacecraft in the formation, the \( i \)-th spacecraft is paired serially with the \((i - 1)\)-th spacecraft for \( i = \{2, 3, \cdots, n\} \). For positive constants, \( k_{\Omega i}, k_{v_i}, k_{\Omega i}, k_{x_i}, \) and \( k_{v_i} \), for \( i \in \{2, 3, \cdots, n\} \), the control inputs are chosen as follows

\[
\begin{align*}
  u_1 &= -e_b - k_{\Omega i} e_{\Omega i} + \hat{\Omega}_{d1}^T \Omega_{d1} + J_{1i} \dot{\Omega}_{1i}, \\
  u_i &= -e_{xi,i-1} - k_{\Omega i} e_{\Omega i} + \hat{\Omega}_{di} J_{si} (e_{\Omega i} + \Omega_{di}) + J_{di} \dot{\Omega}_{di}, \\
  f_1 &= -k_{x1} e_{x1} - k_{v1} e_{v1} + m_1 \ddot{x}_{1i}, \\
  f_i &= -k_{x1} e_{x,i-1} - k_{v1} e_{v,i-1} + m_i (\ddot{x}_{i-1} + \ddot{x}_{d,i-1}).
\end{align*}
\]

Then the zero equilibrium of tracking errors is exponentially stable.

V. NUMERICAL SIMULATION

Two simulation results are presented: (i) formation tracking control for two spacecraft, and (ii) formation stabilization for four spacecraft.

A. Formation Tracking for Two Spacecraft

Suppose \( n = 2 \). The mass and the inertia matrix are chosen as \( m_1 = m_2 = 30 \text{ kg} \) and \( J_1 = J_2 = \text{diag}[3, 2, 1] \text{ kgm}^2 \). The desired absolute attitude of the leader, namely \( R_{1i}(t) \), is specified in terms of \( 3 - 2 - 1 \) Euler angles \((\alpha(t), \beta(t), \gamma(t))\), where

\[
\alpha(t) = 0, \quad \beta(t) = -0.7 + \cos(0.2t), \quad \gamma(t) = 0.5 + \sin(2t).
\]

The desired relative attitude \( Q_{21}(t) \) is also defined in terms of another set of Euler angles given by

\[
\phi(t) = \sin(0.5t), \quad \theta(t) = 2, \quad \psi(t) = \cos(t) + 1.
\]

The initial attitudes for Spacecraft 1 and Spacecraft 2 are chosen as \( R_1(0) = R_2(0) = I_{3 \times 3} \). The initial angular velocity is chosen to be zero for both spacecraft.

For the translational motion, the desired position vectors are given by

\[
\begin{align*}
  x_1^d &= [\sin(0.04t), 0, -\sin(0.07t)]^T, \\
  x_{21}^d &= [2, -3 + \cos(0.02t), 10]^T.
\end{align*}
\]

The initial positions are chosen as \( x_1 = [0, 0, 0] \) and \( x_2 = [2, -1, 7] \). Control gains are selected to be \( k_{\Omega1} = k_{\Omega2} = 7, \quad k_{x1} = k_{x2} = 25, \quad k_{v1} = k_{v2} = 25.1, \quad k_{x1} = k_{x2} = 49, \) and \( k_{v1} = k_{v2} = 12.6 \).

The corresponding numerical results are illustrated in Fig 2 where the attitude error vectors are defined as

\[
\begin{align*}
  e_{R1} &= \frac{1}{2} (R_1^T R_1 - R_1^T R_{1i}^d), \\
  e_{Q21} &= \frac{1}{2} (Q_{21}^T Q_{21} - Q_{21}^T Q_{21i}^d).
\end{align*}
\]

It is illustrated that tracking errors are nicely converted to zero.

**Fig. 2.** Numerical results for two spacecraft in formation.

B. Formation Control for Four Spacecraft

Next, we consider formation control for \( n = 4 \) spacecraft. The mass and the inertia matrix are same as the previous case. The desired attitudes are chosen as \( R_{1i} = Q_{12}^d = Q_{23}^d = Q_{34}^d = I_{3 \times 3} \). This represents attitude synchronization. The initial attitudes are given as

\[
\begin{align*}
  R_1(0) &= \exp(0.2\pi e_2), \quad R_2(0) = \exp(0.5\pi e_1), \\
  R_3(0) &= \exp(0.4\pi e_1), \quad R_4(0) = \exp(0.8\pi e_3),
\end{align*}
\]

where \( e_1 = [1, 0, 0]^T, \quad e_2 = [0, 1, 0]^T, \) and \( e_3 = [0, 0, 1]^T \). Also, the initial angular velocity is chosen to be zero for each spacecraft.

The desired position trajectories are chosen as

\[
\begin{align*}
  x_1^d &= [-100, 0, 0]^T, \\
  x_{21}^d &= [100, 100, 0]^T, \quad x_{32}^d = x_{43}^d = [0, -200, 0]^T.
\end{align*}
\]

The initial conditions are \( x_1 = [-200, 0, 0]^T, \quad x_2 = [-100, -50, 0]^T, \quad x_3 = [0, 0, 20]^T, \quad x_4 = [100, 100, -1]^T, \)

\( v_1 = [0, 0, 0]^T, \quad v_2 = [0, 10, 0]^T \) and \( v_3 = v_4 = [0, 10, 0]^T \).

The corresponding numerical results is illustrated by Figure 4 where the attitude error vectors are defined as

\[
\begin{align*}
  e_{R1} &= \frac{1}{2} (R_1^T R_1 - R_1^T R_{1i}^d), \\
  e_{Q_{i-1,i}} &= \frac{1}{2} (Q_{i-1,i}^T Q_{i-1,i} - Q_{i-1,i}^T Q_{i-1,i}^d).
\end{align*}
\]

\( e_{R1}, e_{Q_{i-1,i}} \in \mathbb{R}^3 \).
APPENDIX

A. Lemmas

Lemma 1: The properties stated in Proposition 1 of [15] are summarized and extended as follows: For non-negative constants $f_1, f_2, f_3$, let $F = \text{diag}[f_1, f_2, f_3] \in \mathbb{R}^{3 \times 3}$, and let $P \in \text{SO}(3)$. Define

$$
\Phi = \frac{1}{2} \text{tr}[F(\mathbb{I}_{3 \times 3} - P)],
$$

$$
e_P = \frac{1}{2}(FP - P^T F)^\top.
$$

Then, $\Phi$ is bounded by the square of the norm of $e_P$ as

$$
\frac{h_1}{h_2 + h_3} \|e_P\|^2 \leq \Phi \leq \frac{h_1 h_4}{h_5(h_1 - \phi)} \|e_P\|^2.
$$

If $\Phi < \phi < h_1$ for a constant $\phi$, where $h_i$ are given by

$$
\begin{align*}
    h_1 &= \min\{f_1 + f_2, f_3 + f_1, f_3 + f_2\}, \\
    h_2 &= \max\{(f_1 - f_2)^2, (f_2 - f_3)^2, (f_3 - f_1)^2\}, \\
    h_3 &= \max\{(f_1 + f_2)^2, (f_2 + f_3)^2, (f_3 + f_1)^2\}, \\
    h_4 &= \max\{f_1 + f_2, f_2 + f_3, f_3 + f_1\}, \\
    h_5 &= \min\{(f_1 + f_2)^2, (f_2 + f_3)^2, (f_3 + f_1)^2\}.
\end{align*}
$$

Proof: From Rodrigues' formula, we have $P = \exp(y)$, for $y \in \mathbb{R}^3$. According to [15], using the MATLAB symbolic computation tool, we find

$$
\|e_P\|^2 = \frac{(1 - \cos \|y\|)^2}{4\|y\|^4} \sum_{(i,j,k) \in \mathcal{E}} (f_i - f_j)^2 y_i^2 y_j^2
$$

Using these two equations it follows that

$$
\frac{\|e_P\|^2}{\Phi} \leq \frac{h_2 + h_3}{h_1}, \quad \frac{\Phi}{\|e_P\|^2} \leq \frac{h_1 h_4}{h_5(h_1 - \phi)},
$$

which shows (70). Lemma 1 is purely the citation from [15].

Lemma 2: Consider

$$
\Phi' = \frac{1}{2} \text{tr}[F'(I_{3 \times 3} - P')],
$$

$$
e_P' = \frac{1}{2}(F'P' - P'^T F')^\top,
$$

where $F' = \text{diag}[f_1', f_2', f_3'] \in \mathbb{R}^{3 \times 3}$, for positive scalars $f_i', i = \{1, 2, 3\}$, and $P' = A^TPA$, for any rotation matrix $A \in \text{SO}(3)$. We say that $\Phi'$ and $e_P'$ are bounded respectively in the form of

(i) \( \frac{h_1}{h_2 + h_3} \|e_P\|^2 \leq \Phi' \leq \frac{h_1 h_4}{h_5(h_1 - \phi)} \|e_P\|^2 \),

(ii) \( \|e_P\| \leq \sqrt{\frac{2h_1 + h_2}{h_5}} \|e_P\| \),
where \( h_i \) are given by
\[
\begin{align*}
&h_2' = \max\{(f_i' - f_j')^2, (f_2' - f_3')^2, (f_3' - f_1')^2\}, \\
&h_3' = \max\{(f_i' + f_j')^2, (f_2' + f_3')^2, (f_3' + f_1')^2\}, \\
&h_4' = \max\{(f_i' + f_j')^2, (f_2' + f_3')^2, (f_3' + f_1')^2\}, \\
&h_5' = \min\{(f_i' + f_j')^2, (f_2' + f_3')^2, (f_3' + f_1')^2\}.
\end{align*}
\]

**Proof:** We start from finding the boundedness of \( \Phi' \), employing (71) leads to
\[
P' = A^T P A = A^T \exp(y) A = \exp(A^T y).
\]

From Rodrigues’ formula, we know \( P' = \exp(y') \) for \( y' \in \mathbb{R}^3 \). Evidently, we have
\[
y' = A^T y, \quad \|y'\| = \|y\|.
\]

We therefore write
\[
\Phi' = \frac{(1 - \cos \|y'\|)}{2\|y'\|^2} \sum_{(i,j,k) \in \mathcal{E}} (f_i' + f_j') y_k'^2 \\
= \frac{(1 - \cos \|y\|)}{2\|y\|^2} \sum_{(i,j,k) \in \mathcal{E}} (f_i' + f_j') y_k^2.
\]

(75)

Comparing (71) with (75) yields
\[
\frac{\|e_P\|^2}{\Phi} \leq \frac{h_2' + h_3'}{h_1}, \quad \Phi \leq \frac{h_1 h_4'}{h_5(h_1 - \phi)},
\]
which shows (i).

Similar to (71), we can have
\[
\|e_P'\|^2 = \frac{(1 - \cos \|x'\|)}{4\|x'\|^4} \sum_{(i,j,k) \in \mathcal{E}} (f_i' - f_j')^2 x_i'^2 x_j'^2 \\
+ \frac{\sin^2 \|x'\|}{4\|x'\|^2} \sum_{(i,j,k) \in \mathcal{E}} (f_i' + f_j')^2 x_k'^2.
\]

By comparing the magnitude \( e_P \) and \( e_P' \)
\[
\frac{\|e_P\|^2}{\|e_P'\|^2} \leq 1 + \cos \|x\| \frac{\max\{(f_i - f_j)^2\} x_i^2 x_j^2}{\min\{(f_i' - f_j')^2\} x_k'^2} \\
+ \frac{\max\{(f_i + f_j)^2\} \|x\|^2}{\min\{(f_i' + f_j')^2\} \|x'\|^2} = (1 + \cos \|x\|) \frac{h_2}{h_5} + \frac{h_5}{h_5} \\
\leq 2 \frac{h_2 + h_5}{h_5},
\]
which leads to (ii). It shows that \( P' = A^T P A \) for any rotation matrix \( A \in \text{SO}(3) \), we can still bound the new \( \Phi' \) and \( e_P' \) in terms of original \( \|e_P\| \).

**B. Proof of Proposition 7**

Property (ii), (iii) has been verified in [14]. Also, it has been shown that
\[
K_1 = U_1 G_1 U_1^T,
\]
(76)

where \( U_1 \in \text{SO}(3) \) and \( G_1 = \text{diag}[k_{b_A}, k_{b_B}, 0] \) such that
\[
\text{tr}[K_1] = \text{tr}[G_1] = k_{b_A} + k_{b_B} \triangleq \bar{k}_b.
\]

(77)

From (33), we can tell that the magnitude of \( e_{b_i} \) is bounded, since \( b_i \) and \( b_i' \) are unit vectors, for \( i = \{A, B\} \).
That is, \( \|e_{b_i}\| \leq 1 \), which leads to
\[
\|e_{b_B}\| \leq k_{b_A} \|e_{b_A}\| + k_{b_B} \|e_{b_B}\| \leq k_{b_A} + k_{b_B} = \bar{k}_b.
\]
which shows (iii).

From (36), Differentiating \( \Psi_1 \) gives us
\[
\dot{\Psi}_1 = -\text{tr}[K_1 \dot{R}_1] \triangleq \text{tr}[\dot{K}_1 (I_{3 \times 3} - R_1^d)].
\]

(79)

From (4) and (23), the first term at right hand side can be written as
\[
-\text{tr}[K_1 \dot{R}_1] = -\text{tr}[\Omega_1 (R_1^d)^T K_1] + \text{tr}([-\hat{R}_1^d R_1^d]^T K_1 R_1].
\]

Applying one of the hat map properties (6) leads to
\[
-\text{tr}[K_1 \dot{R}_1] = \Omega_1^T (R_1^d)^T K_1 R_1 - R_1^T K_1 R_1^d - R_1^T K_1 R_1^d) \}
\]
\[
= (\Omega_1 - \Omega_1^T)^T e_{b} = e_{\Omega_1} e_{b}.
\]

(80)

Next, we can write
\[
\text{tr}[-\dot{K}_1 (I_{3 \times 3} - R_1^d)] = \text{tr}[-\dot{K}_1 (I_{3 \times 3} - R_1^d)]
\]
\[
= \text{tr}[\dot{G}_1] - \text{tr}[-\hat{G}_1 U_1^T \dot{R}_1 U_1]
\]
\[
= \text{tr}[\dot{G}_1] - \text{tr}[\hat{G}_1 (I_{3 \times 3} - \hat{R}_1^d U_1 U_1^T)] \triangleq \dot{\Phi}_1,
\]

(81)

where \( \dot{K}_1 \equiv \dot{U}_1 \dot{G}_1 \dot{U}_1^T \) is symmetric. We have \( \dot{U}_1 \in \text{SO}(3) \) and \( \dot{G}_1 \equiv \text{diag}[\dot{g}_1, \dot{g}_2, \dot{g}_3] \) \( \in \mathbb{R}^{3 \times 3} \) where \( \dot{g}_i \) are positive constants. Then, Property (ii) of Lemma 2 is applied here. That is, we let \( F' = 2\dot{G}_1 \) and \( P' = \dot{U}_1 \dot{R}_1^d U_1 \) to obtain
\[
\frac{h_1 h_4'}{h_2 + h_3} \|e_{b}\|^2 \leq \dot{\Phi}_1 \leq \frac{h_1 h_4'}{h_2 + h_3} \|e_{b}\|^2.
\]

(82)

If \( \dot{\phi}_1 < \dot{h}_1 < \dot{h}_1 \) for a constant \( \phi_1 \), where \( \dot{h}_i \) are given by
\[
\dot{h}_2 = 4\max\{\dot{g}_1 - \dot{g}_2, \dot{g}_2 - \dot{g}_3, \dot{g}_3 - \dot{g}_1\}^2, \\
\dot{h}_3 = 4\max\{\dot{g}_1 + \dot{g}_2, \dot{g}_2 + \dot{g}_3, \dot{g}_3 + \dot{g}_1\}^2, \\
\dot{h}_4 = 2\max\{\dot{g}_1 + \dot{g}_2, \dot{g}_2 + \dot{g}_3, \dot{g}_3 + \dot{g}_1\}.
\]

Summation of (80) and (82) allows us to write
\[
\dot{\Psi}_1 \leq e_{\Omega_1} e_{b} + \Gamma_1 \|e_{b}\|,
\]
where \( \Gamma_1 = \frac{h_1 h_4'}{h_2 + h_3} \). This shows (iv).

Differentiating (37), we are able to write
\[
\dot{e}_{b} = \dot{e}_{b} + \dot{e}_{b}^c,
\]
\[
\dot{e}_{b}^c = R_1^d K_1 R_1 + R_1^d K_1 R_1 - R_1^T K_1 R_1^d - R_1^T K_1 R_1^d,
\]
\[
\dot{e}_{b}^c = R_1^d K_1 R_1 - R_1^T K_1 R_1^d.
\]

Using kinematics equations of the spacecraft, we know that
\[
e_{b}^c = (\Omega_1 R_1^d K_1 R_1 - R_1^d K_1 R_1) + \Omega_1^T R_1^d K_1 R_1.
\]

(83)

Inserting \( \Omega_1 = e_{\Omega_1} + \Omega_1^c \) and rearrangement lead us to
\[
\dot{e}_{b}^c = (e_{\Omega_1} R_1^d K_1 R_1 + R_1^d K_1 R_1 e_{\Omega_1}) + e_{b}^d - \Omega_1^d e_{b}.
\]

(83)
Further, in view of (7) and (5), we obtain
\[ e_\xi^k = E_{\Omega_1} e_\Omega + e_b \times \Omega^d, \]
where \( E_{\Omega_1} = \{\text{tr}[R^T_1 K_1 R^T_1], 1 \times 3 - R^T_1 K_1 R^T_1 \} \in \mathbb{R}^{3 \times 3} \). In particular, the Frobenius norm of \( E_{\Omega_1} \) is given by
\[ \|E_{\Omega_1}\|_F = \sqrt{\text{tr}[E_{\Omega_1}^T E_{\Omega_1}]} = \sqrt{\text{tr}[R^T_1 K_1 R^T_1]^2 - \text{tr}[K_1^2]} \leq \frac{1}{\sqrt{2}} \text{tr}[G_1] = \frac{1}{\sqrt{2}} k_b, \]
where (77) is applied. From (29), we can write
\[ \|e_\xi^k\| \leq \frac{1}{\sqrt{2}} k_b \|e_\Omega\| + B_{\Omega_d} \|e_b\|. \]

As for \( e_\xi^k \), we know that
\[ e_\xi^k = R^T_1 (\hat{U}_1 \hat{G}_1 \hat{U}_1^T) R_1 - \hat{U}_1 (\hat{U}_1 \hat{G}_1 \hat{U}_1^T) R^T_1, \]
and this yields
\[ \|e_\xi^k\| = \|\hat{U}_1 R^T_1 (e_\xi^k) R^T_1 \hat{U}_1\| \]
\[ = \|\hat{G}_1 \hat{U}_1 R^T_1 R_1 \hat{U}_1 - \hat{U}_1 R^T_1 R_1 \hat{U}_1 \hat{G}_1\| \]
\[ \approx \|\hat{G}_1 \hat{U}_1 - \hat{U}_1 \hat{G}_1\|, \]
where \( \hat{U}_1 = \hat{U}_1 R_1 R^T_1 \hat{U}_1 = \hat{U}_1 R_1 \hat{U}_1 \). Notice that we have applied \( \|A\| = \|RA\| = \|AR\| \), for any matrix \( A \in \mathbb{R}^{3 \times 3} \) and \( R \in \text{SO}(3) \). Also we know the vec map does not change the magnitude of the vector, which implies
\[ \|\hat{G}_1 \hat{U}_1 - \hat{U}_1 \hat{G}_1\| = \|\hat{G}_1 \hat{U}_1 - \hat{U}_1 \hat{G}_1\|^\gamma = \|e_\xi^k\|. \]

Using property (ii) of Lemma 2, we know that
\[ \|e_\xi^k\| \leq B_1 \|e_b\|, \]
where \( B_1 = 2 \sqrt{\frac{\|e_p\|}{h_3}} \). The summation of (85) and (86) results in
\[ \|e_b\| \leq \frac{1}{\sqrt{2}} k_b \|e_\Omega\| + (B_{\Omega_d} + B_1) \|e_b\|, \]
which shows (v).

C. Proof of Proposition 2

The configuration error function (40) can be written as
\[ \Psi_{21} = k^a_{21} [1 + b_{21} \cdot (Q^d_2 Q^d_2)] + k^b_{21} [1 + b_{213} \cdot (Q^d_2 Q^d_{213})] \]
\[ = k^a_{21} + k^b_{21} + \text{tr}[k^a_{21} Q^d_2 Q^d_2 b_{21}] + k^b_{21} Q^d_2 Q^d_{213} b_{213}], \]
employing the geometric constraints \( s_{12} = -s_{21} \) and \( s_{213} = -s_{213} \), we obtain
\[ \Psi_{21} = k^a_{21} + k^b_{21} - \text{tr}[Q^d_2 R^T_1 (k^a_{21} s_{21} s_{21}^T + k^b_{21} s_{213} s_{213}^T) R_1], \]
\[ \approx k^a_{21} + k^b_{21} - \text{tr}[Q^d_2 R^T_1 K_21 R_1], \]
where we define a symmetric matrix.
\[ K_{21} = k^a_{21} s_{21} s_{21}^T + k^b_{21} s_{213} s_{213}^T. \]

According to the spectral theory, the matrix \( K_{21} \) can be decomposed into \( K_{21} = U_{21} G_{12} U^T_{21} \), where \( G_{12} \) is the diagonal matrix given by \( G_{12} = \text{diag}[k^a_{12}, k^b_{12}, 0] \in \mathbb{R}^{3} \), and \( U_{21} \) is an orthonormal matrix defined as \( U_{21} = [s_{12}, s_{213}, \frac{s_{12} \times s_{213}}{\sqrt{s_{12}^2 s_{213}^2}}] \in \text{SO}(3) \). Hence, the substitution of \( k^a_{12} + k^b_{12} = \text{tr}[G_{12}] = \text{tr}[K_{21}] \) leads us to
\[ \Psi_{21} = \text{tr}[K_{21} (I_{3 \times 3} - R_1 Q^d_2 R^T_1)], \]
\[ = \text{tr}[G_{21} (I_{3 \times 3} - U^T_1 R_1 Q^d_2 R^T_1 U_{21})], \]

These are alternative expressions of \( \Psi_{21} \). In addition, (41) can be written as
\[ \dot{e}_{21} = k^a_{21} [Q^d_2 R^d_2 b_{21}] + k^b_{21} [Q^d_2 (Q^d_2 b_{213}) \times b_{213}] \]
\[ + k^b_{21} [b_{213} Q^d_2 Q^d_2 b_{213}] + k^b_{21} [Q^d_2 Q^d_{213} b_{213} Q^d_2 Q^d_{213} b_{213}] \]
\[ = k^a_{21} [Q^d_2 R^d_2 b_{21}] + k^b_{21} [Q^d_2 R^d_2 b_{213} Q^d_2 Q^d_{213} b_{213}], \]
\[ = Q^d_2 R^d_2 K_21 R_2 - R^T_2 K_21 R_1 Q^d_2, \]

one of the hat properties (5) is applied. Together with (90), property (i) is proved.

We apply Lemma 1 here, that is, in view of (91) and (68), let \( F = 2G_{12} \) and \( P = U^T_1 R_1 Q^d_2 R^T_1 U_{21} \), we obtain \( \Psi_{12} = \Phi \) and \( \psi = \phi \), which results in
\[ \frac{h_1}{h_2 + h_3} \|e_p\|^2 \leq \|e_p\|^2. \]

Furthermore, substituting the new expression of \( F \) and \( P \) into (69), we obtain
\[ \dot{e}_p = G_{21} U^T_1 R_1 Q^d_2 R^T_1 U_{21} - U^T_2 R_2 Q^d_2 R^T_1 U_{21} G_{21} \]
\[ = U^T_1 (K_21 R_1 Q^d_2 R^T_1 - R_1 Q^d_2 R^T_1 K_21) U_{21} \]
\[ = U^T_2 R_2 Q^d_2 R^T_1 (\dot{e}_{21}) Q^T_2 R^T_2 U_{21}. \]

Employing (8) yields to
\[ \dot{e}_p = (U^T_2 R_2 Q^T_2 (\dot{e}_{21})^\dagger)^\dagger. \]

This implies \( \|e_p\| = \|\dot{e}_{21}\| \) since \( U_{21}, R_2, Q^d_2 \) all belongs to \( \text{SO}(3) \). This shows (ii).

The time-derivative of (90) is given by
\[ \dot{\Psi}_{21} = \text{tr}[\dot{K}_{21} (I_{3 \times 3} - Q^c_{21})] - \text{tr}[K_{21} \dot{Q}_{21}], \]

where \( \dot{Q}_{21} = R_1 Q^d_2 R^T_1 \in \text{SO}(3) \). Notice that \( K_{21} \) is symmetric, since
\[ \dot{K}_{21} = k^a_{21} \mu_{213} s_{213} s_{213}^T + k^a_{21} s_{21} s_{21}^T \mu_{213} + k^b_{21} s_{213} s_{213}^T \mu_{213} = K_{21}, \]
and it can be decomposed to \( K_{21} = \hat{U}_{21} \hat{G}_{21} \hat{U}_{21}^T \) where \( \hat{U}_{21} \in \text{SO}(3) \) and \( \hat{G}_{21} = \text{diag}[\hat{g}_1, \hat{g}_2, \hat{g}_3] \) for three positive scalars. Hence, the first term on the right of (94) can be written as
\[ \text{tr}[\dot{K}_{21} (I_{3 \times 3} - Q^c_{21})] = \text{tr}[\hat{G}_{21} (I_{3 \times 3} - \hat{U}_{21} \hat{G}_{21} \hat{U}_{21}^T)] = \hat{\Psi}_{21}. \]
Here we apply the property (ii) of Lemma. That is, we let \( F' = 2\hat{G}_{21} \) and \( P' = U_{21}^{T}Q_{21}^{d}U_{21} \) to obtain
\[
\frac{h_{1}}{h_{2} + h_{3}} \| e_{21} \|^{2} \leq \frac{h_{1} h_{4}}{h_{5} (h_{1} - \psi)} \| e_{21} \|^{2}. \tag{95}
\]
If \( \hat{\Phi} < \phi < \hat{\Theta} \) for a constant \( \phi \), where \( \hat{\Theta} \) is given by
\[
\hat{\Theta} = 4 \max \{ (\hat{\gamma}_{1} - \hat{\gamma}_{2})^{2}, \ (\hat{\gamma}_{2} - \hat{\gamma}_{3})^{2}, \ (\hat{\gamma}_{3} - \hat{\gamma}_{1})^{2} \},
\]
then
\[
\hat{\Theta} = 4 \max \{ (\hat{\gamma}_{1}^{2} + \hat{\gamma}_{2}^{2}), \ (\hat{\gamma}_{2}^{2} + \hat{\gamma}_{3}^{2}), \ (\hat{\gamma}_{3}^{2} + \hat{\gamma}_{1}^{2}) \},
\]
and
\[
\hat{\Theta} = 2 \max \{ (\hat{\gamma}_{1} + \hat{\gamma}_{2}), \ (\hat{\gamma}_{2} + \hat{\gamma}_{3}), \ (\hat{\gamma}_{3} + \hat{\gamma}_{1}) \}.
\]
In addition, to find out the second term on the right of (94), we first find out the time derivative of \( Q_{21}^{d} \),
\[
\dot{Q}_{21}^{d} = \frac{d}{dt} (R_{1} Q_{21}^{d} R_{1}^{T})
\]
\[
= (R_{1} \Omega) Q_{21}^{d} R_{2}^{T} + R_{1} (Q_{21}^{d} \dot{\Omega}) R_{2}^{T}
+ R_{1} Q_{21}^{d} (\dot{\Omega}_{1} R_{1}^{T}),
\]
where (27) and (8) are applied. Then we can write
\[
-t r [K_{21} \dot{Q}_{21}^{d}] = t r \left[ K_{21} R_{1} Q_{21}^{d} (\dot{e}_{\Omega_{21}}) R_{1}^{T} \right]
= \epsilon_{\Omega_{21}} (Q_{21}^{d} R_{1}^{T} K_{21} R_{2} - R_{2}^{T} K_{21} R_{1} Q_{21}^{d}^{T})^{2}
= \epsilon_{\Omega_{21}} \epsilon_{21}^{T} \epsilon_{21}, \tag{96}
\]
where (6) and (92) are applied in the process of rearrangement. Next, substituting (95) and (96) to (94) results in
\[
\hat{\Phi}_{21} \leq \epsilon_{\Omega_{21}}^{T} \epsilon_{21}^{T} + \Gamma_{21} \| e_{21} \|^{2}, \tag{97}
\]
where \( \Gamma_{21} \triangleq \sqrt{\frac{h_{1} h_{4}}{h_{5} (h_{1} - \psi)}} \). This is the proof of (iii).

Lastly we show (iv). Differentiating (92) with respect to time, we obtain
\[
\dot{\epsilon}_{21} = \epsilon_{a} + \epsilon_{b}, \tag{98}
\]
where
\[
\hat{\epsilon}_{a} = \dot{Q}_{21}^{d} R_{1}^{T} K_{21} R_{2} + Q_{21}^{d} R_{1}^{T} K_{21} R_{2} + Q_{21}^{d} R_{1}^{T} K_{21} R_{2}^{T}
- R_{2}^{T} K_{21} R_{1} Q_{21}^{d} - R_{2}^{T} K_{21} R_{1} Q_{21}^{d}^{T} R_{2}^{T} K_{21} R_{1} Q_{21}^{d}^{T},
\]
\[
\hat{\epsilon}_{b} = Q_{21}^{d} R_{1}^{T} K_{21} R_{2} - R_{2}^{T} K_{21} R_{1} Q_{21}^{d}.
\]
Employing the kinematic equations (4) (27) and one of the hat map properties (8) into \( \epsilon_{a} \) leads to
\[
\hat{\epsilon}_{a} = - (\dot{\Omega}_{2} Q_{21}^{d} R_{1}^{T} K_{12} K_{21} R_{2} + R_{2}^{T} K_{21} R_{1} Q_{21}^{d} R_{1}^{T} K_{21} R_{2}),
\]
and the substitution of \( e_{\Omega_{21}} = \Omega_{2} - \Omega_{2}^{d} \) allows us to write
\[
\hat{\epsilon}_{a} = (Q_{21}^{d} R_{1}^{T} K_{21} R_{2} - R_{2}^{T} K_{21} R_{1} Q_{21}^{d} R_{1}^{T} K_{21} R_{2}),
\]
\[
+ (\dot{\epsilon}_{\Omega_{21}} R_{1}^{T} K_{21} R_{1} Q_{21}^{d} + Q_{21}^{d} R_{1}^{T} K_{21} R_{2} e_{\Omega_{21}})
\]
\[
= \epsilon_{21} \Omega_{2}^{d} - \Omega_{2}^{d} \epsilon_{21}.
\]
Apply (5) and (7) here. Then removing the hat map on both side of the equation, we obtain
\[
\epsilon_{a} = e_{21} \times \Omega_{2}^{d} + \{ t r [R_{2}^{T} K_{21} R_{1} Q_{21}^{d} R_{1}^{T} K_{21} R_{2} e_{\Omega_{21}}] \}
\]
\[
= e_{21} \times \Omega_{2}^{d} + E_{\Omega_{21}} e_{\Omega_{21}}. \tag{99}
\]
where
\[
E_{\Omega_{21}} = \{ t r [R_{2}^{T} K_{21} R_{1} Q_{21}^{d} R_{1}^{T} K_{21} R_{2} e_{\Omega_{21}}] \}
\]
Moreover, the Frobenius norm of \( E_{\Omega_{21}} \) is given by
\[
\| E_{\Omega_{21}} \|_{F} = \frac{1}{2} \sqrt{\| E_{\Omega_{21}}^{T} E_{\Omega_{21}} \|^{2} + 2 \| K_{21} \|^{2}}. \tag{100}
\]
Using the fact that \( K_{21} = U_{21} G_{21} U_{21}^{T} \), we find
\[
\| K_{21} \| = \| U_{21}^{T} R_{2}^{d} U_{21} R_{2}^{d} G_{21} U_{21}^{T} \| = \| (G_{21} U_{21}^{T} R_{2}^{d} U_{21}^{T}) \|^{2}.
\]
Let \( U_{21}^{T} R_{2}^{d} U_{21} = \exp(z) \in SO(3) \) from Rodrigues' formula. Using the MATLAB symbolic tool, we find
\[
\| K_{21} \| = \| U_{21}^{T} R_{2}^{d} U_{21}^{T} U_{21} G_{21} U_{21} \|^{2} = \| G_{21} U_{21}^{T} R_{2}^{d} U_{21}^{T} U_{21} \|^{2}.
\]
Substituting this into (101) we get
\[
\| E_{\Omega_{21}} \| = \| K_{21} \| = \| k_{21}^{a} + k_{21}^{b} \|.
\]
As for \( \epsilon_{b} \), we can rewrite it as follows
\[
\hat{\epsilon}_{b} = Q_{21}^{d} R_{1}^{T} K_{21} R_{2}^{T} - R_{2}^{T} K_{21} R_{1} Q_{21}^{d},
\]
\[
\| \hat{\epsilon}_{b} \| = \| (Q_{21}^{d} R_{1}^{T} K_{21} R_{2} - R_{2}^{T} K_{21} R_{1} Q_{21}^{d}) \|^{2} + \text{further rearrangement}, \tag{102}
\]
we get
\[
\| \epsilon_{b} \| = \| U_{21}^{T} R_{2}^{d} U_{21}^{T} U_{21} G_{21} U_{21}^{T} R_{2}^{d} - R_{2}^{T} K_{21} R_{1} Q_{21}^{d} \|^{2}
\]
\[
\| \hat{A} \| = \| (G_{21} U_{21} G_{21} U_{21}^{T} U_{21}^{T} U_{21} G_{21} U_{21}^{T}) \|^{2},
\]
where we have applied \( \| A \| = \| R A \| = \| A R \| \), for any matrix \( A \in \mathbb{R}^{3 \times 3} \) and \( R \in SO(3) \), and also let \( U_{21} = U \in T Q_{21}^{d} Q_{21}^{d} R_{2} \in SO(3) \).

Use the property (ii) of Lemma 2 that we have developed, that is, let \( \epsilon_{b} = 2 \epsilon' \), \( \hat{G}_{21} = F' \) and \( \hat{U}_{21} = P' \). We claim that
\[
\| \epsilon_{b} \| \leq \frac{1}{2} \sqrt{2 \| \hat{H}_{21} \| \| e_{21} \| + B_{21} \| e_{21} \|}. \tag{103}
\]
Next, inserting (102), (103) into (98) gives rise to

\[
\|\dot{e}_{21}\| \leq \|e_a\| + \|e_b\| = \frac{1}{\sqrt{2}} (k^{\alpha}_{12} + k^{\beta}_{12}) \|e_{\Omega_d}\| + (B_{\Omega_d} + B_{21}) \|e_{21}\|. 
\]

(104)

which shows (iv) directly.

REFERENCES


