Robust Global Exponential Attitude Tracking Controls on SO(3)

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Abstract—Four types of attitude control systems are developed for the rotational dynamics of a rigid body on the special orthogonal group. First, a smooth control system is constructed to track a given desired attitude trajectory, while guaranteeing almost semi-global exponential stability. It is extended to achieve global exponential stability by using a hybrid control scheme based on multiple configuration error functions. They are further extended to obtain robustness with respect to a fixed disturbance using an integral term. The resulting robust global exponential stability for attitude tracking problems on the special orthogonal group is the unique contribution of this paper. These are illustrated by numerical examples.

I. INTRODUCTION

Recent studies on attitude control systems can be summarized into two approaches, namely smooth attitude controls for almost global asymptotic stability [1], [2], and hybrid attitude controls for global asymptotic stability [3], [4]. In the former approaches, the region of attraction only excludes a domain whose measure is zero. This can be considered as the strongest stability property for smooth control systems, given the topological restriction on attitude controls [5]. The latter approaches formulate hybrid attitude control systems with a hysteresis-based switching algorithm to achieve global asymptotic stability, while avoiding chattering. But these results are based on either LaSalle’s principle or hybrid invariance principles, and therefore, they cannot be uniformly applied to attitude tracking problems or they only guarantee asymptotic stability. Furthermore, robustness with respect to uncertainties has not been well addressed.

Attitude control systems can also be categorized with the choice of attitude representation. It is well known that minimal attitude representations, such as Euler angles, suffer from singularities [6]. Quaternions do not have singularities but, as the three-sphere double-covers the special orthogonal group, one attitude may be represented by two antipodal points. This ambiguity should be carefully resolved in quaternion-based attitude control systems, otherwise they may exhibit unwinding [5]. To avoid these, an additional mechanism to lift measurements of attitude onto the unit-quaternion space is introduced [3].

In this paper, we develop four types of attitude control systems to track a given desired attitude trajectory. A smooth attitude control system is developed for almost semi-global exponential stability, and a hybrid control system with a new form of direction-based configuration error functions is introduced for global exponential stability with simpler controller structures. They are extended with a unique integral control term to achieve robust global exponential stability.

The proposed attitude control systems have the following distinct features. First, they provide stronger exponential stability. The attitude control systems in the aforementioned papers rely on the invariance principle, or an exogenous system is introduced to reformulate a tracking problem into stabilization of an autonomous system [3], [4]. As a result, they are applied only to attitude stabilization, or they cannot yield stability properties stronger than asymptotic stability. The stability analysis of this paper is based on Lyapunov stability theorem for time-varying systems to show exponential stability.

Second, a new intuitive form of attitude configuration error functions is introduced to simplify the design of hybrid attitude control systems. Configuration error functions in the prior literature, such as [4] are based on compositions with smooth operations representing stretched rotations, and it is not straightforward to obtain proper controller parameters such as a hysteresis gap for stability. The proposed family of configuration error functions is constructed by comparing the desired directions with the current directions, and they yield an explicit condition on controller parameters for global exponential stability.

Third, a special form of integral term is proposed in the control input for robustness with respect to disturbances. Nonlinear PID-like attitude control systems have been studied in [7], [8], [9], [10]. But, either they have singularities [7], [8], or they are based on the invariance principle that is valid only for attitude stabilization [9], [10]. The robust attitude control systems presented in this paper yield global exponential stability for attitude tracking.

Another distinct feature is that attitude control systems are developed on the special orthogonal group. Therefore, singularities or complexities associated with minimal representations are avoided. Also, the ambiguity of quaternions does not have to be addressed by an additional mechanism to avoid the unwinding. In short, the proposed attitude control systems have simpler controller structures, but they provide stronger stability properties as well as robustness.

This paper is organized as follows. Attitude tracking problem is formulated at Section II, and tracking control systems are developed at Section III without considering disturbances. Robust tracking controllers are introduced at Section IV. Each of Sections III and IV is composed of a smooth controller for almost global attractiveness and a hybrid controller for global attractiveness. They are followed by numerical simulations at Section V. Due to page limit, all of proofs are relegated to [11].
II. Problem Formulation

A. Attitude Dynamics on SO(3)

Consider the attitude dynamics of a rigid body. Define an inertial reference frame and a body-fixed frame. Its configuration manifold is the special orthogonal group:

\[ \text{SO}(3) = \{ R \in \mathbb{R}^{3 \times 3} \mid R^T R = I, \det[R] = 1 \} \]

where a rotation matrix \( R \in \text{SO}(3) \) represents the transformation of a vector from the body-fixed frame to the inertial reference frame. The equations of motion are given by

\[ J \ddot{\Omega} + \Omega \times J\Omega = u + \Delta, \]

\[ \dot{R} = R\hat{\Omega}, \]

where \( J \in \mathbb{R}^{3 \times 3} \) is the inertia matrix, and \( \Omega \in \mathbb{R}^3 \) is the angular velocity represented with respect to the body-fixed frame. The control moment and the unknown, but fixed uncertainty are denoted by \( u \in \mathbb{R}^3 \) and \( \Delta \in \mathbb{R}^3 \), respectively.

At (2), the hat map \( \wedge : \mathbb{R}^3 \to \mathfrak{so}(3) \) represents the transformation of a vector in \( \mathbb{R}^3 \) to a \( 3 \times 3 \) skew-symmetric matrix such that \( \hat{xy} = x \times y \) for any \( x, y \in \mathbb{R}^3 \). The inverse of the hat map is denoted by the vee map \( \vee : \mathfrak{so}(3) \to \mathbb{R}^3 \). Throughout this paper, the dot product of two vectors is denoted by \( x \cdot y = x^T y \) for any \( x, y \in \mathbb{R}^3 \), and the maximum eigenvalue and the minimum eigenvalue of \( J \) are denoted by \( \lambda_M \) and \( \lambda_m \), respectively.

Several properties of the hat and vee maps are summarized as

\[ x \cdot \hat{y}z = y \cdot \hat{x}z, \quad \hat{x}yz = (x \cdot z)y - (x \cdot y)z, \]

\[ \hat{xy} = \hat{yx} = [-y x]^T, \]

\[ \text{tr}(A\hat{x}) = \frac{1}{2}\text{tr}[[A - A^T]^\vee] = -x^T(A - A^T)^\vee, \]

\[ \hat{x}A + A^T\hat{x} = (\{\text{tr}[A] I_{3 \times 3} - A\} x)^\vee, \]

\[ R\hat{x}R^T = (Rx)^\vee, \quad R(x \cdot y) = Rx \times Ry \]

for any \( x, y, z \in \mathbb{R}^3 \), \( A \in \mathbb{R}^{3 \times 3} \) and \( R \in \text{SO}(3) \).

The two-sphere is the manifold of unit-vectors in \( \mathbb{R}^3 \), i.e., \( S^2 = \{ q \in \mathbb{R}^3 \mid ||q|| = 1 \} \). Let \( b_1, b_2 \in S^2 \) be the unit-vectors from the mass center of the rigid body toward two distinct, characteristic points on the rigid body, represented with respect to the body-fixed frame. They may represent the direction of the optical axis for an onboard vision-based sensor, or the direction of solar panel of a spacecraft. Due to the rigid body assumption, the unit-vectors are fixed, i.e., \( b_1 = b_2 = 0 \). Without loss of generality, we assume that \( b_1 \) is normal to \( b_2 \), i.e., \( b_1 \cdot b_2 = 0 \). If \( b_1 \cdot b_2 \neq 0 \), we choose a fictitious direction \( b_3 \) as \( b_3 = \frac{b_1 \times b_2}{||b_1 \times b_2||} \), and rename it as \( b_2 \).

Let \( r_1(t), r_2(t) \in S^2 \) be the representations of \( b_1, b_2 \) with respect to the inertial frame. We have

\[ r_1(t) = R(t)b_1, \quad r_2(t) = R(t)b_2. \]

Using (7) and (2), the kinematic equations for \( r_1, r_2 \) are

\[ \dot{r}_i = R\hat{\Omega}b_i = (R\hat{\Omega})^\vee Rb_i = \omega \times r_i, \]

for \( i \in \{1, 2\} \), where \( \omega = R\hat{\Omega} \in \mathbb{R}^3 \) is the angular velocity of the rigid body, represented with respect to the inertial frame. The attitude kinematic equation (2) can be rewritten as \( \dot{R} = \hat{\omega}R \) from (7).

B. Attitude Tracking Problem

Suppose that a smooth desired attitude trajectory is given by \( R_d(t) \), and it satisfies the following kinematic equation:

\[ \dot{R}_d(t) = \hat{\omega}_d(t)R_d(t), \]

where \( \omega_d \in \mathbb{R}^3 \) is the desired angular velocity expressed in the inertial frame, which is assumed to be bounded.

While it is possible to develop a control system directly in terms of \( R_d \), we transform the desired attitude into the desired directions for \( r_1 \) and \( r_2 \) as follows:

\[ r_{1d} = R_d b_1, \quad r_{2d} = R_d b_2. \]

This formulation is to construct a new form of configuration error functions for hybrid control systems developed later. In some applications, such as antenna pointing of satellites, the desired attitude trajectories are directly described by (11).

From (10) and (11), the desired directions satisfy the following kinematic equation:

\[ \dot{r}_{1d} = \omega_d \times r_{1d}, \quad \dot{r}_{2d} = \omega_d \times r_{2d}, \]

and they are consistent with the rigid body assumption, i.e., \( b_1 \cdot b_2 = r_1 \cdot r_2 = r_{1d} \cdot r_{2d} = 0 \). The goal is to design the control input \( u \) such that the attitude \( \dot{R}(t) \) (or the characteristic directions \( r_1(t), r_2(t) \)) asymptotically tracks the desired value \( R_d(t) \) (or \( r_{1d}(t), r_{2d}(t) \)).

III. Global Attitude Tracking with No Disturbance

In this section, we consider a case where there is no disturbance, i.e., \( \Delta = 0 \). A smooth control system is first developed for almost semi-global exponentially stability, and a hybrid control system with new set of configuration error functions are proposed for global attitude tracking.

A. Almost Global Attitude Tracking

Error variables are defined to represent the difference between the desired directions \( r_{1d}, r_{2d} \) and the current directions \( r_1 = Rb_1, r_2 = Rb_2 \). Define the \( i \)-th configuration error function as

\[ \Psi_i(Rb_i) = \frac{1}{2}||Rb_i - r_{id}||^2 = 1 - Rb_i \cdot r_{id}, \]

which represents \( 1 - \cos \theta_i \), where \( \theta_i \) is the angle between \( Rb_i \) and \( r_{id} \) for \( i \in \{1, 2\} \). Therefore, it is positive definite about \( Rb_i = r_{id} \), and the critical points are given by \( Rb_i = \pm r_{id} \). The derivative of \( \Psi_i \) with respect to \( R \) along the direction of \( \delta R = R\hat{\eta} \) for \( \eta \in \mathbb{R}^3 \) is given by

\[ \frac{d}{d\epsilon}|_{\epsilon=0} \Psi_i(R \exp(\epsilon \hat{\eta}) b_i) = -R\hat{\eta} b_i \cdot r_{id} = \eta \cdot (R^T r_{id} \times b_i). \]

From this, the configuration error vector is defined as

\[ e_{r_i} = R^T r_{id} \times b_i, \]

for \( i \in \{1, 2\} \). For positive constants \( k_1 \neq k_2 \), we also define the complete configuration error function and error vector as

\[ \Psi(R) = k_1 \Psi_1(Rb_1) + k_2 \Psi_2(Rb_2), \]

for \( i \in \{1, 2\} \).
\[ e_r = k_1 e_{r_1} + k_2 e_{r_2}. \]  
\[ e_\Omega = \Omega - R^T \omega_d. \]

The angular velocity error vector is defined as  
\[ e_r = k_1 e_{r_1} + k_2 e_{r_2}. \]  
\[ e_\Omega = \Omega - R^T \omega_d. \]

**Proposition 1:** The error variables (13)-(17), representing the difference between the solution of the equations of motion (1) and (2), and the given desired trajectory (11) with (12), satisfy the following properties, for \( i \in \{1, 2\} \):

(i) \[ \frac{\partial}{\partial t} \Psi_i(R_b) = e_{r_i} \cdot e_\Omega, \text{ and } \dot{\Psi}(R) = e_r \cdot e_\Omega. \]

(ii) \[ \| \dot{e}_{r_i} \| \leq \| e_{\Omega i} \|, \text{ and } \| e_r \| \leq (k_1 + k_2)\| e_\Omega \|. \]

(iii) If \( \Psi \leq \psi < h_1 \), then \( \Psi \) is quadratic with respect to \( \| e_r \| \), i.e., the following inequality is satisfied:

\[ \frac{h_1}{h_2 + h_3} \| e_r \|^2 \leq \psi \leq \frac{h_1 h_4}{h_5 (h_1 - \psi)} \| e_r \|^2, \]  
where constants \( h_1, h_2, h_3, h_4, h_5 \in \mathbb{R} \) are defined as

\[ h_1 = 2 \min\{k_2, k_1\}, \quad h_2 = 4 \max\{(k_1 - k_2)^2, k_2^2, k_1^2\}, \quad h_3 = 4 \max\{(k_1 + k_2)^2, k_2^2, k_1^2\}, \quad h_4 = 2(k_1 + k_2), \quad h_5 = 4 \min\{(k_1 + k_2)^2, k_2^2, k_1^2\}. \]

**Proof:** See [11].

Using these properties, we develop a control system to follow the given desired trajectory as follows.

**Proposition 2:** Consider the dynamic system (1), (2) with \( \Delta = 0 \). A desired trajectory is given by (12). For \( k_1, k_2, k_\omega > 0 \) with \( k_1 \neq k_2 \) a control input is chosen as

\[ u = -e_r - k_\Omega e_\Omega + (R^T \omega_d) \wedge JR^T \omega_d + JR^T \dot{\omega}_d. \]

Then, the following properties hold:

(i) There are four equilibrium configurations for \( (R_\omega, \omega) \), given by \{ \( (R_d, \omega_d) \), \( (\exp(\pi r_{1d}) R_d, \omega_d) \), \( (\exp(\pi r_{2d}) R_d, \omega_d) \), \( (\exp(\pi (r_{1d} \times r_{2d})) R_d, \omega_d) \) \}.

(ii) The desired equilibrium \( (R_d, \omega_d) \) is almost globally asymptotically stable and almost semi-globally exponentially stable, where an estimate to the region of the initial conditions for exponential stability is given by

\[ \Psi(R(0)) \leq \psi < 2 \min\{k_2, k_1\}, \]  
\[ \| e_\Omega(0) \|^2 \leq \frac{2}{\lambda_M} (\psi - \Psi(R(0))). \]

(iii) The three undesired equilibria are unstable.

**Proof:** See [11].

Compares with other attitude control systems achieving almost global asymptotic stability for attitude stabilization of time-invariant systems on \( SO(3) \), such as [2], this proposition guarantees stronger almost semi-globally exponential stability for attitude tracking of time-varying systems. Unlike attitude control systems based on quaternions, such as [13], the proposed control system is directly developed on \( SO(3) \). Therefore, it completely avoids unwinding behaviors of quaternion based attitude control systems [5].

The fact that the region of attraction does not cover the entire configuration manifold globally is not a major issue in practice, as the probability that a given initial condition exactly lies in the stable manifolds to the unstable equilibria is zero, provided that the initial condition is randomly chosen. But, the existence of such stable manifolds may have strong effects on the dynamics of the controlled system [14]. In particular, the proportional terms of the control input, namely \( e_{r_1}, e_{r_2} \) approach zero as the attitude becomes closer to one of the three undesired equilibria, thereby causing a slow convergence rate for large attitude errors. In the following subsection, discontinuities are introduced in the control input to improve convergence rates, in addition to obtaining global exponential stability.

**B. Hybrid Control for Global Attitude Tracking**

Recently, hybrid control systems for global attitude stabilization are developed in terms of quaternions [3], and rotation matrices on \( SO(3) \) [4]. The key idea is switching between different form of configuration error functions, referred to as synergistic potential functions, such that the attitude is expelled from the vicinity of undesired equilibria. The switching logic is defined with a hysteresis model to improve robustness with respect to measurement noises. This paper follows the same framework, but a new form of synergistic configuration error functions is provided to simply controller structures and controller design procedure.

The given control system also provides stronger global exponential stability that is uniformly applied to time-varying systems for tracking problems.

For control analysis, we introduce a mathematical formulation of hybrid systems [15]. Let \( \mathcal{M} \) be the set of discrete modes, and let \( \mathcal{Q} \) be the domain of continuous states. Given a state \( (m, \xi) \in \mathcal{M} \times \mathcal{Q} \), a hybrid system is defined by

\[ \dot{\xi} = F(m, \xi), \quad (m, \xi) \in \mathcal{C}, \]  
\[ m^+ = G(m, \xi), \quad (m, \xi) \in \mathcal{D}, \]

where the flow map \( F : \mathcal{M} \times \mathcal{Q} \to \mathcal{Q} \) describes the evolution of the continuous state \( \xi \); the flow set \( \mathcal{C} \subset \mathcal{M} \times \mathbb{R}^n \) defines where the continuous state evolves; the jump map \( G : \mathcal{M} \to \mathcal{M} \) governs the discrete dynamics; the jump set \( \mathcal{D} \subset \mathcal{M} \times \mathcal{Q} \) defines where discrete jumps are permitted. This represents a class of hybrid systems, where the continuous state does not change over jumps.

For the presented hybrid attitude control system, there are a nominal mode and two expelling modes. The control input at the nominal mode is equal to (19) which is constructed by the following configuration error functions given at (13):

\[ \Psi_{N_1}(R_b) = 1 - R_b \cdot r_{1d}, \]  
\[ \Psi_{N_2}(R_b) = 1 - R_b \cdot r_{2d}, \]

where the subscripts \( N \) are used to explicitly denote that the configuration error functions are for the nominal mode, i.e., \( \Psi_{N_i} \triangleq \Psi_i \). When the attitude becomes closer to the critical points of the nominal configuration error functions, they are switched to the following expelling configuration error functions:

\[ \Psi_{E_1}(R_b) = \alpha + \beta R_b \cdot (r_{1d} \times r_{2d}), \]  
\[ \Psi_{E_2}(R_b) = \alpha + \beta R_b \cdot (r_{1d} \times r_{2d}), \]

for constant \( \alpha, \beta \) satisfying \( 1 < \alpha < 2 \) and \( |\beta| < \alpha - 1 \).
For example, if the attitude becomes close to the critical point of the first nominal error function $\Psi_{N_1}$, where $Rb_1 = -r_{1,d}$, the expelling configuration error $\Psi_{E_1}$ is engaged such that $Rb_1$ is steered towards a direction normal to $-r_{1,d}$, namely $-\frac{\beta}{2}(r_{1,d} \times r_{2,d})$, to rotate the rigid body away from the undesired critical point. Similarly, the second expelling configuration error function $\Psi_{E_2}$ is engaged near the critical points of the second nominal configuration error function $\Psi_{N_2}$. As a result, there are three discrete modes, namely $\mathcal{M} = \{I, II, III\}$, and the configuration error function for each mode is given by

$$
\Psi_I(R) = k_1 \Psi_{N_1}(Rb_1) + k_2 \Psi_{N_2}(Rb_2),
$$

$$
\Psi_{II}(R) = k_1 \Psi_{N_1}(Rb_1) + k_2 \Psi_{E_2}(Rb_2),
$$

$$
\Psi_{III}(R) = k_1 \Psi_{E_1}(Rb_1) + k_2 \Psi_{N_2}(Rb_2).
$$

In short, the nominal control input is constructed from the nominal error function $\Psi_I$. If the attitude is in the vicinity of the undesired critical points of $\Psi_{N_1}$ or $\Psi_{N_2}$, the control input is switched into the mode $II$ or $II$, respectively.

The switching logic is formally specified as follows. Define a variable $\rho$ representing the minimum configuration error for the given attitude:

$$
\rho(R) = \min_{m \in \mathcal{M}} \{\Psi_m(R)\}. \tag{31}
$$

The discrete mode is switched to avoid the undesired critical points of the nominal configuration error functions $\Psi_{N_1}, \Psi_{N_2}$. Observing that the values of $\Psi_{N_1}, \Psi_{N_2}$ are maximized at the undesired critical points, the jump map is chosen such that the discrete mode is switched to the new mode where the configuration error is minimum:

$$
G(R) = \arg\min_{m \in \mathcal{M}} \{\Psi_m(R)\} = \{m \in \mathcal{M} : \Psi_m = \rho\}. \tag{32}
$$

It is possible to switch whenever a new discrete mode with a smaller value of configuration error function is available, or equivalently, when $G(q) \neq q$ or $\Psi_m - \rho > 0$. But, the resulting controlled system may yield chattering due to measurement noises. Instead, a hysteresis gap $\delta$ is introduced for robustness, and a switching occurs if the difference between the current configuration error and the minimum value is greater than the hysteresis gap. More explicitly, the jump set and the flow set are given by

$$
D = \{(R, \Omega, m) : \Psi_m - \rho \geq \delta\}, \tag{33}
$$

$$
C = \{(R, \Omega, m) : \Psi_m - \rho \leq \delta\}, \tag{34}
$$

for a positive constant $\delta$ that is specified later at (39).

The control input at each mode is constructed from the corresponding configuration error function by following the same procedure described in the previous section:

$$
u = -e_H - k_1 e_\Omega + (R^T \omega_d) \times J R^T \Omega + J R^T \omega_d. \tag{35}\$$

where the hybrid configuration error vectors are defined as

$$
e_H = k_1 e_{H_1} + k_2 e_{H_2}, \tag{36}\$$

$$
e_{H_1} = \begin{cases} e_r = R^T r_{1,d} \times b_1 & \text{if } m = I, II, \\ -\beta R^T (r_{1,d} \times r_{2,d}) \times b_1 & \text{if } m = III, \end{cases} \tag{37}\$$

$$
e_{H_2} = \begin{cases} e_r = R^T r_{2,d} \times b_2 & \text{if } m = I, III, \\ -\beta R^T (r_{1,d} \times r_{2,d}) \times b_2 & \text{if } m = II. \end{cases} \tag{38}\$$

**Proposition 3:** Consider a hybrid control system defined by (28)-(38). For given constants $k_1, k_2, \alpha, \beta$ satisfying $k_1, k_2 > 0$, $k_1 \neq k_2$, $1 < \alpha < 2$ and $|\beta| < \alpha - 1$, choose the hysteresis gap $\delta$ such that

$$0 < \delta < \min\{k_1, k_2\} \min\{2 - \alpha, \alpha - |\beta| - 1\}. \tag{39}\$$

Then, the desired equilibrium $(R_d, \omega_d)$ is globally exponentially stable.

**Proof:** See [11].

The unique feature of the proposition is that it provides a stronger global exponential stability for a tracking problem on $SO(3)$, compared with the existing results in [4] yielding global asymptotic stability. Another interesting feature is that the construction of the expelling configuration error functions are simpler. In [4], an expelling configuration error function is constructed by *angular wrapping*, where the nominal configuration error function is composed with a diffeomorphism that represents stretched rotations. The resulting control system design involves nontrivial derivatives and it is relatively difficult to compute the required hysteresis gap $\delta$, that is required to implement the given hybrid controller.

In this paper, the construction of expelling configuration error function at (26), (27) is intuitive and straightforward as they are based on the comparison between the desired directions and the actual directions. It is not required that the critical points of $\Psi_{E_1}, \Psi_{E_2}$ include the desired equilibrium. The proposed simpler form of the expelling configuration error functions yields the explicit expression for the hysteresis gap as given at (39). For example, at (39), the upper bound of $\delta$ is maximized along the line of $2\alpha - |\beta| - 3 = 0$ to yield $\delta = 2 - \alpha$. If $\alpha = 1.6$, then we can simply choose $\beta = 0.2$ to obtain $\delta = 0.4$. In short, the presented control system provide a stronger global exponential stability, with simpler controller design procedure.

IV. ROBUST GLOBAL ATTITUDE TRACKING

In this section, we consider a case where there exists unknown, but fixed disturbance $\Delta$. Both of smooth and hybrid attitude control systems are constructed.

A. Robust Almost Global Attitude Tracking

It is well known that steady state errors due to fixed disturbances can be attenuated by an integral term of the error. In this paper, we consider the following integral term,

$$e_I(t) = \int_0^t c e_r(\tau) + e_\Omega(\tau) d\tau, \tag{40}\$$

for a positive constant $c$, or equivalently, it satisfies $\dot{e}_I = c e_r + e_\Omega$ with $e_I(0) = 0$. The smooth attitude control system developed at Section III-A, can be extended with the integral term for robustness as follows.
Proposition 4: Consider the dynamic system (1), (2) on SO(3). A desired trajectory is given by (12). For \( k_1, k_2, k_1, k_1 > 0 \) with \( k_1 \neq k_2 \) a control input is chosen as
\[
\dot{u} = -e_r - k_1 e_\Omega - k_1 e_\Omega + (R^T \dot{\omega}_d)^\wedge J R^T \dot{\omega}_d + J R^T \dot{\omega}_d. 
\] (41)

Then, the controlled system satisfies the following properties:

(i) There are four equilibrium configurations for \((R, \Omega)\), given by the property (i) of Proposition 2.

(ii) The desired equilibrium \((R_d, \omega_d)\) is almost globally asymptotically stable, and locally exponentially stable with respect to \(e_r, e_\Omega\). The integral term \(e_I\) is globally uniformly bounded.

(iii) The three undesired equilibria are unstable.


Nonlinear PID-like controllers have been developed for attitude stabilization in terms of modified Rodriguez parameters [7] and quaternions [9], and for attitude tracking in terms of Euler-angles [8]. The proposed control system is developed on SO(3), therefore it avoids singularities of Euler-angles and Rodriguez parameters, as well as unwinding of quaternions. It also provides almost global asymptotic stability for attitude tracking problems with fixed uncertainties.

The proposed integral term is unique in the sense that the angular velocity error term is also integrated at (40). As the time-derivative of the error vector, namely \(e_r\), is linear with respect to the angular velocity error \(e_\Omega\), this has the effects of increasing the proportional term of the control input. Effectively, the proportional gain of the control input is given by \( k_1, k_2 \) multiplied by \(1 + c k_I\), and the integral gain of the control input is given by \(ck_I\). The constant \(c\) essentially determines the ratio of the integral gain to the proportional gain.

One of the desirable properties of the proposed control input with an integral term is that the resulting time-derivative of the Lyapunov function is exactly same as the attitude tracking system given in the previous section without disturbances. Therefore, the guaranteed performance of the controlled system, such as convergence rates, is not affected by fixed disturbances, and stability proof becomes simpler. The proposed form of the integral term may be used for other second-order mechanical systems.

B. Robust Global Attitude Tracking

The proposed attitude control system with an integral term can be further developed into a hybrid control system to achieve robust global exponential stability. The integral term is redefined in terms of the hybrid error vector (36) as
\[
e_{I_H}(t) = \int_0^t c e_H(\tau) + e_\Omega(\tau) d\tau, \tag{42}
\]
for a positive constant \(c\), and the jump set and the flow set are revised as
\[
\mathcal{D}' = \{(R, \Omega, m) : \Psi_m - \rho \geq \delta \text{ or } \|e_\Omega\| \geq \frac{\delta}{4ck}\}, \tag{43}
\]
where \(k = k_1 + k_2\). The control input is chosen as
\[
u = -e_H - k_1 e_\Omega - k_1 e_\Omega + (R^T \dot{\omega}_d)^\wedge J R^T \dot{\omega}_d + J R^T \dot{\omega}_d. \tag{45}
\]
The other parts of the hybrid control system, such as the configuration error functions and the jump map are identical to Section III-B.

Proposition 5: Consider a hybrid control system defined by (28)-(32), (36)-(38), and (42)-(44). For given constants \(k_1, k_2, \alpha, \beta\) satisfying \(k_1, k_2 > 0, k_1 \neq k_2, 1 < \alpha < 2\) and \(|\beta| < \alpha - 1\), choose the hysteresis gap \(\delta\) such that (39) is satisfied. Then, the desired equilibrium \((R_d, \omega_d)\) is globally exponentially stable with respect to \(e_H\) and \(e_\Omega\), and the integral term \(e_{I_H}\) is globally uniformly bounded.


Global asymptotic stability is achieved for an attitude control system with an integral term in terms of quaternions, based on LaSalle’s principle [10]. To the best knowledge of the author, the given robust global exponential stability for attitude tracking, developed in terms of global attitude representation without need for addressing unwinding, has been unprecedented.

V. NUMERICAL EXAMPLES

Consider a rigid body whose inertia matrix is given by \(J = \text{diag}[3,2,1] \text{kgm}^2\). The desired attitude command is specified as \(R_d(t) = \exp(\psi(t)e_3) \exp(\theta(t)e_2) \exp(\phi(t)e_1)\) in terms of 3-2-1 Euler-angles, where \(\phi(t) = \sin 0.5t, \theta(t) = 0.1(-1 + t), \psi(t) = 1 - \cos t\). The controller parameters are chosen as \(b_1 = [1,0,0]^T, b_2 = [0,1,0]^T, \alpha = 1.9, \beta = 0.8, \delta = 1, k_1 = 10, k_2 = 11, k_\Omega = 4.42, k_I = 0.5\).

The following three cases are considered.

(a) Attitude tracking error \(\|R - R_d\|\)
(b) Attitude error vector \(e_H\)
(c) Angular velocity error \(e_\Omega\)
(d) Control input \(u\)

Fig. 1. Case (I): Small initial attitude error without disturbances (blue, solid: smooth controller, red, dashed: hybrid controller)
Case (I): It is assumed that there is no disturbance, i.e., \( \Delta = 0 \), and the initial conditions are chosen as \( \Psi(0) = I \) and \( \Omega(0) = 0 \). This corresponds to a small initial attitude error, where \( \Psi(0) = 0.05 \). The simulation results for the smooth control system and the hybrid control system without an integral term, developed at Propositions 2 and 3 respectively, are illustrated at Figure 1. They exhibit good tracking performances. As the initial attitude error is small, no jump occurs at the hybrid control system, and the corresponding responses of the hybrid control system are identical to the smooth control system.

Case (II): The second case is same as Case (I), except the initial condition chosen as \( \Psi(0) = \exp(0.9999\pi(r_{1,2} \times r_{2,3}))R_d(0) \), \( \Omega(0) = R(0)^T\omega_d(0) \), which is close to one of the undesired equilibrium. In this case, there is noticeable difference between the smooth controller and the hybrid controller, as illustrated at Figure 2. For the smooth controller, the attitude tracking error does not change until after \( t = 12 \) seconds (while it is not illustrated at the figure, the error eventually converges to zero after \( t = 15 \)). The convergence rate of the hybrid control system is significantly faster.

Case (III): The initial condition is identical to Case (II), representing a large initial attitude error. In this case, a fixed disturbance of \( \Delta = [-1, 2, 1]^T \) is added. Figure 3 shows numerical results for the hybrid control system presented at Proposition 3, and the hybrid control system with an integral term presented at Proposition 5. The given fixed disturbance causes steady-state tracking errors for the hybrid control system developed at Proposition 3, but those errors are completely eliminated by the integral term of the hybrid control system developed at Proposition 5. It also exhibits good convergence properties for the given large initial attitude error, which are comparable to the hybrid control system without disturbances illustrated at Figure 2.

Fig. 2. Case (II): Large initial attitude error without disturbances (blue, solid: smooth controller, red, dashed: hybrid controller)

Fig. 3. Case (II): Large initial attitude error with disturbances (red, dashed: hybrid controller, black, solid: hybrid controller with an integral term)

REFERENCES