Input Optimization for Multi-Antenna Broadcast Channels with Per-Antenna Power Constraints

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***Abstract*— This paper considers a Gaussian multi-antenna broadcast channel with individual power constraints on each antenna, rather than the usual sum power constraint over all antennas. Per-antenna power constraints are more realistic because in practical implementations each antenna has its own power amplifier. The main contribution of this paper is a new derivation of the duality result for this class of broadcast channels that allows the input optimization problem to be solved efficiently. Specifically, we show that uplink-downlink duality is equivalent to Lagrangian duality in minimax optimization, and the dual multiple-access problem has a much lower computational complexity than the original problem. This duality applies to the entire capacity region. Further, we derive a novel application of Newton's method for the dual minimax problem that finds an optimal search direction for both the minimization and the maximization problems at the same time. This new computational method is much more efficient than the previous iterative water-filling-based algorithms and it is applicable to the entire capacity region. Finally, we show that the previous QR-based precoding method can be easily modified to accommodate the per-antenna constraint.**

I. Introduction

A downlink transmission scenario with a base-station equipped with n transmit antennas and K remote users, each equipped with m_k receive antennas, is often modelled as a Gaussian vector broadcast channel:

$$Y_i = H_i X + Z_i, \quad i = 1, \ldots, K, \tag{1}$$

where X is an $n \times 1$ vector and H_i's are $m_i \times n$ channel matrices, and Z_i are additive white Gaussian noise. A sum power constraint is usually imposed on the transmitter

$$\mathbf{E} X^T X \leq P. \tag{2}$$

Recently, a great deal of progress has been made in characterizing its capacity region. Under a sum power constraint, Caire and Shamai [1] showed that the so-called dirty-paper precoding strategy [2] is optimal for the broadcast channel with two transmit antennas. This result has since been generalized using several different approaches to broadcast channel with an arbitrary number of users and an arbitrary number of antennas [4] [5] [6]. In particular, [4] showed that the precoding strategy is equivalent to successive decoding in a generalized decision-feedback equalizer (GDFE) and the sum capacity can be characterized by a minimax expression, while independent work [5] [6] showed that the broadcast channel dirty-paper region can be found via solving a dual multiple-access channel. The dirty-paper region is then conjectured to be indeed the capacity region of MIMO broadcast channel. Toward this end, Weingarten, Steinberg and Shamai [7] introduced a new notion of enhanced broadcast channels and finally substantiated this conjecture. The sum capacity results in [4] and that in [5] and [6], although equivalent, also have subtle differences. The GDFE approach [4] is applicable to broadcast channel with arbitrary input constraints, but only sum capacity is solved. The alternative approach in [5] [6] establishing the duality of broadcast channel dirty-paper region and reciprocal multiple-access channel capacity region, appears to be applicable to broadcast channels with a sum power constraint only. On the other hand, the dual multiple-access channel is much more amendable to numerical computation. So, solving the dual problem gives an efficient numerical solution for the original minimax problem.

The duality of uplink and downlink channels is no accident. One of the main objectives of this paper is to show that uplink-downlink duality is in fact equivalent to Lagrangian duality in optimization. This approach is different from the approach in [9], which is based on a manipulation of the KKT conditions for the minimax problem. This new viewpoint allows the duality theory to be extended beyond the sum power constrained channels. In particular, we focus on a practical scenario where an individual power constraint needs to be satisfied at each transmit antenna:

$$\mathbf{E} X(i) X(i) \leq P_i \ , \ \text{for } i = 1, \ldots, n. \tag{3}$$

where $X(i)$ is the ith component of vector X. The per-antenna power constraint arises from the power consumption limits of physical amplifiers at the transmitter and is more realistic in practical implementations for both wireless and wired applications.

The main results of the paper are as follows. First, a new duality result for broadcast channels with per-antenna power constraints is derived. The new derivation is based on Lagrangian duality in minimax optimization, and uplink-downlink duality is equivalent to convex duality. It turns out that the Lagrangian dual is also a minimax problem, which cor-

[0]This work was supported by Bell Canada University Laboratories, Communications and Information Technology Ontario (CITO), Connaught Fund, Natural Sciences and Engineering Council (NSERC) of Canada, and by the Canada Research Chairs program.

responds to a multiple-access channel with linearly constrained noise. This duality result also generalizes to entire capacity region. Second, we observe that the dual minimax problem is convex-concave and lies in a much lower dimensional space. Thus, the input optimization problem can be much more efficiently solved in the dual domain. Toward this end, we apply a novel form of Newton's method for minimax optimization [8] that finds an efficient search direction for both the maximization and the minimization problems at the same time, as it approaches the minimax saddle-point directly. The new algorithm not only outperforms previous iterative water-filling based approaches [10] [11], it also admits a rigorous convergence analysis. Finally, we apply the new duality result to the downlink transmission channel with high SNR and show that the previous QR-based approach can easily be modified to accommodate the per-antenna power constraint.

II. Duality with Per-Antenna Power Constraints

The sum capacity of a Gaussian vector broadcast channel is the saddle-point of a minimax mutual information expression, where the maximization is over all transmit covariance matrices subject to the input constraint and the minimization is over all noise covariance matrices subject to a diagonal constraint [3] [4]. Under the per-antenna power constraint (3), the sum capacity minimax problem is as follows:

$$\begin{aligned} C = \quad & \min_{S_z} \max_{S_x} \frac{1}{2} \log \frac{|HS_xH^T + S_z|}{|S_z|}. && (4)\\ \text{s.t.} \quad & S_x^{(i,i)} \le P_i \ , \ \text{for } i = 1, \ldots, n. \\ & S_z^{(i,i)} = I_{m_i \times m_i} \ , \ \text{for } i = 1, \ldots, K. \end{aligned}$$

where $H = [H_1^T, \cdots, H_K^T]^T$ and the transmit and noise covariance matrices S_x and S_z also have to be positive semi-definite.

The main result of this section is a derivation of a dual multiple-access channel for this broadcast channel. We use a novel approach that shows that uplink-downlink duality is equivalent to Lagrangian duality in convex optimization. It also provides a natural generalization for the previous duality results on sum power constrained broadcast channels [5] [6].

Theorem 1: The capacity region of a MIMO Gaussian broadcast channel with per-antenna power constraints $(P_1, \ldots, P_n)$ is the same as the capacity region of a dual multiple-access channel with uncertain noise, where the input covariance is constrained by (6) below and the noise covariance is constrained by (8) below. In particular, for the sum capacity point, the Lagrangian dual of the minimax problem (4) is the following minimax problem:

$$\begin{aligned} & \max_{\Sigma_x} \min_{\Sigma_z} \quad \frac{1}{2} \log \frac{|H^T \Sigma_x H + \Sigma_z|}{|\Sigma_z|}. && (5)\\ & \text{s.t.} \quad \text{tr}(\Sigma_x) \le 1, && (6)\\ & \qquad \Sigma_x \ \text{ is block diagonal}, \ \Sigma_x \ge 0 && (7)\\ & \qquad \text{tr}(\Sigma_z P) \le 1, && (8)\\ & \qquad \Sigma_z \ \text{ is diagonal}, \ \Sigma_z \ge 0 && (9) \end{aligned}$$

where P is a diagonal matrix $\text{diag}(P_1, \cdots, P_n)$.

Proof: For simplicity, we will only focus on sum capacity in this paper. However, the duality result applies to entire capacity region. Suppose (S_x,S_z) to be the optimal solution for problem (4), which is not full rank. H is not necessarily invertible. The first step in deriving the duality is to factorize

$$P^{-\frac{1}{2}} S_x P^{-\frac{1}{2}} = U_1^T S_{\tilde{x}} U_1 \quad \text{and} \quad S_z = U_2^T S_{\tilde{z}} U_2, \qquad (10)$$

where $S_{\tilde{x}}$ and $S_{\tilde{z}}$ are full rank matrices, U_1 and U_2 are matrices consisting of orthonormal row vectors. ($U_1 U_1^T = I$ and $U_2 U_2^T = I$) Then we rewrite optimization problem (4) to a minimax problem on $S_{\tilde{x}}$ and $S_{\tilde{z}}$ with normalized individual power constraints:

$$\begin{aligned} C = \quad & \min_{S_{\tilde{x}}} \max_{S_{\tilde{z}}} \frac{1}{2} \log \frac{|\tilde{H} S_{\tilde{x}} \tilde{H}^T + S_{\tilde{z}}|}{|S_{\tilde{z}}|}. && (11)\\ \text{s.t.} \quad & (U_1^T S_{\tilde{x}} U_1)^{(i,i)} \le 1 \ , \ \text{for } i = 1, \ldots, n. \\ & (U_2^T S_{\tilde{z}} U_2)^{(i,i)} = I_{m_i \times m_i} \ , \ \text{for } i = 1, \ldots, K. \end{aligned}$$

where $H = U_2^T \tilde{H} U_1 P^{\frac{1}{2}}$ and $\tilde{H}$ is the equivalent channel. It is not difficult to see that optimization problems (4) and (11) are equivalent. Further, $S_{\tilde{x}}$ and $S_{\tilde{z}}$ have the same rank and $\tilde{H}$ is square and invertible. (Refer to [12] for details.)

With this observation, the Lagrangian dual is derived in two stages. First, form the Lagrangian for the maximization part of (11):

$$L(S_{\tilde{x}}, Q) = \log \frac{|\tilde{H} S_{\tilde{x}} \tilde{H}^T + S_{\tilde{z}}|}{|S_{\tilde{z}}|} - \text{tr}[Q(U_1^T S_{\tilde{x}} U_1 - I)]. \quad (12)$$

where the dual variable Q is a diagonal matrix with non-negative entries. (For simplicity, the coefficient $\frac{1}{2}$ is omitted throughout the derivation.) The dual objective is therefore

$$g(Q) = \max_{S_{\tilde{x}}} L(S_{\tilde{x}}, Q). \qquad (13)$$

where the optimization is over the constraint set $S_{\tilde{x}} \ge 0$. At optimum, $\partial L / \partial S_{\tilde{x}} = 0$. Thus

$$\tilde{H}^T (\tilde{H} S_{\tilde{x}} \tilde{H}^T + S_{\tilde{x}})^{-1} \tilde{H} - U_1 Q U_1^T = 0. \qquad (14)$$

Let $\tilde{Q} = U_1 Q U_1^T$. We claim that Lagrangian multiplier $\tilde{Q}$ is full rank, otherwise some diagonal components of $S_{\tilde{x}}$ become unbounded. We can solve for $S_{\tilde{x}}$:

$$S_{\tilde{x}} = \tilde{Q}^{-1} - \tilde{H}^{-1} S_{\tilde{z}} \tilde{H}^{-T}. \qquad (15)$$

Substitute this into (12), we get the expression for $g(\tilde{Q})$. Further, since strong duality holds, the dual objective reaches a minimum at the optimal value of the primal problem. Thus, maximization problem (11) has a equivalent dual problem $\min_{\tilde{Q}} g(\tilde{Q})$ as follows, in the sense of Lagrangian duality.

$$\begin{aligned} \min_{\tilde{Q}} \quad & -\log|\tilde{H}^{-T} \tilde{Q} \tilde{H}^{-1}| + \text{tr}(\tilde{Q}) - m \\ & + \text{tr}(S_{\tilde{z}} \tilde{H}^{-T} \tilde{Q} \tilde{H}^{-1}) - \log|S_{\tilde{z}}|. && (16)\\ \text{s.t.} \quad & U_1^T \tilde{Q} U_1 \ \text{ is diagonal and } \tilde{Q} \ge 0. \end{aligned}$$

where m is the total number of receive antennas. The dual minimization problem above is easier to solve numerically.

The minimax problem (4) now becomes a double-minimization problem consisting of a dual minimization on $\tilde{Q}$ and the original minimization on $S_{\tilde{z}}$:

$$\begin{aligned} \min_{\tilde{Q}} \min_{S_{\tilde{z}}} \quad & -\log|\tilde{H}^{-T}\tilde{Q}\tilde{H}^{-1}| + \mathrm{tr}(\tilde{Q}) - m \\ & +\mathrm{tr}(S_{\tilde{z}}\tilde{H}^{-T}\tilde{Q}\tilde{H}^{-1}) - \log|S_{\tilde{z}}|. \quad (17)\\ s.t. \quad & U_1^T\tilde{Q}U_1 \text{ is diagonal}, \quad \tilde{Q} \geq 0. \\ & (U_2^T S_{\tilde{z}} U_2)^{(i,i)} = I_{m_i \times m_i}, \text{ for } i = 1, \ldots, K. \end{aligned}$$

The next step is to use the same procedure to find the dual problem for the minimization with respect to constrained $S_{\tilde{z}}$. Considering only the terms containing $S_{\tilde{z}}$:

$$\begin{aligned} \min_{S_{\tilde{z}}} \quad & -\log|S_{\tilde{z}}| + \mathrm{tr}(S_{\tilde{z}}\tilde{H}^{-T}\tilde{Q}\tilde{H}^{-1}). \quad (18)\\ \text{s.t.} \quad & (U_2^T S_{\tilde{z}} U_2)^{(i,i)} = I_{m_i \times m_i}, \text{ for } i = 1, \ldots, K. \end{aligned}$$

Form its Lagrangian:

$$\begin{aligned} L(S_{\tilde{z}}, \Phi_1, \ldots, \Phi_K) = \quad & -\log|S_{\tilde{z}}| + \mathrm{tr}(S_{\tilde{z}}\tilde{H}^{-T}\tilde{Q}\tilde{H}^{-1}) \\ & + \sum_{i=1}^{K} \mathrm{tr}[\Phi_i((U_2^T S_{\tilde{z}} U_2)^{(i,i)} - I)]. \end{aligned}$$

where Φ_i's are the dual variables associated with diagonal constraints on $U_2^T S_{\tilde{z}} U_2$ and Φ_i's are $m_i \times m_i$ positive semi-definite matrices. Let $\tilde{\Phi} = U_2 \mathrm{diag}(\Phi_1, \ldots, \Phi_K) U_2^T$. The optimality condition for S_z is then

$$\partial L / \partial S_{\tilde{z}} = \tilde{H}^{-T}\tilde{Q}\tilde{H}^{-1} - S_{\tilde{z}}^{-1} + \tilde{\Phi} = 0. \quad (19)$$

The dual problem for the primal minimization is therefore

$$\begin{aligned} \max_{\tilde{\Phi}} \quad & \log|\tilde{H}^{-T}\tilde{Q}\tilde{H}^{-1} + \tilde{\Phi}| + m - \mathrm{tr}(\tilde{\Phi}). \quad (20)\\ \text{s.t.} \quad & U_2^T\tilde{\Phi}U_2 \text{ is block diagonal}, \quad \tilde{\Phi} \geq 0. \end{aligned}$$

Again because of the strong duality, the dual problem achieves a maximum value at the minimum value of the primal objective. Replacing the minimization in (17) by its above dual, we derive the following Lagrangian dual problem for the minimax problem (4):

$$\begin{aligned} \max_{\tilde{\Phi}} \min_{\tilde{Q}} \quad & \log\frac{|\tilde{H}^T\tilde{\Phi}\tilde{H} + \tilde{Q}|}{|\tilde{Q}|} + \mathrm{tr}(\tilde{Q}) - \mathrm{tr}(\tilde{\Phi}) \\ \text{s.t.} \quad & U_2^T\tilde{\Phi}U_2 \text{ is block diagonal}, \tilde{\Phi} \geq 0. \\ & U_1^T\tilde{Q}U_1 \text{ is diagonal}, \quad \tilde{Q} \geq 0. \quad (21) \end{aligned}$$

To derive the uplink-downlink duality under per-antenna power constraint, the above result need to be further modified. The key is to introduce another optimization variable λ,

$$\begin{aligned} \max_{\tilde{\Phi}} \max_{\lambda} \min_{\tilde{Q}} \quad & \log\frac{|\tilde{H}^T\tilde{\Phi}\tilde{H} + \tilde{Q}|}{|\tilde{Q}|} + \mathrm{tr}(\tilde{Q}) - \lambda. \\ \text{s.t.} \quad & U_2^T\tilde{\Phi}U_2 \text{ is block diagonal}, \quad tr(\tilde{\Phi}) \leq \lambda. \\ & \lambda \geq 0 \\ & U_1^T\tilde{Q}U_1 \text{ is diagonal}, \quad \tilde{Q} \geq 0. \quad (22) \end{aligned}$$

Finally, consider $\Sigma_x = \frac{1}{\lambda}U_2^T\tilde{\Phi}U_2$, $\Sigma_z = \frac{1}{\lambda}P^{-\frac{1}{2}}U_1^T\tilde{Q}U_1P^{-\frac{1}{2}}$ and λ as the new optimization variable. The optimization problem becomes

$$\begin{aligned} \max_{\Sigma_x} \max_{\lambda} \min_{\Sigma_z} \quad & \log\frac{|H^T\Sigma_x H + \Sigma_z|}{|\Sigma_z|} + \lambda[\mathrm{tr}(\Sigma_z P) - 1]. \\ \text{s.t.} \quad & \Sigma_x \text{ is block diagonal}, \quad tr(\Sigma_x) \leq 1. \\ & \lambda \geq 0 \\ & \Sigma_z \text{ is diagonal}, \quad \Sigma_z \geq 0. \quad (23) \end{aligned}$$

Note that the minimization with respect to Σ_z and the maximization with respect to λ can be combined. In fact, this minimax is just the dual for primal maximization problem on Σ_z with the inequality constraint $\mathrm{tr}(\Sigma_z P) \leq 1$, with λ serves as the dual variable. Again, by strong duality, we can replace the dual problem with the primal problem. This establishes (5) □

Because Σ_x is diagonal, (5) represents the sum capacity of a multiple access channel. The dual multiple access channel has a sum power constraint across all the transmitters and a linear constraint on the diagonal noise. This is a generalization of previous result on broadcast channel with sum power constraint [5] [6]. Further, the input optimization for the broadcast channel can therefore be solved via its dual channel.

III. Input Optimization via Duality

The dual minimax problem is convex-concave and lies in a much lower dimensional space as compared to the original problem. Thus, it is easier to solve. The goal of this section is to derive a Newton's method to compute the capacity region of the dual multiple-access channel. We will proceed with the minimax problem (5) of sum capacity, however, all results directly apply to the entire capacity region. The proposed method differs from the usual Newton's method in two crucial aspects. First, as opposed to solving a pure maximization or a pure minimization problem, the minimax problem demands the maximization and the minimization to be done at the same time. Second, we recognize that the inequality trace constraints in (5) can be replaced by equality constraints. This greatly speeds up the computation. The methods proposed in this section draw heavily from materials in the optimization literature [8].

As a first step, let's assume for now that the optimal Σ_x and Σ_z are strictly positive definite so that the constraints $\Sigma_z \geq 0$ and $\Sigma_x \geq 0$ are superfluous. This situation corresponds to a broadcast channel in which the number of transmit antennas is about the same as the total number of receive antennas. The positivity constraints are always superfluous, for example, in a digital subscriber line application.

Then, write Σ_x and Σ_z as vectors, call them $\Sigma_x^{(v)}$ and $\Sigma_z^{(v)}$, respectively. This is automatic for $\Sigma_z^{(v)}$, which is already diagonal. For block-diagonal $\Sigma_x^{(v)}$ with block sizes $(m_1, \cdots, m_K)$, $m_i(m_i+1)/2$ entries are involved for each block. Now, recognize that the inequality constraints in (5) are always satisfied with equality, i.e. $\mathrm{tr}(\Sigma_x) = 1$ and $\mathrm{tr}(\Sigma_z P) = 1$. To see this, consider all possible Σ_x's in the set $\{\Sigma_x : \mathrm{tr}(\Sigma_x) < 1\}$.

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We can properly allocate the remaining power $(1 - \mathrm{tr}(\Sigma_x))$ based on the current covariance matrix Σ_x. This strictly increases the sum capacity. The same augment works for Σ_z. Thus, solving (5) is equivalent to solving

$$\max_{\Sigma_x^{(v)}} \min_{\Sigma_z^{(v)}} \quad f(\Sigma_x^{(v)}, \Sigma_z^{(v)}) \tag{24}$$
$$s.t. \quad A\Sigma_x^{(v)} = 1, \quad B\Sigma_z^{(v)} = 1.$$

where $f(\Sigma_x^{(v)}, \Sigma_z^{(v)})$ denote the minimax expression and matrices A and B represent the linear constraints on $\Sigma_x^{(v)}$ and $\Sigma_z^{(v)}$. The KKT condition for this minimax problem now becomes:

$$r_1 = \nabla_D f(\Sigma_x^{(v)}, \Sigma_z^{(v)}) + A^T \nu = 0, \tag{25}$$
$$r_2 = \nabla_Z f(\Sigma_x^{(v)}, \Sigma_z^{(v)}) + B^T \mu = 0, \tag{26}$$
$$r_3 = A\Sigma_x^{(v)} - 1 = 0, \tag{27}$$
$$r_4 = B\Sigma_z^{(v)} - 1 = 0. \tag{28}$$

where ν and μ are Lagrangian multipliers associated with the equality constraints $A\Sigma_x^{(v)} = 1$ and $B\Sigma_z^{(v)} = 1$, respectively. For notational convenience, ∇_x is used to denote $\nabla_{\Sigma_x^{(v)}}$, and likewise for ∇_z. Here $r = [r_1 r_2 r_3 r_4]$ is called the residue. The optimality condition is completely satisfied when the residue is driven down to zero.

We can now reformulate the minimax problem based on an unconstrained minimization (i.e. zero-forcing) of r. To proceed with the minimization, we approximate $r(\Sigma_x^{(v)}, \Sigma_z^{(v)}, \nu, \mu)$ as a linear function using its gradient, and solve the equation $r = 0$ as if it is linear. The update of $(\Delta\Sigma_x^{(v)}, \Delta\Sigma_z^{(v)})$ can then be found as:

$$[\Delta\Sigma_x^{(v)} \ \Delta\Sigma_z^{(v)} \ \Delta\nu \ \Delta\mu]^T = -(\nabla r)^{-1} r \tag{29}$$

More explicitly, the above equation can be written as:

$$\begin{bmatrix} \nabla_{xx}^2 f & \nabla_{xz}^2 f & A^T & 0 \\ \nabla_{zx}^2 f & \nabla_{zz}^2 f & 0 & B^T \\ A^T & 0 & 0 & 0 \\ 0 & B^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\Sigma_x^{(v)} \\ \Delta\Sigma_z^{(v)} \\ \nu \\ \mu \end{bmatrix} = \begin{bmatrix} -\nabla_x f \\ -\nabla_z f \\ 1 - A\Sigma_x^{(v)} \\ 1 - B\Sigma_z^{(v)} \end{bmatrix}$$

The search direction for $(\Delta\Sigma_x^{(v)}, \Sigma_z^{(v)})$ is found via a matrix inversion.

The search direction derived above is actually a Newton's direction for both the minimization and the maximization at the same time! This observation is first made in [8] for concave-convex optimization. One way to interpret the search direction defined by (29) is that the minimax problem is being approximated by a quadratic minimax problem at each step, and the Newton's step represents a direction toward the saddle-point of the quadratic approximation. Note that unlike conventional Newton's method, where the value of the objective function can be used to ensure that the behaviour of the algorithm is monotonic, for minimax problems, a different metric is needed. The natural metric in our case is the norm of residue, which can be used in the backtrack line search.

The derivation so far assumes that the optimal Σ_x is strictly positive definite. When the number of remote users exceeds the number of base-station antennas, Σ_x can become semi-definite. In this case, the positivity constraint needs to be taken into account using interior-point method via a logarithmic barrier. More specifically, let $\phi(\Sigma_x) = \log|\Sigma_x|$ and define:

$$f_t = f + \frac{1}{t}\phi, \tag{30}$$

then the earlier derivation follows with f replaced by f_t.

We summarize the Newton's method for the minimax problem as follows:

Given initial Σ_x, Σ_z, $t > 0$, $\gamma > 1$, tolerance $\epsilon > 0$.

1) *Centering step:*
   - *Initialize* $\alpha \in (0, 1/2)$ *and* $\beta \in (0, 1)$.
   - *Compute* Σ_x *and* Σ_z.
   - *Backtracking line search until* $||r(\Sigma_x^*, \Sigma_z^*)||_2 < (1 - \alpha s)||r(\Sigma_x, \Sigma_z)||_2$.
   - *Update* $\Sigma_x := \Sigma_x^*$ *and* $\Sigma_z := \Sigma_z^*$.
   - *Stop if* $||r||_2 < \epsilon$.
2) $\Sigma_x := \Sigma_x^*$ *and* $\Sigma_z := \Sigma_z^*$.
3) *Stop if* $n/t < \epsilon$.
4) *Update* $t := \gamma t$.

Furthermore, if the objective function satisfies a strongly convexity condition, the Newton's method in fact admits an analytic bound on the maximum number of iterations. Although f_t here is not strongly convex by itself, strong convexity can always be ensured by an addition of superfluous constraints

$$||\Sigma_x^{(v)}|| \le l \quad \text{and} \quad ||\Sigma_z^{(v)}|| \le l, \tag{31}$$

where l is a constant chosen sufficiently large as to make the constraints inactive. The proof of this fact is lengthy, and is omitted here. The analytical bound is stated as follows:

Theorem 2: For $l > 1$, the objective of the proposed Newton's method satisfies strong convexity condition and $\nabla^2 f_t$ satisfies the Lipschitz condition:

$$||\nabla^2 f_t(x_1) - \nabla^2 f_t(x_2)||_2 \le L||x_1 - x_2||_2 \tag{32}$$
$$mI \le \nabla_{zz}^2 f_t \le MI, \quad -MI \le \nabla_{xx}^2 f_t \le -mI \tag{33}$$

where $x_i = (\Sigma_{x_i}, \Sigma_{z_i})$. Further, upper bound on the number of iterations is given by L, m, M and Newton's method parameters for each centering step as follows:

$$N \le \log_2(\log_2(\frac{2}{\epsilon K^2 L})) + \frac{||r^{(0)}||K_{(M,m)}^2 L\beta}{\alpha}, \tag{34}$$

where $r^{(0)}$ is the initial residue, and K is a constant depending on m and M.

A numerical example with a base station with 5 transmit antennas and 15 receivers each with three antennas is presented. The iterative algorithm is run with a guaranteed error gap less than 10^{-6}. The interior point method parameter γ is set to 5, and backtracking line search parameter $\alpha = 1/3$ and $\beta = 2/3$. The norm of residue as a function of iteration number is plotted in Fig. 1. Each horizontal portion corresponds to one fixed t. The proposed algorithm is quite effective.

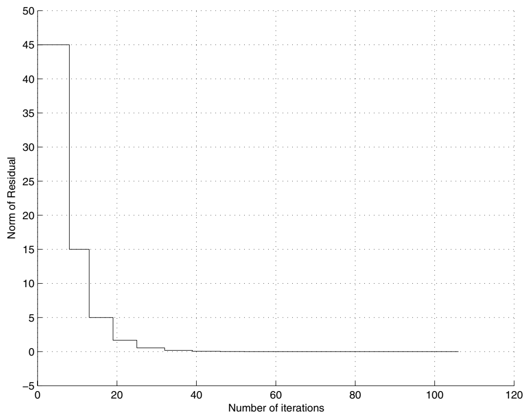

Fig. 1. Convergence of Newton's method for a minimax problem.

IV. EXTENSIONS

The primary motivation for studying uplink-downlink duality is that it allows the spectrum optimization problem to be solved efficiently. In this section, we show how the results of last section extend previous QR-based precoding method. Consider a Gaussian vector channel under high SNR

$$Y = Hx + z. \tag{35}$$

In QR precoding [1] [13], we first form $H^T = QR$, where R is triangular and Q is orthogonal. Then, we let $x = Qu$, where u is the transmit symbol. This effectively triangularizes a full matrix channel. Precoding can then be used to cancel the off-diagonal entries of R. QR precoding works because u and Qu have the same total power. QR precoding is optimal in the high SNR regime.

However, multiplication by an orthogonal matrix does not preserve individual power on each antenna. The goal of this section is to illustrate how to modify QR precoding to preserve pre-antenna power constraint.

The key observation is that optimization (4) with per-antenna power constraint is equivalent to the following optimization problem with linear power constraint,

$$\begin{aligned} \min_{S_z} \max_{S_x} \quad & \frac{1}{2} \log \frac{|HS_xH^T + S_z|}{|S_z|}. \\ \text{s.t.} \quad & \text{tr}(\Sigma_z^* S_x) \leq 1 \\ & S_z^{(i,i)} = I_{m_i \times m_i}\ , \text{ for } i = 1, \dots, K. \end{aligned} \tag{36}$$

where Σ_z^* is the minimizing noise in the dual multiple access channel. Note that (36) and (4) have the same objective, the only difference is the constraint involving Σ_z^*. This relation can be established by finding the KKT conditions for the optimization problem in (4), (5) and (36). These KKT conditions are equivalent based on a simple manipulation. (Refer to [9] for details.)

A modified QR-based precoding methods for broadcast channels with per-antenna constraint is then straightforward. We only need to consider $H_{eq} = H\Sigma_z^{-1/2}$ as the equivalent channel. Under the high SNR condition $S_x \gg S_z$, the minimizing Σ_z^* in (5) has an approximate solution $\Sigma_z^* = P^{-1}$. Recall that P is just the diagonal matrix formed by the power constraint of each transmitter antenna. The QR factorization now becomes

$$P^{1/2}H^T = QR. \tag{37}$$

Thus, the modified QR-based precoding under the per-antenna constraints is the QR factorization of a scaled channel matrix. Transmit precoding would then eliminate the off-diagonal values of the triangular matrix R.

V. CONCLUSION

This paper contains several new results. First, an uplink-downlink duality relation is derived for Gaussian MIMO broadcast channel with per-antenna power constraints. This novel derivation illustrates the equivalence between uplink-downlink duality and Lagrangian duality in convex optimization. It turns out that the dual of a broadcast channel under the per-antenna constraint is also a minimax problem. Second, a new application of Newton's method is used to solve the minimax problem. The numerical algorithm finds an efficient search direction for the maximization and the minimization at the same time, and is much more efficient than previously proposed solutions. Finally, the duality result is applied to channels under high SNR.

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