Encyclopedia Article - Fairness

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Given a vector $\mathbf{x} \in \mathbb{R}_+^n$, where x_i is the resource allocated to user i, *how fair* is it? Consider two allocations among three users: $\mathbf{x} = [1,\ 2,\ 3]$ and $\mathbf{y} = [1,\ 10,\ 100]$. Among the large variety of choices for quantifying fairness, it is possible to have fairness values such as 0.33 or 0.86 for $\mathbf{x}$ and 0.01 or 0.41 for $\mathbf{y}$: $\mathbf{x}$ is viewed as 33 times more fair than $\mathbf{y}$, or just twice as fair as y. How many such "viewpoints" are there? What would disqualify a quantitative metric of fairness? Can they all be constructed from a set of simple statements taken as true for the sake of subsequent inference?

Fairness of $\mathbf{x}$ can be quantified through a fairness measure, which is a function f that maps $\mathbf{x}$ into a real number. These measures are sometimes referred to as diversity indices in statistics. Various fairness measures have been proposed throughout the years. These range from simple ones, e.g., the ratio between the smallest and the largest entries of $\mathbf{x}$, to more sophisticated functions, e.g., Jain's index and the entropy function. Some of these fairness measures map $\mathbf{x}$ to a normalized range between 0 and 1, where 0 denotes the minimum fairness, 1 denotes the maximum fairness (often corresponding to an $\mathbf{x}$ where all x_i are the same), and a larger value indicates more fairness. For example, min-max ratio [1] is given by the maximum ratio of any two user's resource allocation, while Jain's index [2] computes a normalized square mean. How are these fairness measures are related? Is one measure "better" than any other? What other measures of fairness may be useful?

An alternative method that has gained attention in the networking research community since [3, 4] is the optimization-theoretic approach of α-fairness and the associated utility maximization problem. Given a set of feasible allocations, a maximizer of the α-fair utility function satisfies the definition of α-fairness. Two well-known examples are as follows: a maximizer of the log utility function ($\alpha = 1$) is proportionally fair, and a maximizer of the α-fair utility function as $\alpha \to \infty$ is max-min fair. More recently, α-fair utility functions have been connected to divergence measures [5]. In [6, 7], the parameter α was viewed as a fairness measure in the sense that a fairer allocation is one that is the maximizer of an α-fair utility function with larger α – although the exact role of α in trading-off fairness and throughput can sometimes be surprising [8]. While it is often believed that $\alpha \to \infty$ is more fair than $\alpha = 1$, which is in turn more fair than $\alpha = 0$, it remains unclear what it means to say, for example, that $\alpha = 3$ is more fair than $\alpha = 2$.

There are in general two main approaches to fairness evaluation: binary (is it fair according to a particular criterion, e.g., optimization objective, proportional fair, no-envy?) or continuous (quantifying how fair is it, and how fair is it relative to another allocation?). In the latter case, there are three sub-approaches:

(A) A system-wide, global measure: (A1) $f(\mathbf{x})$ where f is our fairness function, or (A2) $f(U_1, U_2, \ldots, U_n)$ where U_i is a utility function for each user that may depend on the entire$\mathbf{x}$.
(B) Individual, global measures: the set of $\{f_i(\mathbf{x})\}$ (each user i cares about the entire allocation $\mathbf{x}$).
(C) Individual, local measures: the set of $\{\tilde{f}_i(x_i)\}$ (user i only cares about her resource via some function $\tilde{f}_i$).

Renyi entropy. Renyi entropy is a family of functionals quantifying the uncertainty or randomness of generalized probability distributions, developed in 1960 [10]. Renyi entropy is derived from a set of five axioms as follows:

1. Symmetry.
2. Continuity.
3. Normalization.
4. Additivity.
5. Mean-value property.

Renyi entropy, a generalization of Shannon entropy, has been further extended by others since the 1960s, e.g., smooth Renyi entropy for information sources [11] and quantum Renyi entropy [12].

Lorenz Curve. Schur-concavity of fairness measures is a critical property for justifying fairness measures by establishing an ordering on the set of Lorenz curves [9]. Let $P_{\mathbf{x}}(y)$ be the cumulative distribution of a resource allocation $\mathbf{x}$. Its Lorenz curve L_x, defined by

$$L_{\mathbf{x}}(u) = \frac{1}{\mu} \cdot \int_{\{P_{\mathbf{x}}(y) \le u\}} y dP_{\mathbf{x}}(y), \quad (1)$$

is a graphical representation of the distribution of $\mathbf{x}$, and has used to characterize the social welfare distributions and relative income differences in economics. In 2001, an axiomatic characterization of Lorenz curve orderings is proposed based on a set of four axioms [9]:

1. Order. (The ordering is transitive and complete.)
2. Dominance. (The ordering is Shur-concavity.)
3. Continuity.
4. Independence.

Cooperative Economic Theories. In economics, a number of theories have been developed to study the collective decisions of groups. Many of these theories have also been uniquely associated with sets of axioms [13], including two well-known axiomatic constructions: the the Nash bargaining solution in

1950 [14] and the Shapley value in 1953 [15].

The Nash bargaining solution was developed from a set of four axioms:

1. Invariance to affine transformation.
2. Pareto optimality.
3. Independence of irrelevant alternatives (IIA).
4. Symmetry.

Nash's axiom of IIA contributes most to his uniqueness result, and it is also often considered as a value statement. It has been shown by many others, that replacing IIA with other value statements may result in solution classes different from the bargaining solution. Given a feasible region of individual utilities, the Nash bargaining solution is also equivalent to a maximization of the proportional fairness utility function.

Another well known solution concept in economic study of groups is the Shapley value [15]. In a coalition game, individual decide whether or not to form coalitions in order to increase the maximum utility of the group, while ensuring that their share of the group utility is maximized. The Shapley value concerns an operator that maps the structure of the game, to a set of allocations of utility in the overall group. Given a coalition game, four axioms uniquely define the Shapley value:

1. Pareto Optimality.
2. Symmetry.
3. Dummy.
4. Additivity.

Ultimatum Game. Cake-Cutting, and Fair Division: In Ultimatum Game [33], player A divides a resource into two parts, one part for herself and the other for player B. Player B can then choose to accept the division, or reject it, in which case neither player receives any resource. Without a prior knowledge about player B's reaction, player A may divide anywhere between [0:5; 0:5] and [1; 0]. Running this game as a social experiment in different cultures have lead to debates about the exact implications of the results on people's perception on fairness: how fair does it take for player B's perception, and player A's guess of that, to accept player A's division? This has also been contrasted with the perception of fairness in the related Dictator Game [28, 29], where player B has no option but to accept the division by player A.

A classic generalization of Ultimatum Game that has received increasing attention in the past decade is the cake-cutting problem. As reviewed in books [27, 32], the cake is a measure space, and each player uses a countably-additive, non-atomic measure to evaluate different parts of the cake. Among the work studying the cake-cutting problem, e.g., [31], the primary focus has been on two criteria: efficiency (Pareto optimality) and fairness (envy-freeness). Achievability results for 4 users or more are still challenging.

Fairness in cake-cutting and fair-division is traditionally defined as envy-freeness. It is a binary summary based on each individuals local evaluation: the allocation of cake is fair if no user wants to trade her piece with another piece. In [30], this restrictive viewpoint on fairness is expanded to include proportional allocation of the left-over piece after each user gets 1=nth of the cake (in her own evaluation). It is shown that Pareto optimality and proportional sense of fairness may not be compatible for 3 players or more.

Rawls' Theory of Justice and Distributive Fairness. In political philosophy, the work of Rawls has been both influential and provacative since the original publication in 1970 [16]. Ralw's theory of justice concerns mostly with liberties, which are not exactly the same as resources. However, one may also read his theory as a qualitative framework for distributive fairness in allocating limited resources among users. The arguments posed by Rawls are based on two fundamental *principles* (axioms stated in English),

1. "Each person is to have an equal right to the most extensive scheme of equal basic liberties compatible with a similar scheme of liberties for others."
2. "Social and economic inequalities should be arranged so that they are both (a) to the greatest benefit of the least advantaged persons, and (b) attached to offices and positions open to all under conditions of equality of opportunity."

The first principle governs the distribution of liberties and has priority over the second principle. One may interpret it as a principle of distributive fairness in allocating limited resources among users. Next, the first part of Rawls second principle concerns with the distribution of opportunity while the second part is the celebrated "difference principle": an approach different from strict egalitarianism (since it is on the absolute value of the least advantaged user rather than the relative value) and utilitarianism (when narrowly interpreted where the utility function does not capture fairness).

Normative Economics and Welfare Theory. Rawls theory also has intricate interactions with normative economics, where many results are analytic in nature [68]. In addition to stochastic dominance, Arrows Impossibility Theorem, and the cooperation game theories of Nash and of Shapley, there are several major branches [20, 23, 25]. Another set is the ethical axioms of transfers. For example, the Pigou-Dalton principle states that inequality decreases via Robin Hood operation that does not reverse relative ranking. The principle of proportional transfers states that what the donor gives and the beneficiary receives should be proportional to their initial positions.

Bergson-Samuelson social welfare function [22, 26] $W(U_1(\mathbf{x}),\ldots,U_n(\mathbf{x}))$ aims at enabling complete and consistent social welfare judgment on top of individual preference-based utility functions U_i. Kolms theory of fair allocation [24] uses the criterion of equity as no-envy, and it is well- known that competitive equilibrium with equal budget is the only Pareto-efficient and envy-free allocation if preferences are sufficiently diverse and form a continuum [21].

Sociology and Psychology: Inequality Indices. Quantifying inequality/injustice/unfairness using individual, local measures, has been pursued in sociology. For example, Jasso in 1980 [**?**] advocated justice evaluation index as log of the ratio between actual allocation and just allocation. Allocation can be done either in quantity or in quality (in which case ranking quantifies the quality allocation). Many properties were derived in theory and experimented with in data about income distribution in different countries [19]. In particular, probability distribution of the index is induced by the probability distribution of the allocation. This index is derived based on two principles and three laws in the paper, including equal allocation maximizes justice and aggregate justice is arithmetic mean of individual ones.

In [18], two other injustice indexes, JI1 and JI2, were developed based on the above. One interesting feature is that JI1 differentiates between under-reward and over-reward as two types of injustice. Another useful feature is the decomposition of the total amount of perceived injustice into injustice due to scarcity and injustice due to inequality. They are further unified with Atkinsons measure of inequality [20]: 1 minus the ratio of geometric mean and arithmetic mean. At the heart of these indices is the approach of taking combinations of arithmetic and geometric means of an allocation to quantify the its spread.

Lan-Chiang fairness theory. Many existing approaches for quantifying fairness are different. On the one hand, α-fair utility functions are continuous and strictly increasing in each entry of $\mathbf{x}$, thus its maximization results in Pareto optimal resource allocations. On the other hand, scale-invariant fairness measures (ones that map $\mathbf{x}$ to the same value as a normalized $\mathbf{x}$) are unaffected by the magnitude of $\mathbf{x}$, and an allocation that does not use all the resources can be as fair as one that does. Can the two approaches be unified? To address the above questions, Lan and Chiang develop an axiomatic approach to fairness measures [17]. It is show that a set of five axioms, each of which simple and intuitive, thus accepted as true for the sake of subsequent inference, can lead to a useful family of fairness measures, i.e.,

1. Continuity.
2. Homogeneity.
3. Saturation.
4. Partition.
5. Starvation.

Starting with these five axioms, a unique family of fairness measures are generated from generator functions g: any increasing and continuous functions that lead to a well-defined "mean" function (i.e., from any Kolmogorov-Nagumo function [**?**, **?**]). Using power functions with exponent β as the generator function, a unique family of fairness measures are derived, i.e.,

$$f_{\beta,\lambda}(\mathbf{x},\mathbf{q}) = \mathrm{sign}(-1-\beta)\cdot\left(\sum_i x_i\right)^{1/\lambda} \cdot\left[\sum_{i=1}^{n} q_i \left(\frac{x_i}{\sum_{j=1}^{n} x_j}\right)^{-\beta}\right]^{\frac{1}{\beta}} \quad (2)$$

This result unifies many existing fairness measures: Generalized Jain's index is a special case of $F_{\beta,\lambda}(\mathbf{x})$ for $1/\lambda = 0$ and $\beta < 1$; inverse of p-norm is another subclass of $F_{\beta,\lambda}(\mathbf{x})$ for $\beta \leq -1$; and α-utility is obtained for $1/\lambda = \beta/(1-\beta)$ and $\beta > 0$. New fairness measures are revealed corresponding to other ranges of β and λ. The degree of homogeneity $1/\lambda$ determines how $F_{\beta,\lambda}(\mathbf{x})$ scales as throughput increases, while β provides tradeoff between "resolution" and "strictness" of the fairness measure. The unification is illustrated in Figure 1, including all known fairness measures that are global (i.e., mapping a given allocation vector to a single scalar and decomposable (i.e., subsystems fairness values can be somehow collectively mapped into the overall systems fairness value).

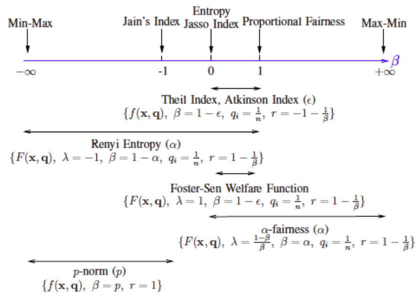

Fig. 1 $F_{\beta,\lambda}$ unifies different fairness measures from diverse disciplines.

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