Lecture Notes in Percolation

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The material presented here is taken from various sources. Lecture 1 is taken from Meester and Roy (1996), Lectures 2, 4 and 5 are basically taken from Grimmett (1989). Lecture 3 is an improved version of Burton and Keane (1989). Lecture 6 is from Dekking and Meester (1990). These references do not always refer to the original publications. For further information, the reader should look in the references cited.

Burton, R.M. and Keane, M.S. (1989). Density and uniqueness in percolation, *Comm. Math. Phys.* **121**, 501–505.

Dekking, F.M. and Meester, R. (1990). On the structure of Mandelbrot's percolation process and other random Cantor sets, *J. Stat. Phys.* **58**, 335-341.

Grimmett, G.R. (1989). *Percolation* (Springer-Verlag, New York).

Meester, R. and Roy, R. (1996). *Continuum percolation* (Cambridge University Press, Cambridge).

1 Lecture 1: Introduction

The set-up is as follows. Each element of the d-dimensional integer lattice $\boldsymbol{Z}^d$ is called a vertex, where $d \geq 1$. Two vertices at a Euclidean distance one apart are called *neighbours*. Each pair of neighbours has an edge between them. The graph obtained this way is, with a slight abuse of notation, denoted by $\boldsymbol{Z}^d$. An edge is often called a *bond*, and the set of all bonds is denoted by $\mathbf{E}$. Bonds can be either *open* or *closed*. A *path* is a finite or infinite alternating sequence $(z_1, e_1, z_2, e_2, \ldots)$ of vertices z_i and bonds e_i such that $z_i \neq z_j$ and $e_i \neq e_j$ whenever $i \neq j$ and such that e_i is the bond between the neighbours z_i and z_{i+1}, for all i. The length of a path is the number of bonds it contains. A *circuit* is a finite path, the only difference being that it starts and ends at the same vertex. An *open (closed) path* is a path whose bonds are all open (closed). Two vertices are said to be *connected* if there is a finite open path from one to the other. An open *cluster* is a set of connected vertices which is maximal with respect to this property. Of course, clusters can be either finite or infinite. The open cluster containing the origin is denoted by C.

Next we introduce probability into this set-up. For $0 \leq p \leq 1$, we equip the space $\Omega = \{0,1\}^{\mathbf{E}}$ with the natural product measure P_p which is defined via $P_p(\omega(e) = 1) = p$ for all $e \in \mathbf{E}$. For any realisation $\omega \in \Omega$, the bond e is said to be open if $\omega(e) = 1$ and closed otherwise. Thus each bond is open with probability p independently of all other bonds.

This is the basic percolation model on the d-dimensional integer lattice. We are interested in infinite clusters, so here are some natural definitions:

Definition 1.1 *The percolation function* $\theta = \theta^{(d)}$ *is defined by*

$$\theta^{(d)}(p) = P_p(|C| = \infty).$$

where $|\cdot|$ *denotes cardinality.*

It seems obvious that $\theta^{(d)}$ is non-decreasing in p. In Lecture 5 it will become clear how to prove this. Based on $\theta^{(d)}$ we can define the following *critical probability*:

Definition 1.2 *The critical probability* $p_c = p_c(d)$ *is defined by*

$$p_c(d) = \inf\{p \,:\, \theta^{(d)}(p) > 0\}.$$

The following result is the beginning of percolation theory, since it asserts the existence of a so called *phase transition.*

Theorem 1.3 *For all* $d \geq 2$ *we have* $0 < p_c(d) < 1$.

Proof The inequality $p_c(d) > 0$ is very simple. Indeed, the number of distinct paths of length n starting at the origin is at most $2d(2d-1)^{n-1}$. (For the first bond, we have $2d$ possibilities; after that we have at most $2d-1$ possibilities for each new bond as we are not allowed to go back to where we came from.) Each of these paths has probability p^n to be open. Thus the expected number of open paths of length n starting at the origin is at most $2d(2d-1)^{n-1}p^n$. If $p < \frac{1}{2d-1}$ then $\sum_{n=1}^{\infty} 2d(2d-1)^{n-1}p^n < \infty$ and hence the expected number of open edges in the component $C(0)$ is finite. This necessarily means that the probability that the component $C(0)$ is finite is equal to one and hence $\theta^{(d)}(p) = 0$ if $p < \frac{1}{2d-1}$. Thus we obtain that $p_c(d) \geq \frac{1}{2d-1}$.

For the other inequality we observe that it suffices to prove it for the case $d = 2$ as $p_c(d)$ is non-increasing in d (see Exercises). We need to introduce the *dual* graph $\boldsymbol{Z}^{2*}$. This is the graph obtained from $\boldsymbol{Z}^2$ by shifting it by the vector $(\frac{1}{2}, \frac{1}{2})$. The set of edges of the dual graph is denoted by $\mathbf{E}^*$. Each edge of $\mathbf{E}$ now crosses exactly one edge in $\mathbf{E}^*$. We declare an edge in $\mathbf{E}^*$ to be open if and only if the edge it crosses in $\mathbf{E}$ is open, and closed otherwise. It is intuitively obvious and a well known fact in graph theory that there is a closed circuit in $\boldsymbol{Z}^{2*}$ surrounding the origin if and only if C is finite. Now we can perform a counting argument as in the first part of this proof. There are at most $n3^n$ distinct circuits of length n surrounding the origin. (This is a rather crude bound: such a circuit has to contain at least one vertex on the x-axis. There are at most n possibilities for this. Starting at this vertex, we have only three possibilities for each new bond.) If for some $N > 0$ (i) all bonds in $[-N, N] \times [-N, N]$ are open and (ii) there is no closed circuit in the dual surrounding $[-N, N] \times [-N, N]$ then C is infinite. The event in (i) certainly has positive probability. Furthermore, a circuit surrounding $[-N, N] \times [-N, N]$ has length at least $4N$. Hence, if $p > \frac{2}{3}$ we can choose N so large that $\sum_{n=4N}^{\infty} n3^n(1-p)^n < 1$ and for such p and N, the event in (ii) also has positive probability. As the events in (i) and (ii) depend on disjoint sets of edges they are independent, and we conclude that $p_c(2) \leq \frac{2}{3}$. $\square$

We continue the discussion with some basic inequalities which are very useful in the analysis of models of this type. For this, we need to introduce some terminology. There is a natural partial order on $\Omega = \{0, 1\}^{\mathbf{E}}$: $\omega \leq \omega'$ if and only if $\omega(e) \leq \omega'(e)$ for all $e \in \mathbf{E}$. An event A in Ω (we assume that Ω is equipped with the usual Borel σ-field) is said to be *increasing* if its indicator function is increasing (a real valued function f on Ω is increasing if $f(\omega) \leq f(\omega')$ whenever $\omega \leq \omega'$). An event A is said to be *decreasing* if its complement is increasing. A typical example of an increasing event is the event that two distinct vertices are connected to each other by an open path. Writing E_p for the expectation operator corresponding to P_p we have the following important inequality:

Theorem 1.4 FKG inequality *Let f_1 and f_2 be both increasing or both decreasing functions. Then*

$$E_p f_1 f_2 \geq E_p f_1 E_p f_2.$$

Taking f_1 and f_2 to be the indicator functions of two increasing (or two decreasing) events A and B respectively, this inequality reduces to

$$P_p(A \cap B) \geq P_p(A) P_p(B).$$

This result is not surprising: conditioned on the existence of an open path connecting two different vertices, another path connecting two other vertices becomes more likely as it can 'use' the bonds of the first path.

Sometimes we need an inequality which goes in the opposite direction. Given the existence of an open path connecting two vertices, we can make it 'harder' for other connections to exist by requiring them to be disjoint from the first connection. This motivates the following definition. Suppose A and B are increasing events which depend only on the state of finitely many bonds. We define $A \Box B$ to be the set of all configurations ω for which there exist disjoint sets of open bonds with the property that the first such set guarantees the occurrence of A and the second guarantees the occurrence of B. More precisely, $A \Box B$ is the set of all configurations ω for which there exist finite and disjoint sets of bonds K_A and K_B such that any configuration

ω' with $\omega'(e) = 1$ for all $e \in K_A$, is in A, and any configuration ω'' with $\omega''(e) = 1$ for all $e \in K_B$, is in B.

Theorem 1.5 BK inequality *Let A and B be two increasing events which depend on the state of only finitely many bonds. Then*

$$P_p(A \Box B) \leq P_p(A) P_p(B).$$

The requirement that A and B are allowed to depend on only finitely many bonds has a technical reason. As we shall see, this will not be important in applications.

Excercise 1.1 Prove that $p_c(d+1) \leq p_c(d)$ for all $d \geq 1$.

Excercise 1.2 Show that the event that $|C| = \infty$ is the same as the event that there is an infinite open path starting at the origin. (It is not enough to show that for all n, there is a path af length at least n from the origin.)

Excercise 1.3 Show that

(i) $A \Box B \subset A \cap B$,

(ii) $(A_1 \cup A_2) \Box B \supset ((A_1 \Box B) \cup (A_2 \Box B))$.

2 Lecture 2: Decay of the cluster radius

We denote the set $\{-n, \ldots, n\}^d$ by B_n. If there is an open path from a set A to a set B we write $A \leftrightarrow B$. Boundary is denoted by ∂. This lecture is devoted to the following important result.

Theorem 2.1 *There exist positive constants ρ and σ, independent of p, and a unique function $\phi(p)$ such that*

$$\rho n^{1-d} e^{-n\phi(p)} \leq P_p(0 \leftrightarrow \partial B_n) \leq \sigma n^{d-1} e^{-n\phi(p)}. \tag{1}$$

The function $\phi(p)$ has the following properties:

(a) $\phi(p) = 0$ for $p > p_c$;

(b) $\phi(p) > 0$ for $p < p_c$;

(c) $\phi(p)$ is continuous and therefore $\phi(p_c) = 0$;

(d) $\phi(p) \to \infty$ when $p \downarrow 0$.

Once we have established (1), (a) follows immediately, since $P_p(0 \leftrightarrow \partial B_n) \geq \theta(p) > 0$ for $p > p_c$ and for all n. The proof of (b) is very hard and we do not give it here. We will later see that at least for $d = 2$, $\theta(p_c) = 0$. Combining this with the above theorem, we see that the critical case is different from supercriticality by the absence of infinite clusters, and different from subcriticality by having no exponential decay of the radius of the cluster of the origin.

Proof Suppose that $0 \leq p \leq 1$ and let $\beta(n) = P_p(0 \leftrightarrow \partial B_n)$. The strategy is to show that the sequence $\log \beta(n)$ is both subadditive and superadditive, subject to small error terms. Then the limit theorem for subadditive sequences will do it for us.

We claim that

$$\beta(n+m) \leq |\partial B_m| \beta(m) \beta(n), \tag{2}$$

and

$$\beta(n+m) \geq \frac{1}{2d|\partial B_m|} \beta(m) \beta(n). \tag{3}$$

Let us see how the analysis continues once we have established these inequalities. Note that $|\partial B_m| \leq 2d(2m+1)^{d-1} = 2d(2+\frac{1}{m})^{d-1} m^{d-1} \leq d3^d m^{d-1}$.

We can therefore write

$$\log\beta(m+n) \le \log\beta(m) + \log\beta(n) + g(m), \tag{4}$$

and

$$\log\beta(m+n) \ge \log\beta(m) + \log\beta(n) - g(m), \tag{5}$$

where $g(r) = \log(d^2 3^{d+1}) + (d-1)\log r$. Suppose wlog that $m \le n$ and add $g(n)$ to both sides of (4), obtaining

$$g(n) + \log\beta(m+n) \le g(m) + \log\beta(m) + \log\beta(n) + g(n). \tag{6}$$

Also, $g(m+n) - g(n) = (d-1)\log(1+\frac{m}{n}) \le (d-1)\log 2$ since $m \le n$. Adding the last two inequalities tells us that the sequence (a_k) defined by

$$a_k = g(k) + (d-1)\log 2 + \log\beta(k),$$

is subadditive, that is, satisfies

$$a_{n+m} \le a_m + a_n.$$

Similarly, one can show that the sequence $(g(k) + (d-1)\log 2 - \log\beta(k))$ satisfies this inequality as well. The well known subadditive limit theorem, together with the fact that $g(k)/k \to 0$ for $k \to \infty$ then gives that

$$\phi(p) = \lim_{k\to\infty}\left(-\frac{1}{k}\log\beta(k)\right)$$

exists, and in addition that

$$g(m) + (d-1)\log 2 + \log\beta(m) \ge -m\phi(p)$$

and

$$g(m) + (d-1)\log 2 - \log\beta(m) \ge m\phi(p) \tag{7}$$

for all m. Combining these inequalities gives

$$\begin{aligned} |m\phi(p) + \log\beta(m)| &\leq g(m) + (d-1)\log 2 \\ &\leq c + (d-1)\log m, \end{aligned}$$

proving (1).

We still need to prove (2) and (3) though. For (2), we observe that it follows easily (but be careful!) from the BK inequality that

$$\begin{aligned} P_p(0 \leftrightarrow \partial B_{m+n}) &\leq \sum_{x\in\partial B_m} P_p(0 \leftrightarrow x) P_p(x \leftrightarrow \partial B_n(x)) \\ &= \sum_{x\in\partial B_m} P_p(0 \leftrightarrow x) P_p(0 \leftrightarrow \partial B_n) \\ &\leq |\partial B_m|\beta(m)\beta(n). \end{aligned}$$

where $B_n(x) = x + B_n$, giving (2).

For (3) we have to be a bit more careful. Let $\gamma(n)$ be the probability that there is an open path from the origin to a particular face of the box B_n. Note that $\gamma(n) \leq \beta(n) \leq 2d\gamma(n)$. Let x be on the boundary of B_m and write U_x for the event that there is an open path from 0 to x in B_m. Then

$$\beta(m) \leq \sum_{x\in\partial B_m} P_p(U_x),$$

so there exists some $x = (x_1, \ldots, x_d)$ with

$$P_p(U_x) \geq \frac{1}{|\partial B_m|}\beta(m).$$

Suppose that $x_k = m$ for this x and let V_x be the event that there is an open path from x to a point y on the boundary of $B_n(x)$ for which $y_k = m + n$. We have $P_p(V_x) = \gamma(n)$ and using the FKG inequality we find

$$\beta(m+n) \geq P_p(U_x \cap V_x) \quad \geq \quad P_p(U_x)P_p(V_x)$$

$$
\begin{aligned}
&\geq \frac{1}{|\partial B_m|}\beta(m)\gamma(n) \\
&\geq \frac{1}{2d|\partial B_m|}\beta(m)\beta(n),
\end{aligned}
$$

as required.

It remains to show that $\phi(p)$ is continuous and that $\phi(p)$ tends to infinity when $p \downarrow 0$. To do this, define $b_m(p) = -m^{-1}\log\beta(m)$ and write (7) in the form

$$
|\phi(p) - b_m(p)| \leq \frac{1}{m}\left(c + (d-1)\log m\right).
$$

Notice now that $b_m(p) \to \phi(p)$ as $m \to \infty$ and this convergence is uniform on $(0,1]$ since the right hand side of the last inequality does not involve p. Also observe that $b_m(p)$ is continuous in p (why?). Therefore, $\phi(p)$ is the uniform limit of continuous functions and hence it is continuous.

Finally, to show that $\phi(p)$ tends to infinity for $p \downarrow 0$, it suffices to glance at the proof of Theorem 1.3. $\square$

Exercise 2.1 Give a rigorous proof of the fact that $\lim_{p\downarrow 0}\phi(p) = \infty$.

Exercise 2.2 Show that $\phi(p)$ is non-increasing (using the fact that $P_p(0 \leftrightarrow \partial B_n)$ is non-decreasing in p).

Exercise 2.3 Show that $E_p(|C|) < \infty$ for $p < p_c$, using the main result of this lecture.

3 Lecture 3: Uniqueness of the infinite cluster

We know that for p large enough, there is a positive probability that the origin is in an infinite cluster. It is easy to see, using Kolmogorov's $0-1$ law, that this is equivalent to the statement that there is at least one infinite cluster *somewhere*. This section is devoted to the following striking theorem:

Theorem 3.1 *For every p, there is P_p-a.s. at most one infinite cluster.*

The proof requires some preliminary results. We call an event *trivial* if it has probability zero or one. We call en event A *translation-invariant* if $A = T_z(A)$ for any $z \in \mathbf{Z}^d$, where T_z denotes the translation over the vector z.

Lemma 3.2 *Any translation-invariant event is trivial.*

Proof We only sketch the proof; you can fill in the details yourself. Let A be translation-invariant, and approximate A by a set A' which depends on the state of only finitely many bonds. For suitable z, A' and $T_z(A')$ are independent since they depend on different bonds. Also, $T_z(A')$ approximates $T_z(A) = A$. So on the one hand we have by independence $P_p(A' \cap T_z(A')) = P_p(A')P_p(T_z(A'))$ which approximates $P_p(A')P_p(A')$. On the other hand we have that $P_p(A' \cap T_z(A'))$ approximates $P(A' \cap A)$ which in turn approximates $P(A')$. Combining these things, we see that $P_p(A')P_p(A')$ approximates $P_p(A')$; therefore, $P_p(A')$ is either zero or one. $\square$

Corollary 3.3 *For all p, the number of infinite clusters is a P_p-a.s. constant (possibly infinity).*

Proof For all $0 \leq N \leq \infty$, the event E_N that the number of infinite clusters is equal to N, is a translation invariant event, and hence by Lemma 3.2 trivial. Therefore, for all p there exists a unique N such that $P_p(E_N) = 1$. $\square$

Lemma 3.4 *For all p, the number of infinite clusters is either $0, 1$ or ∞ a.s.*

Proof Suppose the number of infinite clusters (which is an a.s. constant according to the previous corollary) equals k, where $2 \leq k < \infty$. Then there exists a (nonrandom) number m such that the box B_m is intersected by all these k clusters with positive probability. This means that graph $\mathbf{Z}^d \backslash B_m$ contains k infinite clusters which all touch the boundary of B_m. This latter event, call it A, depends only on the state of the bonds outside B_m. Let B be the event that all bonds inside B_m are open. Then $P_p(A \cap B) = P_p(A)P_p(B) > 0$ by independence. But on the event $A \cap B$, there is only one infinite cluster, a contradiction since we assumed there were k of them. $\square$

Thus we need only to rule out the possibility of having infinitely many infinite clusters.

We define $x \in \mathbf{Z}^d$ to be an *encounterpoint* if (i) x belongs to an infinite cluster $C(x)$, and (ii) the set $C(x)\backslash\{x\}$ has no finite components and exactly three infinite components (the 'branches' of x). Now suppose that there are infinitely many infinite open clusters. (In fact, three would do as will become clear in a moment.) An argument as in the proof of the last lemma (using three intersecting infinite clusters, each of which is connected to the origin within the box B_n as to obtain the three branches of the origin) will show that the origin is an encounterpoint with positive probability ϵ say, and so is any other vertex. Hence the expected number of encounterpoints in the box B_n is $\epsilon(2n+1)^d$

Now it is the case that if a box contains k encounterpoints, there will be at least $k+2$ vertices on the boundary of the box which belong to some branch of these encounterpoints. If you want to convince yourself that this statement is true, just draw a picture; the encounterpoints together with their branches form a tree-like structure resembling a regular tree with degree 3. (Remember that the branches belonging to one encounterpoint are disjoint!) This is not as yet a proof but it would mean that the expected number of points on the boundary which are connected to an encounterpoint in B_n is at least $\epsilon(2n+1)^d + 2$. This however is clearly impossible for large n since the number of vertices on the boundary is of the order of n^{d-1} for large n.

We will now finish the proof by giving a rigorous proof of the fact that if a box contains k encounterpoints, there will be at least $k+2$ vertices on the boundary of the box which belong to some branch of these encounterpoints.

Let C denote a fixed infinite cluster, and let z and z' be two enounterpoints of $C \cap B_n$. Define $Y = C \cap \partial B_n$. The removal of z from C defines a partition $P = \{P_1, P_2, P_3\}$ of Y with $P_i \neq \emptyset$ for all i. For z', the corresponding partition will be denoted $Q = \{Q_1, Q_2, Q_3\}$. Now observe that we can choose the indices such that $P_1 \supseteq Q_2 \cup Q_3$: simply choose P_1 to correspond to the component of $C \backslash \{z\}$ containing z' and Q_1 to correspond to the component of $C \backslash \{z'\}$ containing z. The following lemma implies that the number of encounterpoints belonging to $C \cap B_n$ is at most $|Y| - 2$, and summing over all infinite cluster completes the proof.

Lemma 3.5 *Let Y be set with $|Y| \geq 3$. A partition of Y is a collection $P = \{P_1, P_2, P_3\}$ of three nonempty disjoint subsets of Y whose union is Y. Partitions P and Q are* compatible *if there is an ordering of each such that $P_1 \supseteq Q_2 \cup Q_3$. A collection of partitions is compatible if each pair is compatible. It is now the case that for any compatible collection $\mathcal{P}$ of partitions of Y we have*

$$|\mathcal{P}| \leq |Y| - 2.$$

Proof If $|Y| = 3$ the lemma is clearly true. Suppose the lemma is valid whenever $|Y| < n$. Choose any $P \in \mathcal{P}$, and denote $|P_i| = n_i$. Divide $\mathcal{P}\backslash\{P\}$ into three classes $\mathcal{P}_i$ with $Q \in \mathcal{P}_i$ if $P_i \supseteq Q_2 \cup Q_3$ (after reordering of the Q-indices). Each Q now belongs to exactly one $\mathcal{P}_i$. Now fix an i, say $i = 1$, and define $Y_1 = P_1 \cup \{\Delta\}$, for some $\Delta \notin Y$. For each $Q \in \mathcal{P}_1$ define a partition Q' of Y_1 by setting $Q' = \{Q_1', Q_2, Q_3\}$ with $Q_1' = (Q_1 \cap P_1) \cup \{\Delta\}$. Then $\mathcal{P}_1' = \{Q' : Q \in \mathcal{P}_1\}$ is a compatible collection of partitions of Y_1 and

$$|Y_1| = n_1 + 1 < n_1 + n_2 + n_3 = n.$$

By induction,

$$|\mathcal{P}| = \sum_i |\mathcal{P}_i| + 1 \leq \sum_i ((n_i + 1) - 2) + 1 = n - 2.$$

$\square$

The philosophy of this proof is that it is impossible to imbed a regular tree-like structure into the lattice in a stationary way, the point being that there are not enough vertices to accommodate such a tree. The proof given above exploits this idea using the fact that the size of the boundary of a box is of smaller order than the volume of the box. There are thus certain graphs where this approach does not work, for example on a regular tree, where the boundary of a box centered at the origin is of the same order as the volume of the box.

Exercise 3.1 Show that $\theta(p) > 0$ is equivalent to the statement that under P_p there exists an infinite cluster somewhere a.s.

Exercise 3.2 Provide the details in the proof of Lemma 3.2.

Exercise 3.3 Show that for all $x, y \in \mathbf{Z}^d$ and all p we have $P_p(x \leftrightarrow y) \geq \theta(p)^2$ (using uniqueness).

4 Lecture 4: $p_c = \frac{1}{2}$ in two dimensions

This is the result which was already anticipated in the 1960's and was the problem which attracted many people to the field. What follows is a modern proof. It might appear to be not so hard, but notice that we strongly rely on the fact that $\phi(p) > 0$ for $p < p_c$ (see Lecture 2), and this fact was only proved in the 1980's. We start with a simple device, called the square root trick:

Lemma 4.1 *If $A_1, A_2, \ldots, A_m$ are increasing events with equal probability, then*

$$P_p(A_1) \geq 1 - \left(1 - P_p(\bigcup_{i=1}^{m} A_i)\right)^{1/m}.$$

Proof Just write

$$\begin{aligned} 1 - P_p\left(\bigcup_{i=1}^{m} A_i\right) &= P_p\left(\bigcap_{i=1}^{m} A_i^c\right) \\ &\geq \prod_{i=1}^{m} P_p(A_i^c) \\ &= (1 - P_p(A_1))^m, \end{aligned}$$

where the inequality follows from the FKG inequality. □

Lemma 4.2 *In two dimensions we have $\theta(\frac{1}{2}) = 0$ and therefore $p_c \geq \frac{1}{2}$.*

Proof Suppose $\theta(\frac{1}{2}) > 0$. For any integer n, let $A^r(n)$ (respectively $A^l(n)$, $A^t(n)$, $A^b(n)$) be the event that some vertex of the right (left, top, bottom) of the square $T(n) = [0,n]^2$ is the starting point of an infinite open path which uses no other vertices from $T(n)$ and whose second vertex is to the right (left, top, bottom). Clearly the $A^i(n)$'s ($i = t, b, r, l$) all have the same probability and since the probability of their union tends to 1, it follows from the square root trick (with $m = 4$) that $P_{\frac{1}{2}}(A^i(n))$ tends to 1 as well ($i = t, b, r, l$). In particular, $P_{\frac{1}{2}}(A^r(n) \cap A^l(n))$ tends to 1. Since we have $p = \frac{1}{2}$, similar statements are true for closed paths, so if we denote by $B^i(n)$ the corresponding events for closed paths we find that in particular, $P_{\frac{1}{2}}(B^t(n) \cap B^b(n))$ tends to 1.

Now look at open paths from the boundary of $T(n)$ and for closed paths from the boundary of $T(n)+(\frac{1}{2}, \frac{1}{2})$ in the dual graph, in the same realisation. From what we observed above, we see that the event that there are infinite open paths in the square lattice from the left and right side of $T(n)$ and infinite closed paths in the dual from the top and bottom of $T(n) + (\frac{1}{2}, \frac{1}{2})$ has probability tending to 1. But, we now from the previous lecture that there can be at most one infinite open cluster in the lattice, and at most one infinite closed one in the dual as well. However, it is geometrically impossible to connect both the two open paths and the two closed paths just described. This is the desired contradiction. $\square$

Lemma 4.3 *In two dimensions,* $p_c \leq \frac{1}{2}$.

Proof The proof of this result is another application of the concept of duality. Consider the rectangles $R(n) = [0, n+1] \times [0, n]$ and $S(n) = (\frac{1}{2}, -\frac{1}{2}) + ([0, n] \times [0, n+1])$. It is easy to see that if we do percolation in two dimensions, then there is either an open left to right crossing of $R(n)$, or a closed top to bottom crossing of $S(n)$ in the dual. Call these events $A(n)$ and $B(n)$ respectively. If $p = \frac{1}{2}$, symmetry gives us that $A(n)$ and $B(n)$ must have the same probability, and since they are mutually exclusive, it follows that their probability is equal to $\frac{1}{2}$, independent of n.

Suppose now that $p_c > \frac{1}{2}$. This implies that $\frac{1}{2}$ is subcritical and therefore $\phi(\frac{1}{2}) > 0$. This is to say that the probability that there is an open path from the origin to the line $L_n = \{(x, y) : x = n\}$ decays exponentially in n, so for some positive α we have $P_p(0 \leftrightarrow L_n) \leq e^{-\alpha n}$. Therefore,

$$\begin{aligned} P_{\frac{1}{2}}(A(n)) &\leq \sum_{k=0}^{n} P_{\frac{1}{2}}((0,k) \leftrightarrow L_n) \\ &\leq (n+1)e^{-\alpha n} \to 0 \end{aligned}$$

as $n \to \infty$, contradicting the fact that $P_{\frac{1}{2}}(A(n)) = \frac{1}{2}$ for all n. $\square$

Exercise 4.1 Use duality and Lemma 4.2 to prove that for $p = \frac{1}{2}$, the probability that the origin and the point $(1, 0)$ are connected by an open path is equal to $\frac{3}{4}$.

Exercise 4.2 Show, using the proof of the last lemma, that in two dimensions, $E_{p_c}(|C|) = \infty$.

5 Lecture 5: Properties of $\theta(p)$

In this section we collect some properties of the percolation function $\theta(p)$. For this it turns out to be convenient to couple all measures P_p on the same probability space $(\Omega, \mathcal{F}, P)$ as follows. Let $X(e)$, $e \in \mathbf{E}$ be a collection of independent random variables uniformly distributed on $[0, 1]$. For $0 \leq p \leq 1$ we define $\eta_p(e)$ to be 1 if $X(e) < p$ and 0 otherwise. Call an edge p-open of $\eta_p(e) = 1$. Write C_p for the p-open cluster containing the origin and note that

$$C_p \subseteq C_q \text{ for } p \leq q. \tag{8}$$

Observe also that

$$\theta(p) = P(|C_p| = \infty). \tag{9}$$

The power of this construction is that it allows us to compare different percolation processes for different values of p. For example, it follows immediately from (8) and (9) that

Proposition 5.1 *The function $\theta(p)$ is non-decreasing in p.*

The next two result are less obvious:

Lemma 5.2 *$\theta(p)$ is continuous from the right.*

Lemma 5.3 *$\theta(p)$ is continuous from the left, except possibly at $p = p_c$.*

Note that the last lemmas, together with the results from Lecture 4, imply that for $d = 2$, $\theta(p)$ is a continuous function. It is expected that this is the case in any dimension $d \geq 2$.

Proof of Lemma 5.2 We write $\theta_n(p) = P_p(0 \leftrightarrow \partial B_n)$. First observe that $\theta(p) = \lim_{n\to\infty} \theta_n(p)$. Choose $\epsilon > 0$. For n large enough we have

$$\theta_n(p) \leq \theta(p) + \epsilon.$$

Note that θ_n is continuous in p since the event $\{0 \leftrightarrow \partial B_n\}$ depends on only finitely many bonds. Hence we can choose $\delta > 0$ so small that

$$\begin{aligned} \theta_n(p+\delta) &\leq \theta_n(p) + \epsilon \\ &\leq \theta(p) + 2\epsilon. \end{aligned}$$

This yields

$$\theta(p) \leq \theta(p+\delta) \leq \theta_n(p+\delta) \leq \theta(p) + 2\epsilon,$$

completing the proof. $\square$

Proof of Lemma 5.3 In the coupling described at the beginning of this lecture, we have

$$\begin{aligned}\lim_{q\uparrow p}\theta(q) &= P\left(\bigcup_{q<p}(|C_q|=\infty)\right)\\ &= P(|C_q|=\infty \text{ for some } q<p).\end{aligned}$$

It follows that continuity from the left is equivalent to the statement

$$P(|C_p|=\infty, |C_q|<\infty \text{ for all } q<p)=0,$$

and this is what we will prove. For $p<p_c$ there is nothing to prove, so assume that $p>p_c$ and suppose that $|C_p|=\infty$. Choose $p_c<r<p$; it follows from Lecture 3 that a.s. there exists somewhere an infinite r-open cluster I_r. This I_r is a.s. contained in C_p, otherwise there would be two infinite p-open clusters. Hence there is a finite p-open path π from the origin to I_r, and each edge e of π satisfies $X(e)<p$. Therefore, $\mu=\max\{X(e): e\in\pi\}$ satisfies $\mu<p$. So for q such that $\mu<q<p$ we have that $|C_q|=\infty$, proving the result. $\square$.

Exercise 5.1 Show that $P_p(x\leftrightarrow y)$ is a continuous function of p, for all $x,y\in\mathbf{Z}^d$.

Exercise 5.2 Use the proof of Theorem 1.3 to give a simple alternative proof of the fact that θ is continuous (from the left) at $p=1$.

Exercise 5.3 Show that θ is strictly increasing for $p>p_c$. You may want to use the coupling described at the beginning of this lecture and the fact that the expected number of vertices in the box B_n which percolate is $\theta(p)(2n+1)^d$.

6 Lecture 6: Fractal percolation

The main reason to include a lecture devoted to fractal percolation is the fact that the percolation function in this model turns out to be discontinuous at the critical point, in contrast to ordinary percolation (at least in two dimensions).

Here is a verbal description of the model. Choose an integer $N \geq 2$ and divide the unit square into N^2 subsquares. Fix a number $p \in [0,1]$. We keep each subsquare with probability p, and throw it away with probability $1-p$, independently of all other squares. Next we divide the remaining subsquares into N^2 further subsquares, and again keep these with probability p and throw them away with probability $1-p$. We continue this process in the obvious way, and we are interested in the limiting set.

There are several ways of formalising this construction, and here is one of them: Consider all subsquares of I^2 of the form $I^n_{k\ell} = I^n_k \times I^n_\ell$, where

$$I^n_k = [(k-1)N^{-n}, kN^{-n}],$$

for integer n and $1 \leq k, \ell \leq N^n$. We call such a square $I^n_{k\ell}$ a *level-n square*. We define a probability space $(\Omega, \mathcal{B}, \mathcal{P}_p)$ as follows:

$$\Omega = \prod_{n=1}^{\infty} \prod_{k=1}^{N^n} \prod_{\ell=1}^{N^n} \{0,1\},$$

$\mathcal{B}$ is the standard Borel $\sigma-$field generated by cylinder sets and P_p is the product measure

$$P_p = \prod_{n=1}^{\infty} \prod_{k=1}^{N^n} \prod_{\ell=1}^{N^n} \mu^n_{k\ell},$$

where $\mu^n_{k\ell}\{1\} = 1 - \mu^n_{k\ell}\{0\} = p$, for all possible k, ℓ and n.

We write $\omega \in \Omega$ as

$$\omega = (\omega_{11}^1, \omega_{12}^1, \ldots, \omega_{NN}^1, \omega_{11}^2, \ldots, \omega_{N^2N^2}^2, \ldots).$$

For all n, we associate with this ω a set $A_n(\omega)$ as follows. Let $I_{k\ell}^n$ be a level-n square and denote by $I_{k_m\ell_m}^m$ the unique level-m square which contains $I_{k\ell}^n$, where $1 \leq m \leq n$. Let $m_{k\ell}^n(\omega) = \prod_{m=1}^n \omega_{k_m\ell_m}^m$; then $m_{k\ell}^n(\omega)$ is either zero or one. Now $A_n(\omega) \subseteq I^2$ is the set

$$A_n(\omega) = \bigcup_{k,\ell} \{I_{k\ell}^n : m_{k\ell}^n(\omega) = 1\}.$$

The A_n's decrease to a compact subset A of the unit square. We say that A *percolates* if there exists a connected component in A which intersects the left and right side of I^2. (We also say that A is *crossed* from left to right.) It is not difficult to see that A percolates if and only if, for all n, A_n contains a component which intersects the left and right side of I^2. The main result of this lecture is the following theorem. We define the percolation function $\theta(p)$ as the probability that A percolates under P_p.

Theorem 6.1 *There exists a critical value $0 < p_{c,N} < 1$ such that $\theta(p) > 0$ if and only if $p \geq p_{c,N}$.*

The most striking part of this theorem is the fact that the inequality is not strict; at the critical point there is positive probability of percolation, in contrast to the previous independent percolation model. We shall not prove the fact that $p_{c,N}$ is strictly between zero and one. Instead we will concentrate on the discontinuity of $\theta(p)$ at criticality. For ease of notation we will take $N = 3$ from now on. It turns out to be useful to consider the process in the rectangle $[0, 1] \times [0, 2]$, putting two independent copies of the process on top of each other. We denote the percolation function in this model by $\tau(p)$, i.e. $\tau(p)$ is the probability of a crossing from left to right in the limit when we put two copies of the model on top of each other. Of course we have that $\tau(p) \geq \theta(p)$ for all p. The following lemma implies that θ and τ have the same critical point.

Lemma 6.2 *If $\tau(p) > 0$, then $\theta(p) > 0$.*

We postpone the proof of this lemma and first prove Theorem 6.1. A level-n column is a set of the form $[\frac{i}{3^n}, \frac{i+1}{3^n}] \times [0,1]$ or $[\frac{i}{3^n}, \frac{i+1}{3^n}] \times [0,2]$, depending on the current model.

Proof of Theorem 6.1 The first remark is that it is easy to show that θ and τ are right-continuous (see Exercises). It now follows from Lemma 6.2 that it is sufficient to prove that τ is discontinuous at p_c. It might seem strange, but this is much easier than a direct proof of the theorem. For a rectangle R, denote by $\mathcal{H}(R)$ and $\mathcal{V}(R)$ the events that A is crossed horizontally respectively vertically in the limit.

If $\mathcal{H}([0,1]\times[0,2])$ occurs, at least one of the following events must occur:

$$E_1^p = \cup_{i=2}^5 \mathcal{V}(S_i),$$

$$E_2^p = \cup_{i=1}^5 \mathcal{H}(S_i \cup S_{i+1}),$$

where $S_1, \ldots, S_6$ denote the six level-1 squares in a particular level-1 column, ordered from top to bottom. Since there are three level-1 columns, we obtain

$$\tau(p) \leq \{P_p(E_1^p \cup E_2^p)\}^3 \leq \{P_p(E_1^p) + P_p(E_2^p)\}^3.$$

We have

$$P_p(E_1^p) \leq 4P_p(\mathcal{V}(S_1)) \leq 4\theta(p)$$

and

$$P_p(E_2^p) \leq 5P_p(\mathcal{H}(S_1 \cup S_2)) \leq 5\tau(p).$$

Combining these estimates, we obtain

$$\tau(p)^{\frac{1}{3}} \leq 4\theta(p) + 5\tau(p) \leq 9\tau(p).$$

Now suppose $\tau(p) \neq 0$. Then it follows that

$$\tau(p)^{\frac{2}{3}} \geq \frac{1}{9},$$

i.e.

$$\tau(p) \geq \left(\frac{1}{9}\right)^{\frac{3}{2}}.$$

Thus τ is discontinuous at p_c. □

Proof of Lemma 6.2 We need some definitions. Let S_1^n denote a level-n square in $[0, 3^{-n}] \times [0, 2]$ (i.e. in the first level-n column), S_2^n a level-n square in the i-th level-n column, for some $i \in \{1, \dots, 3^{n-1}\}$, T_1^m a level-m square in the j-th column for some $j \in \{2 \cdot 3^{m-1} + 1, \dots, 3^m\}$ and T_2^m a level-m square in the final level-m column. The rows containing these squares are denoted by s_1^n, s_2^n, t_1^m and t_2^m. Note that S_1^n and S_2^n are squares in $[0, \frac{1}{3}] \times [0, 2]$ whilst T_1^m and T_2^m are contained in $[\frac{2}{3}, 1] \times [0, 2]$. We consider the following event:

$C(S_1^n, S_2^n, T_1^m, T_2^m) = \{A$ contains a connected component C which crosses $[0, 1] \times [0, 2]$ horizontally, such that the intersection of C with the first i level-n columns connects the left side of S_1^n to the right side of S_2^n, and the intersection of C with the final $3^m - j + 1$ level-m columns connects the left side of T_1^m to the right side of T_2^m.$\}$

Now $\tau(p) > 0$, so for each choice of n and m we can find S_1^n, S_2^n, T_1^m and T_2^m as above with $P_p(C(S_1^n, S_2^n, T_1^m, T_2^m)) > 0$. Suppose that this is true for a choice S_1^n, S_2^n, T_1^m and T_2^m with $s_1^n \neq s_2^n$ and $t_1^m \neq t_2^m$.

It is now easy to to build a crossing of $[0,3] \times [0,2]$ by pasting these crossings together. Rather than giving a turgid formal proof, we remark that it follows from reflection, translation invariance and the FKG inequality that such a crossing has positive probability. It then follows that $P_p(\mathcal{H}([0,3] \times [0,3])) > 0$ and hence by rescaling, $\theta(p) > 0$.

It remains to prove the existence of squares S_1^n, S_2^n, T_1^m and T_2^m with these properties. First we only require $s_1^n \neq s_2^n$, and suppose such squares do not exist. This implies that there is a level-2 row i_2 say, such that there is a positive probability $q > 0$, say, that a horizontal crossing of $[0,1] \times [0,2]$ intersects the first three level-2 squares of this row. In particular, all first three level-2 squares of this row must be open. Furthermore, there is a level-3 subrow of i_2, i_3 say, such that there is a probability of at least $q3^{-1}$ that this crossing intersects the first nine squares of this row. Again, these first nine squares must be open in that case. Iterating this, we conclude that the probability that there exists a level-n row with all its first 3^{n-1} level-n squares open, is at least $q3^{2-n}$. On the other hand, it is easy to deduce that this probability is at most

$$2 \cdot 3^n P_p(\text{ row 1 has all its first } 3^{n-1} \text{ squares open }) \leq 2 \cdot 3^n\, p^{3^{n-1}}.$$

Hence

$$q3^{2-n} \leq 2 \cdot 3^n p^{3^{n-1}},$$

which is impossible for n large enough. Now fix S_1^n and S_2^n, $s_1^n \neq s_2^n$ with

$$P_p(C(S_1^n, S_2^n, T_1^m, T_2^m)) > 0.$$

Using the same argument, it follows that we can find T_1^m and $T_2^m, t_1^m \neq t_2^m$, such that this event still has positive probability. $\square$

Exercise 6.1 Show that $\theta(p)$ is continuous from the right.

Exercise 6.2 Show that $p_{c,N} > 0$ for all $N \geq 2$, using a branching process argument.

Exercise 6.3 Show along the lines of the proof of Theorem 6.1 that if A contains a connected component larger than a single point with positive P_p probability, then in fact $\theta(p) > 0$.

Exercise 6.4 Is it obvious that we are allowed to use the FKG inequality in fractal percolation?

Exercise 6.5 It is not known whether θ is continuous for $p > p_c$. Where does the proof of Lecture 5 break down?