# A SET THEORETIC APPROACH TO IMAGE RECONSTRUCTION FROM PROJECTIONS

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## ABSTRACT

The reconstruction of an image from incomplete view data requires the use of several constraints not derived from ray sum (projection data) measurements. The constraints can be incorporated through the method of (sequential and parallel) projections onto the constraint sets. These methods for the use of information regarding the noise and the image are implemented and compared in this paper. It is shown that the use of noise statistics decreases the mean square error in the image and that the method of parallel projections results in smaller error than the method of sequential projections if a sufficient number of iterations is permitted.

## 1. INTRODUCTION

Algebraic methods were among the first used to solve the tomographic problem. These methods approach the tomographic problem as one where the solution is constrained to satisfy a set of linear equations. The algebraic reconstructive technique (ART) [1] seeks a solution by sequential projection of an initial estimate onto the sets of solutions of the linear equations, each of these sets is a hyperplane. The simultaneous iterative reconstructive technique (SIRT) [1] approaches the solution through parallel projection onto the hyperplanes and provides far better results in the presence of noise.

The method of sequential projections onto closed convex constraint sets for the general image recovery problem was proposed in [2] and implemented without using noise information on tomographic images in [3]. A set-theoretic framework for the use of prior information obtained from the noise statistics for the general signal recovery problem was provided in [4], and demonstrated to be very useful with certain types of

signals. The use of (set-theoretic) parallel algorithms to solve the general constrained signal recovery problem was presented in [5], and these algorithms were shown to be more resilient to noise than (set-theoretic) sequential methods. The foundations of set-theoretic estimation are discussed in detail in [6], where the method of sequential projections is referred to as MOSP and that of parallel projections as MOPP.

This paper presents the use of parallel set-theoretic methods for the tomographic problem. It also suggests the use of noise information along the lines of [4] in the image reconstruction process. It is shown that the problems of noisy projection data can be partly overcome using MOPP (even without noise information) or MOSP with information about the noise variance.

## 2. SET THEORETIC FORMULATION

The tomographic problem may be formulated in set-theoretic terms as follows. The linear model for image reconstruction may be written as,

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \mathbf{n}$$

where $\mathbf{g}$ is the projection data vector of size $(M \times 1)$, $\mathbf{f}$ is the original image vector of size $(N^2 \times 1)$, $\mathbf{H}$ is the weight matrix of size $(M \times N^2)$, $\mathbf{n}$ is the signal-independent, gaussian, zero-mean noise vector of size $(M \times 1)$.

The required image, $\tilde{\mathbf{f}}$, is assumed square. $\mathbf{f}(i)$ represents the $i^{th}$ pixel of any image $\mathbf{f} \in \Re^{N \times N}$. The residual [4], $\mathbf{r} = \mathbf{g} - \mathbf{H}\tilde{\mathbf{f}}$, is the 'noise estimate'. Constraints on the residual may be expressed as constraints on the required image. The required image may be chosen to belong to one or many of the following constraint sets in the $N^2$-dimensional vector space over the reals:

1. Set of all images with non-negative pixels

$$C_1 = \{\mathbf{f} s.t. \mathbf{f}(i) \geq 0 \ \forall i\}$$

2. Set of all images whose pixel values lie in a pre-
   scribed closed interval,

$$C_2 = \{ \mathbf{f} s.t. \alpha \leq \mathbf{f}(i) \leq \beta \; \forall i \}$$

3. Set of images with known points,

$$C_3 = \{ \mathbf{f} s.t. \mathbf{f}(i) = \gamma_i \forall i \in I \subset \{1....N^2\}, \}$$

4. Set of all images that provide a residual of rea-
   sonable mean [4]

$$C_m = \{ \mathbf{f} \mid | \sum_i (\mathbf{g}_i - [\mathbf{Hf}]_i) | \leq \delta_m \}$$

where $\mathbf{g}_i$ and $[\mathbf{Hf}]_i$ are the $i^{\text{th}}$ elements of vectors
$\mathbf{g}$ and $\mathbf{Hf}$ respectively. The bound $\delta_m$ represents
the confidence limits on the sample mean. The size
of the set is determined by $\delta_m$.

5. Set of all images that provide a residual of rea-
   sonable variance [4]

$$C_v = \{ \mathbf{f} \mid \| \mathbf{g} - \mathbf{Hf} \|^2 \leq \delta_v \}$$

The bound $\delta_v$ represents the confidence limits on
the sample variance. The size of the set is deter-
mined by $\delta_v$.

It is simple to check that each of the above sets is con-
vex and closed. In MOSP, an initial estimate is pro-
jected onto each of the above closed convex sets in a
cyclic fashion. In MOPP, all sets are activated at each
iteration, i.e., projections are made on all sets during
each iteration [5]. The estimate is a weighted-average
of all the projections. Both methods converge to a
point in the intersection of the constraint sets if the
intersection is non-empty. If the intersection is empty,
MOSP does not converge and oscillates among the con-
straint sets. MOPP, however, converges to a minimum
weigthed-square solution, i.e. it converges to a point
whose distances from the constraint sets are such that
the weighted sum of the squares of these distances is
minimum [6].

In the image reconstruction problem, non-
intersecting constraint sets often arise because of noisy
projection data which gives rise to a set of inconsis-
tent linear equations. ART and SIRT are specific cases
of MOSP and MOPP respectively, when the only con-
straint sets are the hyperplanes defined by the projec-
tion data. It is well-known that SIRT has far better
noise characteristics than does ART. ART deals with
noise by allowing variation in the hyperplane, but this
treats each noise variable separately. Using the sets $C_v$
and $C_m$ in addition to the constraints used for ART

will allow a method in which the noise can be treated
as a random process.

Illustrative results that demonstrate the use of con-
straint sets other than non-negativity and the ray-sum
linear equations are presented in the next section. The
next section also presents an example of the better be-
haviour of MOPP in noisy conditions.

## 3. SIMULATIONS AND RESULTS

Simulations were mainly carried out on three images :
(i) An elliptical image of size $10 \times 10$ with uniform grey
level in the background, with 130 equations in 100 un-
knowns.
(ii) A sagittal section of the human brain of size $64 \times
64$, 5520 equations in 4096 unknowns.
(iii) A $16 \times 16$ image of alternating white and black
squares of value 0 and 255.
Apart from these images another image of the human
head was used to confirm the results.
Ray sums were calculated and simulated zero-mean,
gaussian, independent, identically distributed noise vari-
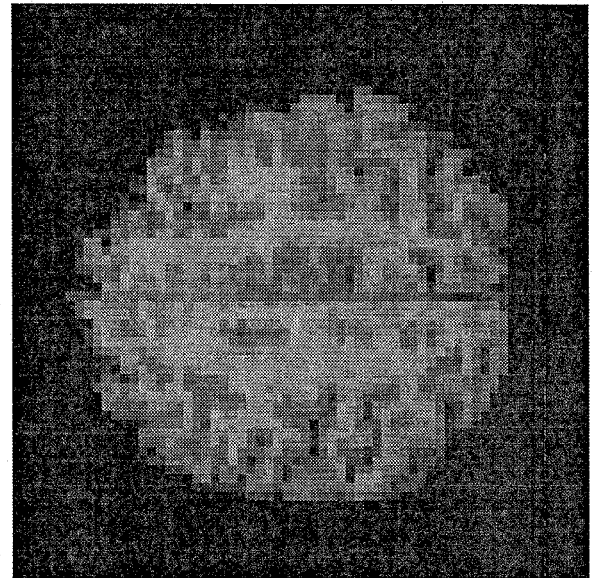ables added.



Figure 1: Original Image.

Fig. 1 shows the original $64 \times 64$ brain image. Fig. 2
shows its reconstruction using ART. The mean-square
error in this case is 117.61, corresponding to an RMS
error of 10.84. The maximum value for the image is
255. Fig. 3 shows the reconstructed image with use
of $C_2$, the set of images with pixels bounded above by
255 and below by 0, in MOSP. The mean square error

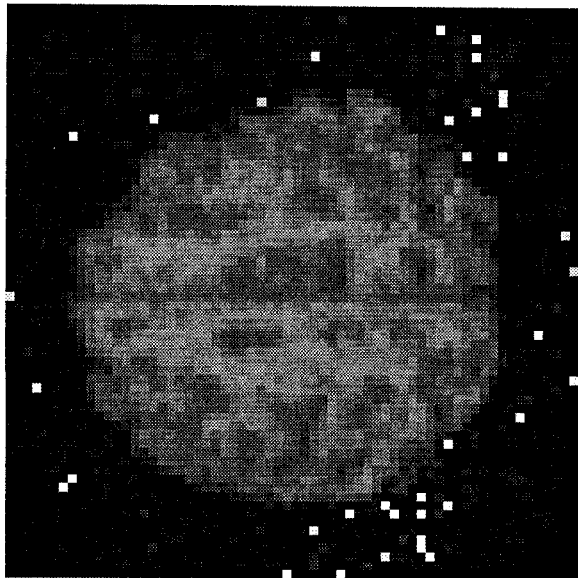in this case is 80.97, corresponding to an RMS error of almost 9.
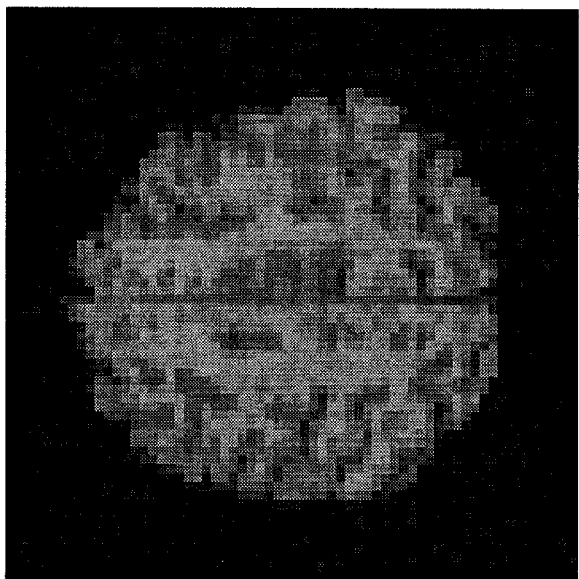


Figure 2: ART Reconstruction.



Figure 3: MOSP Reconstruction with $C_v$.

The set of images with bounded pixels is very effective since the mean square error has reduced remarkably. The above images are representative and the reduction in error is true for both noisy as well as non-

noisy cases. The above error-reduction was also noted in ART with non-negativity, but the above formulation provides far more flexibility.

Gaussian noise of mean 0 and variance 1 is added to the projection data and the image reconstructed using $C_v$, the residual variance set, in addition to the linear equations and non-negativity. Table 1 lists results for projections including $C_v$ for the ellipse image. MSE1 is the mean square error for $\delta_v = 2.2493$, and MSE2 is the mean square error for $\delta_v = 1.7649$. Lagrange's multipliers for the projection operator are found by solving the constraint equation by the Newton-Raphson method [4]. Projecting onto $C_v$ is computationally highly intensive [4].

Table 1: MSE for Projections onto $C_v$, for ellipse.

| #Iterations | Newton Iterations | MSE1 | MSE2 |
|---|---|---|---|
| 50 | 2 | 0.25 | 0.18 |
| 20 | 2 | 0.25 | 0.18 |
| 10 | 2 | 0.32 | 0.24 |
| 10 | 5 | 0.15 | 0.13 |

The size of the set plays an important role during reconstruction. As the size of the set decreases, error also decreases. This is to be expected and explains the pattern of error observed in Table 1. Further, for the $10 \times 10$ image, the MSE using $C_1$ is 0.005084, while using the set $C_2$, of images whose pixels are bounded above by 1 and below by 0, the reconstruction is almost perfect. The first set is unbounded, the second is bounded.

The $16 \times 16$ checker-board image was reconstructed using 1) ART and non-negativity, 2) ART, non-negativity and the noise variance constraint as MOSP, and 3) the noise variance constraint. The MOSP results of 2) were far better than the those of 1), and the results of 3) were comparable to those of 1). This indicates the usefullness of the variance constraint.

MOSP works quite well in non-noisy data situations, but its performance deteriorates in noisy situations when the variance constraint is not used. MOPP performs well in the presence of noise without the variance constraint, but it requires a large number of iterations to assure convergence. This is evident from the following graph (Fig. 4) for the ellipse.

After 50 iterations the MSE in MOPP is less than that in MOSP. The main disadvantage of MOPP, however, is its poor convergence rate. For the $64 \times 64$ brain image, the mean square difference between consecutive iterates for 30 iterations is 0.00954 in MOSP

and 0.063154 in MOPP. The final solution for MOSP depends upon the sequence with which the projections are made onto the sets. This is not the case with MOPP where the projections are made on all the sets during each iteration and the weighted average of all projections is considered as the next estimate.
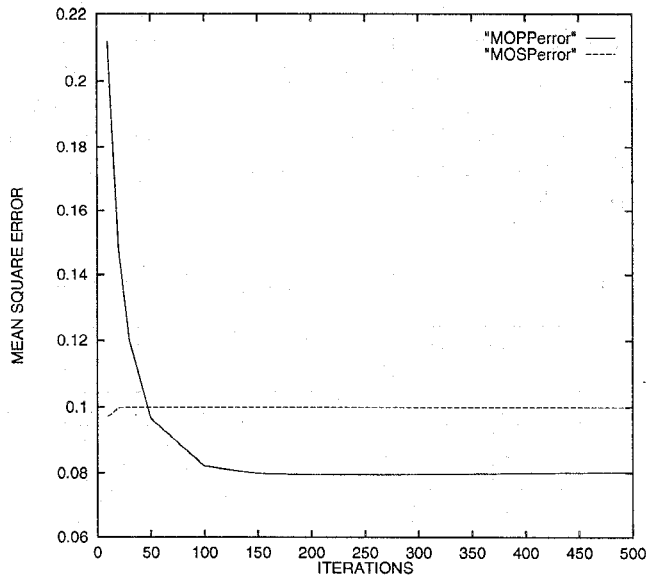


Figure 4: MSE V/s No of iterations in MOSP and MOPP for projection data with noise.

## 4. CONCLUSIONS

The use of MOSP and MOPP for tomographic reconstruction has been demonstrated. The methods provide means of using information of the noise statistics to reconstruct images from noisy projection data. While more computationally intense than the non-iterative methods, these methods provide a means of incorporating prior information into the reconstruction procedure that the non-iterative methods do not.

## 5. REFERENCES

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