# A Bayes Approach to Step-Stress Accelerated Life Testing

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***Key Words*** — **Ordered Dirichlet distribution, Overstress testing, Environmental testing**

***Summary & Conclusions*** — **This article develops a Bayes model for step-stress accelerated life testing. The failure times at each stress level are exponentially distributed, but the specification of strict adherence to a time transformation function is not required. Rather, prior information is used to define indirectly a multivariate prior distribution for the failure rates at the various stress levels. Our prior distribution preserves the natural ordering of the failure rates in both the prior and posterior estimates. Methods are developed for Bayes point estimates as well as for making probability statements for use-stress life parameters. The approach is illustrated with an example.**

## 1. INTRODUCTION

When inference concerning the life length of high reliability items is required, often the life test is conducted in a more severe environment than occurs in actual use. Such a test is called an *accelerated life test* and is usually undertaken to save the time and cost of testing. The statistical problem then, is making inference about the life length of the item operating under use conditions based on failure data obtained under the more severe environment. There is a large body of literature on this topic (see for example [1,2] for up-to-date surveys of non-Bayes and Bayes approaches). In addition, the procedure is popular enough to have been codified by the U. S. Department of Defense in the MIL-STD-781C Handbook under the title *Environmental Tests*. Most of the available literature, however, is confined to the topic of *constant stress accelerated life tests* where each test item is subjected to a fixed test environment or stress level, but the test environment may differ from item to item. Here, we concern ourselves with *step-stress accelerated life tests* where each test item is subjected to a pattern of stress levels, each for fixed period of time. The main advantage of step-stress testing over constant stress testing, is that it can further reduce test time and the variability of the failure times [3].

Other authors have considered this problem, but have either approached the solution from a non-Bayes point of view [3,5-7,9] or from the simplifying assumption that only a single step change in stress is allowed [4,5,8,9]. In addition, all of the aforementioned works are developed under the assumption that a single stress constitutes the operating environment and that the relationship between stress and the parameters of the failure distribution of the item can be specified by a given function known as the *time transformation function* or *acceleration function*.

The assumption of only one change in stress is too restrictive and the disadvantage of non-Bayes techniques is that they often require large sample sizes due to their inability to incorporate available prior information into the analysis. Though Bayes techniques are not prevalent, they are being used in reliability more and more [10] and have considerable representation in the area of accelerated life testing [2,11].

The operating environment can be characterized by several stresses such as temperature dwell or shock, pressure, voltage, vibrations and cycling time. If such is the case, it may be difficult to specify an appropriate single or multivariate time transformation function which will be adhered to throughout the analysis. Rather, there are some well known single and multivariate time transformation functions [12,13] which can serve better as a guide for prior estimation rather than a given truth.

Herein we consider the analysis of step-stress accelerated life tests from a Bayes point of view. Our analysis applies where the underlying failure distributions at each accelerated level of stress is exponential, and the life test can only be periodically monitored. We discuss the incorporation of prior information into the analysis and develop Bayes point estimates and credibility intervals for the use-stress life parameters.

*Acronyms*[1]

| | |
|---|---|
| $MART | $tress Model for Accelerated Reliability Testing |
| VDC | DC input Voltage |

*Notation*

| | |
|---|---|
| $E_i$ | set of stress conditions constituting test environment i, $i = 0, \ldots, m$; $E_0 =$ use-stress. |
| $[0, T^*]$ | time interval over which testing is performed. |
| $(t_i, t_{i+1}]$ | subintervals of $[0, T^*]$, $i = 0, \ldots, m$, with $0 \equiv t_0 < t_1 < \cdots < t_m < t_{m+1} \equiv T^*$. |
| $L_i$ | $t_{i+1} - t_i$, the duration of time that a test item is subjected to $E_i$, $i = 0, \ldots, m$. |
| $\rho_i$ | length of time for ramping between $E_i$ and $E_{i+1}$, $i = 0, \ldots, m$, $\rho_{m+1} \equiv 0$. |
| $\lambda_i$ | constant failure rate at $E_i$, $i = 0, \ldots, m$. |
| $\underset{\sim}{\lambda}$ | $(\lambda_0, \ldots\lambda_m)$ |
| $\Delta\lambda_i$ | $\lambda_i - \lambda_{i-1}$, $i = 1, \ldots, m$. |

[1] The singular & plural of an acronym are always spelled the same.

| | |
|---|---|
| $u_i$ | $e^{-c\lambda_i}$, transformed failure rate at $E_i$, $i = 0, \ldots, m$; $u_{-1} \equiv 1$, $u_{m+1} \equiv 0$. |
| c | numerical constant used in failure rate transformation to avoid numerical difficulties when dealing with very small $\lambda_i$. [2] |
| $\underset{\sim}{u}$ | $(u_0, ..., u_m)$ |
| $s_i$ | total number of items which fail in $(t_i, t_{i+1}]$, $i = 0, \ldots, m$. |
| $\underset{\sim}{s}$ | $(s_0, ...s_m)$ |
| $r_i$ | indices for ith binomial expansion, $i = 0, \ldots, m$. |
| $\underset{\sim}{r}$ | $(r_0, \ldots, r_m)$ |
| $n_0$ | total number of items initially on test. |
| $n_i$ | $n_0 - \sum_{j=0}^{i-1} s_j$ number of items at risk of failure at time $t_i$, $i = 1, \ldots, m$. |
| T | failure time of an item on test. |
| $\beta, \alpha_j$ | prior parameters, j=0, ..., m. |
| $\alpha_{i\bullet}$ | $\sum_{j=0}^{i} \alpha_j$, $i = 0, \ldots, m$, |
| $b_j$ | $\beta\alpha_j$, j=0, ..., m. |
| $a_j(\underset{\sim}{r})$ | $\left(\frac{2L_j - \rho_j}{2c}\right)\left(n_j - s_j + r_j\right) + \left(\frac{\rho_{j+1}}{2c}\right)\left(n_{j+1} - s_{j+1} + r_{j+1}\right)$, j=0, ..., m. |

*Assumptions*

1. A step-stress accelerated life test is to be performed on a few highly reliable items.
2. The life of items at each environment $E_i$ can be described by an exponential distribution where the failure rate is an increasing function of the applied stress levels.
3. It is possible to find some prior information on the failure rate at the use-stress level and the accelerated stress levels from such sources as Mil-Handbooks, field data, environmental stress screening results, growth or verification testing, etc.

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[2] A procedure was developed for selecting c to avoid these numerical difficulties. The analysis is stable within a large range of  values close to the chosen c value.

4. Each test item is subjected to the same increasing step stress pattern. Testing proceeds by starting at the use-stress for a fixed time period and then increasing the stress to higher levels, each for a fixed time period. Thus $E_1$ is a more severe environment than $E_0$, $E_2$ is a more severe environment than $E_1$, and so on.
5. The failure of a test item can only be determined at times $t_i$, i=1, …, m+1. All failed items are removed from testing and all surviving items proceed to testing under the next stress level.

## 2. THE PRIOR DISTRIBUTION

### *2.1. Motivation of the Prior distribution*

Given assumptions 2 and 4, it follows that

$$\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_m . \tag{1}$$

and given the relationship between $\lambda_i$ and $u_i$ it also follows that

$$1 > u_0 \geq u_1 \geq \cdots \geq u_m > 0. \tag{2}$$

Concentrating on the variables, $u_i$, $i = 0, \ldots, m$, we wish to define a prior distribution that is mathematically tractable, is defined over the region specified in (2), and imposes no other unnecessary restrictions on the $u_i$. The multivariate ordered Dirichlet distribution,

$$\Pi\{(u_0, \ldots, u_m)\} = \frac{\Gamma(\beta)}{\prod_{j=0}^{m+1} \Gamma(\beta\alpha_j)} \prod_{j=0}^{m+1} \left(u_{j-1} - u_j\right)^{\beta\alpha_j - 1} \tag{3}$$

where $\beta > 0$, $\alpha_j > 0$, $j = 1, \ldots, m$, and $\sum_{j=0}^{m+1} \alpha_j = 1$, is such a distribution. The distribution is defined over (2) and thus imposes no extraneous restrictions. An additional advantage of the ordered Dirichlet distribution is that due to its mathematical properties, the incorporation of expert judgment is facilitated.

Typically, to define the parameters for the prior distribution of the quantities of interest, expert judgment concerning these quantities are equated to the theoretical expression for central tendency such as mean, median, or mode. In addition, some quantification of the quality of the expert judgment is often given by specifying a variance or a probability interval for the prior quantity. Solving these equations leads to the desired parameter estimates.

Specific quantities of interest for the problem at hand are the failure rates and reliability functions for each stress environment. From the joint distribution (3), the prior marginal distribution for any $u_i$ is obtained as a beta distribution, $u_i \sim \text{Beta}\left(\beta(1-\alpha_{i\bullet}),\, \beta(\alpha_{i\bullet})\right)$, given as

$$\Pi\{u_i\} = \frac{\Gamma(\beta)}{\Gamma\left(\beta(1-\alpha_{i\bullet})\right)\Gamma\left(\beta(\alpha_{i\bullet})\right)}\left(u_i\right)^{\beta(1-\alpha_{i\bullet})-1}\left(1-u_i\right)^{\beta(\alpha_{i\bullet})-1}. \tag{4}$$

This distributions can be used to make prior probability statements concerning both the failure rates and reliabilities at the different stress levels due to the one-to-one relationships of these quantities to the $u_i$. Specifically, it follows that $\lambda_i = -\frac{\ln(u_i)}{c}$, $R_i(t|u_i) = (u_i)^{t/c}$ and prior probability statements are easily obtained, for example

$$\Pr\{R_* \le R(t|u_i) \le R^*\} = \Pr\{(R_*)^{c/t} \le u_i \le (R^*)^{c/t}\} \tag{5}$$

and

$$\Pr\{\lambda_* \le \lambda_i \le \lambda^*\} = \Pr\{e^{-c\lambda^*} \le u_i \le e^{-c\lambda_*}\} \tag{6}$$

To aid in the incorporation of prior subjective information, it is often advisable to assess several related quantities to establish coherence on the part of the expert judgment In addition to providing a means for analyzing the failure rates and reliabilities at each stress, the following distributional relationship may also be proven

$$\frac{u_j}{u_i} \sim \text{Beta}\left(\beta(1-\alpha_{j\bullet}),\beta(\alpha_{j\bullet}-\alpha_i)\right) \qquad i < j\,. \tag{7}$$

This distribution can be used for cross validation of the elicited prior information as it can be used to make probability statements concerning the difference of the failure rates for different stress environments

$$\Pr\{\lambda_* \le \lambda_i - \lambda_j \le \lambda^*\} = \Pr\{e^{c\lambda^*} \le u_j/u_i \le e^{c\lambda_*}\} \tag{8}$$

or the ratio of reliabilities for the different stress environments

$$\Pr\{\, R_* \leq \frac{R(t|u_j)}{R(t|u_i)} \leq R^* \} = \Pr\{(R_*)^{c/t} \leq u_j/u_i \leq (R^*)^{c/t}\}. \quad (9)$$

*2.2. Specification of the Prior Parameters*

Interpreting an expert's judgment as a median estimate results in less biased estimates then interpreting this estimate as a mean or a modal estimate of the underlying distribution of the quantity of interest [14]. Thus from assumption 3, prior estimates $\lambda_0^*, \ldots, \lambda_m^*$ are obtained and interpreted as marginal median values. It follows that

$$\Pr\{\lambda_i \leq \lambda_i^*\} = 0.5 \Leftrightarrow \Pr\{e^{-c\lambda_i} \geq e^{-c\lambda_i^*}\} = 0.5 \Leftrightarrow \Pr\{u_i \geq e^{-c\lambda_i^*}\} = 0.5.$$

Thus $u_i^* = e^{-c\lambda_i^*}$ is the median of (4). Using the probability statements (5) or (6) another quantile of the failure rate at use-stress can be elicited from the experts. Let this quantile be denoted as $\lambda_0^q$, where

$$\Pr\{\lambda_0 \leq \lambda_0^q\} = q \Leftrightarrow \Pr\{u_0 \geq e^{-c\lambda_0^q}\} = q\,.$$

A suitable suggestion [14] when dealing with expert judgment is to elicit an upper (lower) bound on a quantity of interest and equate this to the 95-th (5-th) quantile. Using the two distinct quantile estimates of $u_0$, $e^{-c\lambda_0^*}$ and $e^{-c\lambda_0^q}$, and using the fact that $u_0 \sim \text{Beta}\Big(\beta(1-\alpha_{0\bullet}), \beta(\alpha_{0\bullet})\Big)$, it follows that we can solve for $\beta$ and $\alpha_{0\bullet} = \alpha_0$ using a bisection method [15]. Next, having obtained $\beta$, $\alpha_{i\bullet}$, i=1, …, m can be solved in a similar manner. The parameters $\alpha_i$, i=1, …, m are calculated using $\alpha_i = \alpha_{i\bullet} - \alpha_{i-1\bullet}$, i=1, …, m. Finally, $\alpha_{m+1}$ follows as $1 - \alpha_{m\bullet}$.

## 3. DETERMINING THE LIKELIHOOD FUNCTION

Due to assumption 2 and 4, the failure rate of a test item, $h(t)$, is the step function shown in Figure 1 for the case of 5 test intervals. It is also possible, especially when the stress environment involves temperature, that the stress conditions cannot be changed instantaneously and thus one must account for a gradual change from one set of stress conditions to the next. This is known as a ramping phenomenon [16]. The failure rate in this scenario is depicted in Figure 2 for the case where ramping is a linear function of time. For the development of the likelihood, the case of linear ramping will be assumed though other ramping functions are easily accommodated. The case in Figure 1, for example, is accommodated by setting $\rho_j = 0$, $j = 0, \ldots, m$.

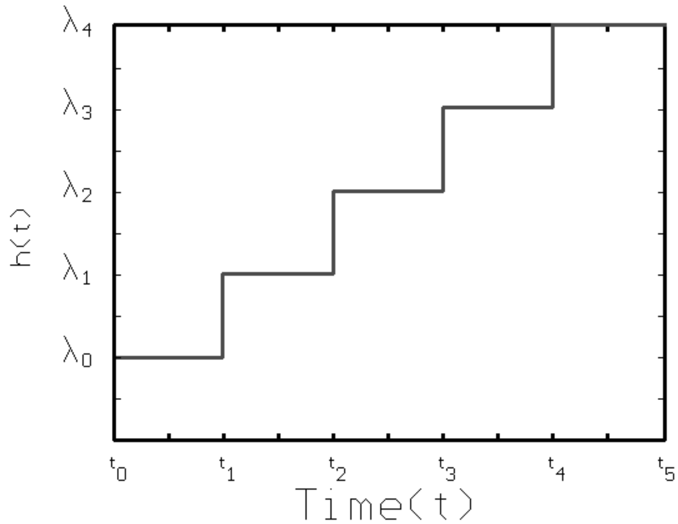


Figure 1. Failure Rate for Items Subjected to a Step-Stress Accelerated Life Test.

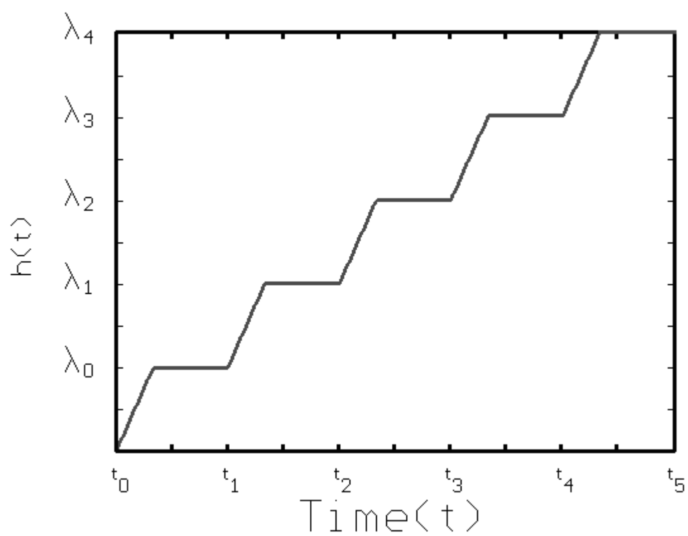


Figure 2. Failure Rate for Items Subjected to a Step-Stress Accelerated Life Test with Ramping Between Stress Levels.

In a general formulation, Miner's Rule [3] is observed. In the case of no ramping and the exponentiality assumption, however, Miner's Rules is vacuous as at the beginning of any new step the test items are as good as new. In the case of linear ramping, the demonstration of Miner's Rule is exhibited within the ramping periods.

The initial failure rate in Figures 1 and 2 is $\lambda_0$ the operational (use-stress) failure rate and this is assumed throughout the analysis presented below. However, with suitable modification to notation, this initial failure rate could also be from a known environment about which there is prior information.

Given assumption 5, the likelihood of the number of failures, $\underset{\sim}{s}$ , given $n_0$ items initially on test is

$$L\left(\underset{\sim}{\lambda} \mid n_0,\underset{\sim}{s}\right) \propto \prod_{i=0}^{m} \left[\Pr\left\{T > t_{i+1} \mid T > t_i\right\}\right]^{n_i - s_i} \left[1 - \Pr\left\{T > t_{i+1} \mid T > t_i\right\}\right]^{s_i}. \qquad (10)$$

It then follows with the well know relationship between the reliability function and the failure rate function that

$$\Pr\left\{T > t_{i+1} \mid T > t_i\right\} = \exp\left(-\int_{t_i}^{t_{i+1}} h(u)du\right) = \exp\left(-\left(\lambda_j L_j - \frac{\Delta\lambda_j}{2}\rho_j\right)\right). \qquad (11)$$

The likelihood can then be expressed in terms of $\underset{\sim}{u}$ as

$$L\{\underset{\sim}{u} \mid n,\underset{\sim}{s}\} = \prod_{i=0}^{m}\left[\left(u_i\right)^{\frac{2L_i-\rho_i}{2c}}\left(u_{i-1}\right)^{\frac{\rho_i}{2c}}\right]^{n_i-s_i}\left[1 - \left(u_i\right)^{\frac{2L_i-\rho_i}{2c}}\left(u_{i-1}\right)^{\frac{\rho_i}{2c}}\right]^{s_i} \qquad (124)$$

Note that if failures can be observed in other sub-intervals (i.e. another partition of $[0,T^*]$ ), then the likelihood may be easily rewritten to reflect this with no additional prior information needed and no real complication of the posterior analysis.

## 4. POSTERIOR ANALYSIS

The posterior distribution of $\underset{\sim}{u}$ is obtained as the product of the prior distribution (3) and the likelihood (12) and is proportional to

$$\prod_{i=0}^{m}\left[\left(u_i\right)^{\frac{2L_i-\rho_i}{2c}}\left(u_{i-1}\right)^{\frac{\rho_i}{2c}}\right]^{n_i-s_i}\left[1-\left(u_i\right)^{\frac{2L_i-\rho_i}{2c}}\left(u_{i-1}\right)^{\frac{\rho_i}{2c}}\right]^{s_i}\prod_{j=0}^{m+1}\left(u_{j-1}-u_j\right)^{\beta\alpha_j-1} \qquad (13)$$

Expand the second term in a binomial series and gather terms:

$$\sum_{r_0=0}^{s_0}\cdots\sum_{r_m=0}^{s_m}(-1)^{\sum_{j=0}^{m}r_j}\left[\prod_{j=0}^{m}\binom{s_j}{r_j}\left(u_j\right)^{a_j(\underset{\sim}{r})}\left(u_{j-1}-u_j\right)^{b_j-1}\right]\left(u_m\right)^{b_{m+1}-1} \qquad (14)$$

The distribution in (14) is a weighted mixture of distributions of the form

$$\frac{\left[\prod_{j=0}^{m}u_j^{a_j(\underset{\sim}{r})}\left(u_{j-1}-u_j\right)^{b_j-1}\right]u_m^{b_{m+1}-1}}{\prod_{j=0}^{m}B\left(\sum_{z=j}^{m}a_z(\underset{\sim}{r})+b_{z+1},\,b_j\right)}\,. \qquad (15)$$

This distribution is the *Generalized Dirichlet distribution* [17].

Of primary interest in accelerated life testing is the distribution of the use-stress parameters which are related to $u_0$. Using the identity in (15) and integrating (14) over $u_1, \ldots, u_m$ subject to (2), the posterior marginal distribution of $u_0$ is:

$$\Pi\{u_0 \mid n,\underset{\sim}{s}\} = \frac{1}{\mathcal{K}}\sum_{r_0=0}^{s_0}\ldots\sum_{r_m=0}^{s_m}\left[(-1)^{\sum_{j=0}^{m}r_j}\prod_{j=0}^{m}\binom{s_j}{r_j}\prod_{j=1}^{m}B\left(\sum_{z=j}^{m}a_z(\underset{\sim}{r})+b_{z+1},\,b_j\right)\right]*$$

$$\left[\left(u_0\right)^{\sum_{z=0}^{m}\left(a_z(\underset{\sim}{r})+b_{z+1}\right)-1}\left(1-u_0\right)^{b_0-1}\right]. \qquad (16)$$

where

$$\mathcal{K} \equiv \sum_{r_0=0}^{s_0}\ldots\sum_{r_m=0}^{s_m}\left[(-1)^{\sum_{j=0}^{m}r_j}\prod_{j=0}^{m}\binom{s_j}{r_j}B\left(\sum_{z=j}^{m}a_z(\underset{\sim}{r})+b_{z+1},\,b_j\right)\right] \qquad (17)$$

is the constant of integration.

Equation (16) is just a weighted mixture of Beta densities and thus, the posterior inference for the operational stress failure rate and reliability function can be obtained in manner similar to that for prior inference. That is, (5) and (6) hold aposteriori, but the posterior distribution of $u_0$ is a mixture of beta densities. Thus posterior point estimates and credibility intervals can be obtained using (16) with the appropriate transformation.

Unfortunately, marginal densities like (16) can only be derived for $u_0$ and $u_m$. However, moments for $u_i$ for i = 1, ..., m − 1 can be derived as

$$E[u_i^k \mid n, \underset{\sim}{s}\,] = \tag{18}$$

$$\frac{1}{\mathcal{K}} \sum_{r_0=0}^{s_0} \cdots \sum_{r_m=0}^{s_m} (-1)^{\sum_{j=0}^{m} r_j} \left[ \prod_{j=0}^{m} \binom{s_j}{r_j} \prod_{j=0}^{i} B\Big(k + \sum_{z=j}^{m} a_z(\underset{\sim}{r}) + b_{z+1}, b_j\Big) \prod_{j=i+1}^{m} B\Big(\sum_{z=j}^{m} a_z(\underset{\sim}{r}) + b_{z+1}, b_j\Big) \right]$$

Using the moments one can infer $u_i$ for i = 1, ..., m − 1. The software package $MART [18] generates the first five moments using (18) and fits a 2-member Beta mixture to these five moments to obtain posterior median estimates for the steps i = 1, ..., m − 1.

## 5. EXAMPLE PROBLEM

Three case studies have been conducted in various industries of highly reliable systems. For proprietary reasons, however, the results of these case studies cannot be used as an example for this paper. This example is fictitious, however, the example does resemble the case studies in setup and complexity. The analysis has been executed using $MART .

The test system is a new design of an electronic system of a radar manufacturer. The following questions were posed to qualify the design:

A. The environmental specifications requires the system to withstand 13.0 VDC at 125° F for one hour with a probability of at least 0.999. This requirement was developed from suspected worst case environment.

B. The customer wants to know whether the s-expected survival probability of a 1000 hour mission at use-stress exceeds 0.95 .

C. The manufacturer's program manager would like to obtain insight on warranty returns over the life of the system.

A test-design team including reliability and manufacturing engineers was formed and developed a 5-step accelerated test plan to answer questions A-C. The design team developed the step-stress accelerated life test design presented in Table 1. All times presented are in hours. Step 1 is the use-stress environment.

TABLE 1

Example Step-Stress Design.

| STEP | STEP TIME | RAMP TIME | TEMP. (°F) | VOLT. (VDC) |
|---|---|---|---|---|
| 1 | 120 | 1 | 100 | 10.0 |
| 2 | 120 | 1 | 125 | 13.0 |
| 3 | 120 | 1 | 160 | 15.0 |
| 4 | 120 | 1 | 200 | 17.0 |
| 5 | 120 | 1 | 250 | 19.0 |

The test design team chose the system design engineer as the lone expert having a vast experience with similar designs used in similar environments. A facilitator was appointed by the design team to obtain the prior failure rate estimates of the new system design in each step of the accelerated life test and the uncertainty in the expert estimates. These estimates were obtained indirectly and converted to numbers of failures per $10^6$ hours. The prior failure rate estimates in each step of the accelerated life test and 95% quantile of the failure rate at use-stress are given in Table 2. All failure rate estimates in this table are in failures per $10^6$ hours.

As discussed in section 2.2, the values in Table 2 are interpreted as prior median estimates of the failure rate. It follows that the point estimate for the life of the system at use-stress is given by the prior median estimate of $(50.36*10^{-6})^{-1}$= 19857 hours or approximately 2.25 years. The total length of the proposed life test equals 600 hours or 25 days. The graph of the prior failure rate associated with the test plan and the estimates in Table 2 is given in Figure 3.

TABLE 2

Prior Step Failure Rate Estimates

| STEP | PRIOR FAIL. RATE |
|---|---|
| 1 | 50.36 |
| 2 | 109.95 |
| 3 | 573.23 |
| 4 | 1428.83 |
| 5 | 3780.97 |
| 95% Quantile of Failure Rate at Use-Stress: 1315.20 | |

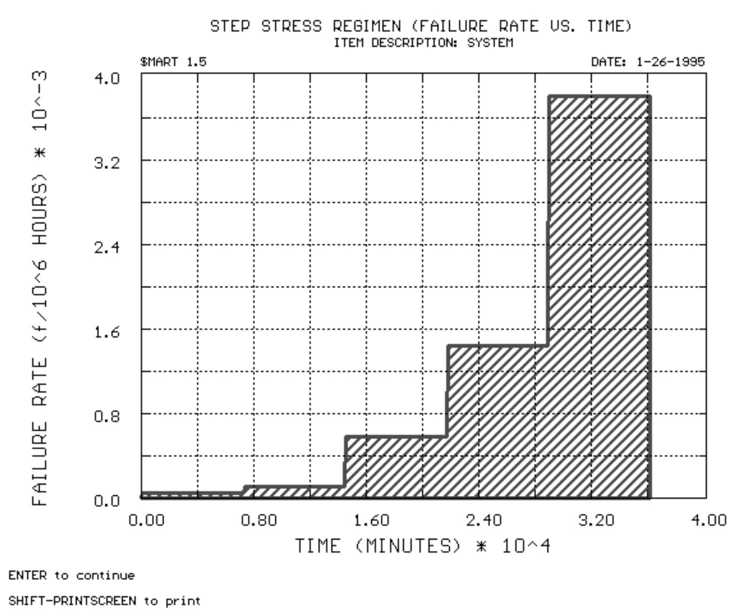


Figure 3. Prior Failure Rate for Items Subjected to Step-Stress Plan Given in Table 1.

As indicated in section 2, the prior parameters of (3) can be obtained from Table 2 and the 95% quantile of the use-stress failure rate estimated by the expert. The value of c is determined to be 841.61. The resulting parameter values for $\beta$ and $\alpha_i$ are

$$\beta = 1.6589 \; ; \; \alpha_0 = 0.1525 \; ; \; \alpha_1 = 0.0481; \; \alpha_2 = 0.2196;$$

$$\alpha_3 = 0.2165 \; ; \; \alpha_4 = 0.2108 \; ; \; \alpha_5 = 0.1525$$

Having obtained the prior parameters of (3), the marginal prior distribution of the failure rate at use-stress level is obtained as discussed in section 4. Figure 4 displays the prior cdf for the failure rate at the use-stress level for this example. In addition, the prior median and the 0.95 prior quantile are indicated.

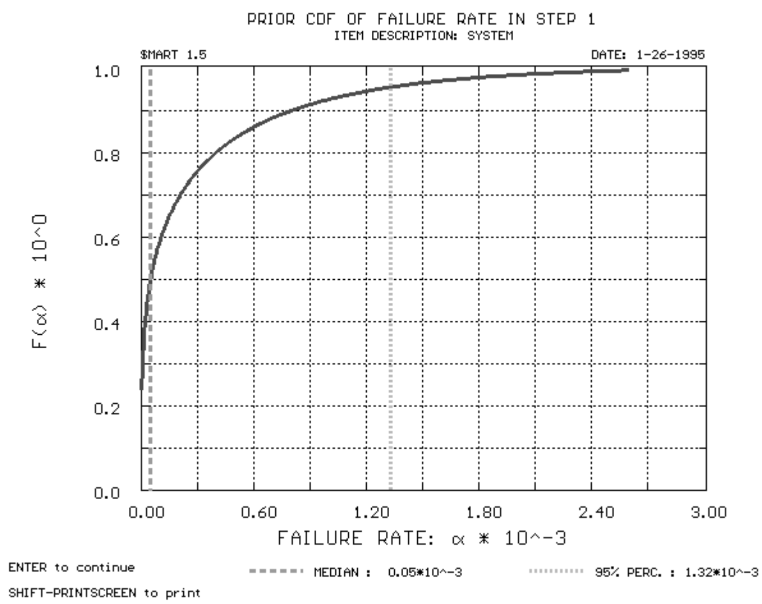


Figure 4. Prior cdf for the Failure Rate at the Use-Stress Level.

In this test, 12 proof systems were subjected to the simultaneous voltage and temperature step stresses as indicated in Table 1. The test results in Table 3 were obtained. The posterior marginal distribution at the use-stress level given this data can be calculated as discussed in section 4.[3] Figure 5 contains a comparison plot of the prior and the posterior cdf. In addition, the posterior median and 95% posterior quantile may be calculated at the use-stress level.

TABLE 3

Step-Stress Test Results

| STEP | NUMBER AT RISK($n_i$) | NUMBER OF FAILURES($s_i$) |
|---|---|---|
| 1 | 12 | 0 |
| 2 | 12 | 0 |
| 3 | 12 | 0 |
| 4 | 12 | 0 |
| 5 | 12 | 1 |

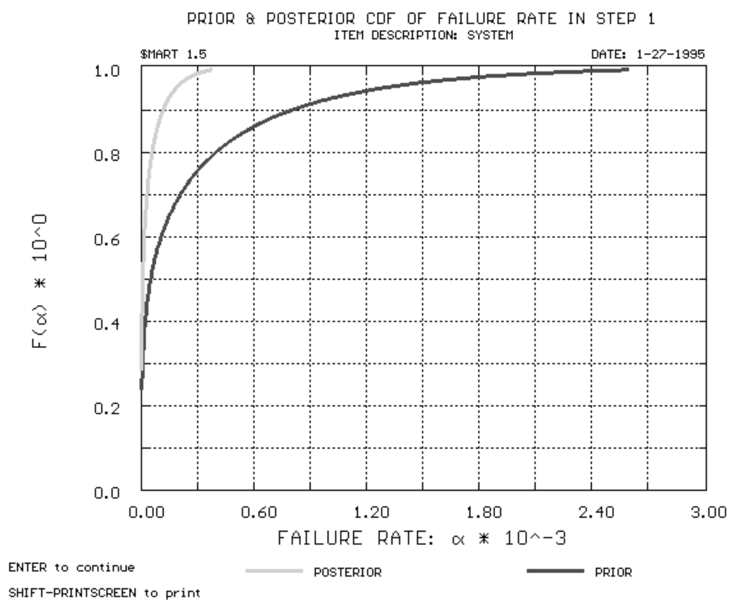


Figure 5. Prior and Posterior cdf for the Failure Rate at the Use-stress Level.

[3] Due to the small sample size and lack of failures in the test results, a non-Bayes approach for inference would be difficult.

The posterior median failure rate at use-stress equals 7.43 per $10^6$ hours. The 95% posterior quantile equals 190.30 per $10^6$ hours. Given the test data, it may thus be concluded that the prior failure rate at the use-stress level was overestimated.

$MART was used to obtain posterior median estimates for the remaining steps in the accelerated life test. Table 4 contains a comparison of the prior median and posterior median values of the failure rates in each step. Again all failure rate estimates in this table are in failures per $10^6$ hours.

TABLE 4

Prior and Posterior Failure Rate Estimates

| STEP | PRIOR FAIL. RATE | POST FAIL. RATE |
|---|---|---|
| 1 | 50.36 | 7.43 |
| 2 | 109.95 | 16.60 |
| 3 | 573.23 | 98.58 |
| 4 | 1428.83 | 261.42 |
| 5 | 3780.97 | 785.55 |
| | 95% Quantile: 1315.20 | 95% Quantile: 190.30 |

Question A is answered using Table 4. Realizing that the conditions described in Question A coincide with the environmental conditions in Step 2, it follows using the constant failure rate assumption and the posterior median failure rate in Step 2 that an estimate of

$$\Pr\left\{ \text{An item survives one hour in } E_2 \right\} = e^{(-16.60*10^{-6})} \approx 0.99998.$$

Therefore, Question A is answered affirmatively in light of the current accumulated information (expert judgment and the test results ).

Using (5), the posterior distribution of the system survivability for a mission of a 1000 hours at use-stress can be obtained. From this distribution, the posterior s-expected mission survivability as well as one sided credibility limits for the mission survivability are obtained and given in Table 5.

Questions B and C are answered using Table 5 where it follows that the s-expected survival probability equals 0.9638. Hence, Question B is answered affirmatively. To answer question C, from Table 5 it follows that:

$$\Pr\left\{ \Pr\left\{ \text{system life at use-stress} > 1000 \text{ hours} \right\} > 0.8873 \right\} = 0.90 \Leftrightarrow$$
$$\Pr\left\{ \Pr\left\{ \text{system life at use-stress} < 1000 \text{ hours} \right\} < 0.1137 \right\} = 0.10.$$

Thus, there is a 10% chance that less than approximately 11% of the systems will fail in 1000 hours of normal use. Similarly, there is only a 5 % chance that less than approximately 17% of the systems will fail in 1000 hours at use-stress.

TABLE 5

Posterior One-Sided Credibility Limits for 1000 Hour Mission Survivability

$\Pr\left\{\text{Mission Survival Probability} > \text{Lower Limit}\right\} = P$

| **Lower Limit** | **P** |
|---|---|
| 0.9926 | 0.50 |
| 0.9585 | 0.75 |
| 0.8873 | 0.90 |
| 0.8267 | 0.95 |
| 0.6896 | 0.99 |
| S-Expected Survival Probability After 1000 Hours in Step 1: 0.9638 | |

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