1. (5 pts. each) Describe a regular expression of the language accepted by each of the following deterministic finite state automaton.

(a)

Sol: \((a \cup ba \cup bba)^* \cup b \cup bb\)

(b)

Sol: \((a(ab)^*b)^*\)

2. (5 pts. each) Give a deterministic finite state automaton accepting each of the following languages over the alphabet \(\{0, 1\}\). (You only need to show a state diagram.)

(a) The set of all strings such that every block of three consecutive symbols contains at least two 0's.
(b) The set of all strings not containing 101 as a substring.

3. (5 pts. each) Consider the language \( L = (ab \cup aba)^* \).

(a) Give a deterministic finite state automaton accepting \( L \).

(b) Give a non-deterministic finite state automaton with three states that is equivalent to your DFA in (a).
4. (10 pts. each) Prove or disprove the following for regular expressions $r$, $s$, and $t$.

(a) $(r \cup s)^* = r^* \cup s^*$.

Sol: Let $r = aa$ and $s = b$, and $w = aabaa$. We then note that $w \in (aa \cup b)^*$, but $w \notin (aa)^* \cup b^*$.

(b) $(rs \cup r)^* = r(sr \cup r)^*$.

Sol: Note that $l.h.s. = \cup_{k \geq 0}(rs \cup r)^k r$ and $r.h.s. = \cup_{k \geq 0} r(sr \cup r)^k$. We will next prove that $(rs \cup r)^k r = r(sr \cup r)^k$ for each $k \geq 0$ using induction on $k$.

As a basis case when $k = 0$, $l.h.s. = r$ and $r.h.s. = r$; hence, the claim holds. Now assume that $(rs \cup r)^k r = r(sr \cup r)^k$ for each $0 \leq k \leq l$, and consider the case when $k = l + 1$. We then have $l.h.s. = (rs \cup r)^{l+1} r = (rs \cup r)(rs \cup r)^l r$ which then implies by the induction hypothesis that $l.h.s. = (rs \cup r)(sr \cup r)^l$. Since $(rs \cup r)r = r(sr \cup r)$, $l.h.s. = r(sr \cup r)^{l+1}$, which completes the proof of induction. Therefore, $(rs \cup r)^k r = r(sr \cup r)^k$ for each $k \geq 0$. The claim is now proved.

5. (10 pts. each) Prove or disprove that following.

(a) $L = \{xwx^R \mid x, w \in \{0,1\}^+\}$ is regular. (Note that $x^R$ is $x$ written backward.)

Sol: Note that $L = (0(0 \cup 1)^* 0 \cup 1(0 \cup 1)^*)^*$; hence, it is regular.

(b) $L = \{0^m 1^n 0^{m+n} \mid m, n \geq 1\}$ is regular.

Sol: It is not regular. Suppose it was, and consider a string $w = 0^p 10^{p+1}$, where $p$ is the pumping length. Then, $w$ can be written as $xyz$ where $x = 0^p$, $y = 0^p$, and $z = 0^p 10^p$ with $b > 0$. We then note that $xy^0z \notin L$ contradicting to the Pumping lemma.

6. (10 pts. each) For each of the following languages, show that it is not regular. (Note that you may directly apply the Pumping lemma or may use the closure properties of the regular languages.)

(a) $L = \{a^n b^n a^k \mid k \geq n + l\}$

Sol: It is not regular. Suppose it was, and consider a string $w = a^p ba^{p+1}$ where $p$ is the pumping length. It is easy to see that $xy^0z \notin L$. Hence, $L$ is not regular.

(b) $L = \{a^n b^n a^k \mid n = l \text{ or } l \neq k\}$.

Sol: It is not regular. Suppose it was, and consider a string $w = a^p b^p a^p \in L$. $w$ can then be written as $xyz$ where $x = a^l$, $y = a^l$, and $z = a^k b^p a^p$. We then note that $xy^0z = a^l a^k b^p a^p \notin L$. Hence, $L$ is not regular.

7. (10 pts.) Show that regular languages are closed under difference, i.e., if $L_1$ and $L_2$ are regular languages, then $L_1 - L_2$ is necessarily regular.

Sol: Note that $L_1 - L_2 = L_1 \cap \bar{L}_2$. Since regular languages are closed under complement, $\bar{L}_2$ is regular. Since regular languages are closed under intersection, $L_1 \cap \bar{L}_2$ is also regular.